THE GRAVITATIONAL AND ELECTROMAGNETIC FIELDS DUE TO ROTATING BODIES IN GENERAL RELATIVITY

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ABSTRACT

A metric which is supposed to describe the field due to bodies in stationary rotation about their axes is considered, and it is shown that in canonical co-ordinates this metric is incompatible with the perfect fluid hypothesis. The metric in this case is, however, compatible with a stationary electromagnetic field which, in the opinion of the authors, seems to support Blackett's hypothesis. Further, in this case we get a consistent set of equations for determining the metric potentials in empty space-time. A class of exact solutions of these equations is obtained. It is also shown that if canonical co-ordinates are not used the metric can represent perfect fluid distributions.

1. INTRODUCTION

Astronomical observations show that a large number of heavenly bodies are in a state of stationary rotation about their axes. Any solution of the field equations of General Relativity which is to describe their field must, therefore, take into account the effects of their rotation. One of the most interesting effects of rotation is that the radial character of the field is destroyed. i.e. a test particle if left to itself anywhere in the field does not move radially.

The first attempt to obtain the gravitational field due to a rotating body was made by Thirring (1918, 1921) who used Einstein's field equations in the linear approximation and showed that a rotating thin spherical shell produces near its centre forces analogous to the Coriolis and centrifugal forces of classical mechanics. He assumed the shell to consist of incoherent material. To the order of approximation considered by him, the conservation law $T^{\mu\nu}_{;\nu} = 0$ is not satisfied. If, however, we assume that the conservation law is satisfied the incoherent particles will not move in circular paths but will follow geodesics. Hence the particles of the shell, if set in rotation, will fly apart. This inconsistency in Thirring's work was resolved, on a suggestion of C. Lanczos, by Bass and Pirani (1955) who supplemented the energy tensor of incoherent material by a term representing the elastic interaction between the particles of the shell.

Bach (1922) considered the field due to a slowly rotating sphere by successive approximations taking the Schwarzschild solution as his zeroth approximation. K. Lanczos (1924) considered a special case of stationary fields and

applied his results to cosmological problems. Andress (1930) and Akeley (1931*a*, *b*) have considered the more general case of gravitational fields produced by matter in stationary rotation and obtained approximate solutions of the problem. Lewis (1932) obtained the field due to a rotating infinite cylinder. He also obtained two methods of successive approximations for constructing solutions of a more general type which behave in an assigned manner at infinity and on a surface of revolution enclosing the rotating matter to which the field is due.

Clark (1950) tried to solve the empty gravitational field equations, using successive approximations, with forms of $g_{\mu\nu}$ appropriate to the gravitational field of a rotating body. The forms contain disposable constants, but these cannot be so chosen as to satisfy the equations. He then examined the value of $T_{\mu\nu}$ given by the selected $g_{\mu\nu}$ and this he was able to interpret, though not uniquely, as the energy tensor of an electromagnetic field. This result in his opinion established Blackett's (1947) hypothesis that 'a rotating body produces an electromagnetic field'. This assertion of Clark was objected to by Synge (1951) 'in view of the special forms considered and his own uncertainty as to what rotation means in relativity'.

In the present paper the authors consider a metric which is supposed to describe the field due to bodies in stationary rotation. This metric was earlier considered, among others, by Andress and Lewis. We show that:

(i) if, following Lewis, canonical co-ordinates be introduced the metric becomes incompatible with the perfect fluid hypothesis. However, from considerations of the eigenvalues of the Einstein tensor, it is shown that the eigenvalues can be reduced to the form k, k, $-k$, $-k$ and thus the metric in canonical co-ordinates is compatible with a stationary electromagnetic field. In the opinion of the authors, this result seems to support Blackett's hypothesis. Further, in this case we get a consistent set of equations for determining $g_{\mu\nu}$ in empty space-time. It is also shown that by choosing the forms of $g_{\mu\nu}$ suitably the problem of solving four equations in three unknowns reduces to that of two equations in two unknowns;

(ii) if canonical co-ordinates are not used all the four eigenvalues of the Einstein tensor are different and therefore the metric can, in this case, be used to represent perfect fluid distributions.

A class of exact solutions of the empty gravitational field equations is obtained.

2. Nature of the Field

A metric which is supposed to describe the field due to bodies in stationary rotation about their axes is

$$ds^2 = -e^{2\lambda}(d\rho^2+dz^2)-C d\phi^2+D\,dt^2+2E d\phi\,dt, \tag{2.1}$$

where λ, C, D and E are functions of ρ and z only. Here ρ, z, ϕ and t correspond respectively to x^1, x^2, x^3 and x^4.

The expressions for the Γ-symbols and the Ricci tensor corresponding to (2.1) are given in the appendix for convenience. Following Lewis (1932) we make use of the canonical co-ordinates in the sense of Weyl. The choice of these co-ordinates is possible only in matter-free space as can easily be verified by a procedure similar to that of Synge (1960). Consequently in domains occupied by matter the canonical co-ordinates cannot be used. We shall reach this very conclusion by a consideration of the eigenvalues of the Einstein tensor in the sequel. In canonical co-ordinates we have

$$CD+E^2=\rho^2, \tag{2.2}$$

and consequently the expressions for the non-vanishing components of the Ricci tensor reduce to

$$\left.\begin{aligned}
R_{11} &= \lambda_{11}+\lambda_{22}-\frac{\lambda_1}{\rho}-\frac{C_1D_1+E_1^2}{2\rho^2},\\
R_{22} &= \lambda_{11}+\lambda_{22}+\frac{\lambda_1}{\rho}-\frac{C_2D_2+E_2^2}{2\rho^2},\\
R_{33} &= \tfrac{1}{2}e^{-2\lambda}\left[C_{11}+C_{22}-\frac{C_1}{\rho}+\frac{C}{\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)\right],\\
R_{44} &= -\tfrac{1}{2}e^{-2\lambda}\left[D_{11}+D_{22}-\frac{D_1}{\rho}+\frac{D}{\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)\right],\\
R_{34} &= -\tfrac{1}{2}e^{-2\lambda}\left[E_{11}+E_{22}-\frac{E_1}{\rho}+\frac{E}{\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)\right],\\
R_{12} &= -\frac{\lambda_2}{\rho}-\frac{C_1D_2+C_2D_1+2E_1E_2}{4\rho^2};
\end{aligned}\right\} \tag{2.3}$$

whereas the curvature scalar is given by

$$R=-e^{-2\lambda}\left[2\lambda_{11}+2\lambda_{22}-\frac{C_1D_1+C_2D_2+E_1^2+E_2^2}{2\rho^2}\right]. \tag{2.4}$$

Throughout the paper the lower suffixes 1 and 2 after the unknown functions imply a partial differentiation with respect to ρ and z respectively.

The expressions for the Einstein tensor are found from (2·3) and (2·4) to be

$$\left.\begin{aligned}
G_{11} &= -G_{22}=-\frac{\lambda_1}{\rho}-\frac{C_1D_1+E_1^2-C_2D_2-E_2^2}{4\rho^2},\\
G_{33} &= \tfrac{1}{2}e^{-2\lambda}\left[-2C(\lambda_{11}+\lambda_{22})+C_{11}+C_{22}-\frac{C_1}{\rho}+\frac{3C}{2\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)\right],\\
G_{44} &= \tfrac{1}{2}e^{-2\lambda}\left[2D(\lambda_{11}+\lambda_{22})-D_{11}-D_{22}+\frac{D_1}{\rho}-\frac{3D}{2\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)\right],\\
G_{34} &= \tfrac{1}{2}e^{-2\lambda}\left[2E(\lambda_{11}+\lambda_{22})-E_{11}-E_{22}+\frac{E_1}{\rho}-\frac{3E}{2\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)\right],\\
G_{12} &= -\frac{\lambda_2}{\rho}-\frac{C_1D_2+C_2D_1+2E_1E_2}{4\rho^2}.
\end{aligned}\right\} \tag{2.5}$$

Let us now consider the determinantal equation

$$|G_{\mu\nu}-sg_{\mu\nu}| = 0. \tag{2.6}$$

We find that two of the eigenvalues of $G_{\mu\nu}$ with respect to $g_{\mu\nu}$ are given by

$$s_i = \pm e^{-2\lambda}\left[G_{22}^2+G_{12}^2\right]^{\frac{1}{2}}, \qquad (i = 1, 2) \tag{2.7}$$

whereas the other two are given by the quadratic equation

$$s^2(g_{33}g_{44}-g_{34}^2)-s(g_{33}G_{44}+g_{44}G_{33}-2g_{34}G_{34})+G_{33}G_{44}-G_{34}^2 = 0. \tag{2.8}$$

This equation can also be written as

$$s^2+Rs-\frac{1}{\rho^2}(G_{33}G_{44}-G_{34}^2) = 0, \tag{2.9}$$

where R is the curvature scalar defined by (2.4). It follows from (2.7) and (2.9) that, in general, two of the eigenvalues of $G_{\mu\nu}$ are equal and opposite while the other two are different. Consequently the metric (2.1) in canonical co-ordinates cannot represent a perfect fluid distribution. However, if we do not utilize the canonical co-ordinate system, that is in the absence of the condition (2.2), all the eigenvalues of the Einstein tensor are, in general, different as can easily be verified from the expressions of the Ricci tensor given in the appendix. It follows, therefore, that the metric (2.1) in the absence of (2.2) can be used to describe the space-time in the domains occupied by matter. It seems as if the use of canonical co-ordinates for describing the field due to axially symmetric bodies, rotating or otherwise, rules out the compatibility of such fields with the perfect fluid hypothesis.

In the case of an electromagnetic field

$$R = 0, \tag{2.10}$$

and, therefore,

$$S_j = \pm\frac{1}{\rho}(G_{33}G_{44}-G_{34}^2)^{\frac{1}{2}}, \ (j = 3, 4). \tag{2.11}$$

Thus in this case the other two eigenvalues are also equal and opposite. In fact the eigenvalues are:

$$k, -k, m, -m, \tag{2.12}$$

where

$$k = e^{-2\lambda}(G_{22}^2+G_{12}^2)^{\frac{1}{2}}, \quad m = \frac{1}{\rho}(G_{33}\,G_{44}-G_{34}^2)^{\frac{1}{2}}. \tag{2.13}$$

If, in addition, we put

$$k = m, \tag{2.14}$$

they become

$$k, k, -k, -k. \tag{2.15}$$

which characterize an electromagnetic field. Since in order to reduce the eigenvalues to the form (2.15) we have imposed only three conditions, i.e. (2.2), (2.10) and (2.14), on the four unknowns it follows that this reduction is always possible. We can, thus, say that the metric (2.1) in canonical co-ordinates is always compatible with the stationary electromagnetic field. This investigation, in fact, supports Blackett's hypothesis that 'a rotating body produces an electromagnetic field'. It may be noted here that for obtaining this result we have not assumed the form of the metric potentials as was the case with Clark.

3. The Field Equations for Empty Space-time

The field equations for empty space-time are found from (2.3) to be given by

$$\lambda_{11}+\lambda_{22}-\frac{\lambda_1}{\rho}=\frac{C_1D_1+E_1^2}{2\rho^2}, \tag{3.1}$$

$$\lambda_{11}+\lambda_{22}+\frac{\lambda_1}{\rho}=\frac{C_2D_2+E_2^2}{2\rho^2}, \tag{3.2}$$

$$\lambda_2=-\frac{1}{4\rho}(C_1D_2+C_2D_1+2E_1E_2), \tag{3.3}$$

$$C_{11}+C_{22}-\frac{C_1}{\rho}+\frac{C}{\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)=0, \tag{3.4}$$

$$D_{11}+D_{22}-\frac{D_1}{\rho}+\frac{D}{\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)=0, \tag{3.5}$$

$$E_{11}+E_{22}-\frac{E_1}{\rho}+\frac{E}{\rho^2}(C_1D_1+C_2D_2+E_1^2+E_2^2)=0. \tag{3.6}$$

From (3.1) and (3.2) we get

$$\lambda_{11}+\lambda_{22}=\frac{C_1D_1+C_2D_2+E_1^2+E_2^2}{4\rho^2}, \tag{3.7}$$

$$\lambda_1=\frac{C_2D_2+E_2^2-C_1D_1-E_1^2}{4\rho}. \tag{3.8}$$

It can easily be verified that the condition of integrability of (3.3) and (3.8), i.e.

$$\lambda_{21}=\lambda_{12},$$

is satisfied in view of (2.2). Also (3.7) is consistent with (3·3) and (3.8) in view of (2.2). Thus the problem of solving the equations (2.2), (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) reduces to that of determining C, D and E from (2.2), (3.4), (3.5) and (3.6). λ will then be given by (3.3) and (3.8) subject to an additive constant. However, we have got four equations to be satisfied by the three unknowns C, D and E and unless they are consistent we cannot

solve them. Our subsequent analysis will show that they are consistent and hence they can be solved.

Let us, for convenience, put

$$C = \rho u,\ D = \rho v,\ E = \rho w, \tag{3.9}$$

where u, v and w are functions of ρ and z. Then the equations to be satisfied by u, v and w are found from (2.2), (3.4), (3.5), (3.6) and (3.9) to be

$$uv+w^2 = 1, \tag{3.10}$$

$$u_{11}+u_{22}+\frac{u_1}{\rho}+\frac{u}{\rho}P = 0, \tag{3.11}$$

$$v_{11}+v_{22}+\frac{v_1}{\rho}+\frac{v}{\rho}P = 0, \tag{3.12}$$

$$w_{11}+w_{22}+\frac{w_1}{\rho}+\frac{w}{\rho}P = 0, \tag{3.13}$$

where P is given by

$$P = u_1v+uv_1+2ww_1+\rho(u_1v_1+u_2v_2+w_1^2+w_2^2). \tag{3.14}$$

We make the further substitution

$$u = e^{-\mu}\cos\theta,\ v = e^{\mu}\cos\theta,\ w = \sin\theta, \tag{3.15}$$

where μ and θ are new functions of ρ and z. The substitution (3.15) is suggested by

(i) the form of the equation (3.10), and

(ii) the fact that the metric (2.1) must reduce to the static axially symmetric metric of Weyl in the absence of rotation that is when E vanishes.

In view of (3.15) P is given by

$$P = \rho[\theta_1^2+\theta_2^2-\cos^2\theta\,(\mu_1^2+\mu_2^2)], \tag{3.16}$$

and consequently (3.11), (3.12) and (3.13) reduce to

$$\cos\theta\left[\mu_{11}+\mu_{22}+\frac{\mu_1}{\rho}-2\tan\theta(\mu_1\theta_1+\mu_2\theta_2)\right] + \sin\theta\left[\theta_{11}+\theta_{22}+\frac{\theta_1}{\rho}-\sin\theta\cos\theta\,(\mu_1^2+\mu_2^2)\right] = 0, \tag{3.17}$$

$$\cos\theta\left[\mu_{11}+\mu_{22}+\frac{\mu_1}{\rho}-2\tan\theta(\mu_1\theta_1+\mu_2\theta_2)\right] - \sin\theta\left[\theta_{11}+\theta_{22}+\frac{\theta_1}{\rho}-\sin\theta\cos\theta\,(\mu_1^2+\mu_2^2)\right] = 0, \tag{3.18}$$

$$\theta_{11}+\theta_{22}+\frac{\theta_1}{\rho}-\sin\theta\cos\theta\,(\mu_1^2+\mu_2^2) = 0. \tag{3.19}$$

From (3.17) and (3.18) we get, in view of (3.19), the single equation:

$$\mu_{11}+\mu_{22}+\frac{\mu_1}{\rho}-2\tan\theta(\mu_1\theta_1+\mu_2\theta_2)=0. \tag{3.20}$$

Thus finally we have reduced the problem of solving four equations in three unknowns to that of solving the two equations (3.19) and (3.20) in the two unknowns θ and μ. Consequently knowing θ and μ from these we can determine C, D and E from (3.9) and (3.15). Also (3.3) and (3.8) can, in view of (3.9) and (3.15), be written as:

$$\lambda_1=-\frac{1}{4\rho}-\frac{\rho}{4}[\theta_1^2-\theta_2^2-\cos^2\theta(\mu_1^2-\mu_2^2)], \tag{3.21}$$

$$\lambda_2=\frac{\rho}{2}[\cos^2\theta\mu_1\mu_2-\theta_1\theta_2], \tag{3.22}$$

from which λ can be determined.

4. Solutions of the Field Equations for Empty Space-time

Consider the field equations (3.19), (3.20), (3.21) and (3.22). If we put

$$\theta=0, \tag{4.1}$$

we find that the metric (2.1) reduces to the static axially symmetric metric of Weyl in canonical co-ordinates. Several exact solutions of the empty gravitational field equations for this metric are well known. One of the authors (Misra 1960, 1961) in two earlier papers obtained several exact solutions of these equations corresponding to the external gravitational fields due to oblate spheroidal and toroidal distributions, using the oblate spheroidal and toroidal co-ordinates associated with the canonical co-ordinates of Weyl.

If, however, we put

$$\mu=0, \tag{4.2}$$

the metric (2.1) reduces to

$$ds^2=-e^{2\lambda}(d\rho^2+dz^2)-\rho\cos\theta(d\phi^2-dt^2)+2\rho\sin\theta\,d\phi\,dt, \tag{2.1a}$$

with the corresponding empty space-time field equations:

$$\lambda_1=-\frac{1}{4\rho}-\frac{\rho}{4}(\theta_1^2-\theta_2^2), \tag{4.3}$$

$$\lambda_2=-\frac{\rho}{2}\theta_1\theta_2, \tag{4.4}$$

$$\theta_{11}+\theta_{22}+\frac{\theta_1}{\rho}=0. \tag{4.5}$$

Equation (4.5) is the Laplace equation in cylindrical polar co-ordinates whose solutions are well known. Thus for every value of θ satisfying (4.5)

we shall have a value of λ given by (4.3) and (4.4). We have thus obtained a class of exact solutions of the empty gravitational field equations corresponding to the metric (2.1*a*).

In the general case, when neither θ nor μ is zero, we are faced with the problem of solving (3.19) and (3.20) for θ and μ. The linear part of these equations is merely the Laplacian of θ and μ but the non-linear part is quite complicated.

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Appendix

The non-vanishing independent Γ-symbols for the metric (2.1) are:

$$\Gamma^1_{11} = \lambda_1, \qquad \Gamma^2_{12} = \lambda_1, \qquad \Gamma^3_{14} = \frac{D_1E - DE_1}{2(CD+E^2)},$$

$$\Gamma^2_{11} = -\lambda_2, \quad \Gamma^3_{13} = \frac{C_1D + EE_1}{2(CD+E^2)}, \quad \Gamma^4_{14} = \frac{CD_1 + EE_1}{2(CD+E^2)},$$

$$\Gamma^1_{12} = \lambda_2, \qquad \Gamma^4_{13} = \frac{CE_1 - C_1E}{2(CD+E^2)}, \quad \Gamma^1_{22} = -\lambda_1,$$

$$\Gamma^2_{22} = \lambda_2, \; \Gamma^4_{24} = \frac{CD_2 + EE_2}{2(CD+E^2)}, \; \Gamma^2_{34} = \frac{E_2}{2}e^{-2\lambda},$$

$$\Gamma^3_{23} = \frac{C_2D + EE_2}{2(CD+E^2)}, \; \Gamma^1_{33} = -\frac{C_1}{2}e^{-2\lambda}, \; \Gamma^1_{44} = \frac{D_1}{2}e^{-2\lambda},$$

$$\Gamma^4_{23} = \frac{CE_2 - C_2E}{2(CD+E^2)}, \; \Gamma^2_{33} = -\frac{C_2}{2}e^{-2\lambda}, \; \Gamma^2_{44} = \frac{D_2}{2}e^{-2\lambda};$$

$$\Gamma^3_{24} = \frac{D_2E - DE_2}{2(CD+E^2)}, \; \Gamma^1_{34} = \frac{E_1}{2}e^{-2\lambda},$$

and the non-vanishing components of the Ricci tensor are:

$$R_{11} = \lambda_{11}+\lambda_{22}+\frac{1}{2(CD+E^2)}[C_{11}D+CD_{11}+2EE_{11}-\lambda_1(C_1D+CD_1+2EE_1)$$
$$+\lambda_2(C_2D+CD_2+2EE_2)]$$
$$-\frac{1}{4(CD+E^2)^2}[C_1^2D^2+C^2D_1^2+2E^2E_1^2-2CDE_1^2-2C_1D_1E^2-4(CD_1+C_1D)EE_1],$$

$$R_{22} = \lambda_{11}+\lambda_{22}+\frac{1}{2(CD+E^2)}[C_{22}D+CD_{22}+2EE_{22}+\lambda_1(C_1D+CD_1+2EE_1)$$
$$-\lambda_2(C_2D+CD_2+2EE_2)]$$
$$-\frac{1}{4(CD+E^2)^2}[C_2^2D^2+C^2D_2^2+2E^2E_2^2-2CDE_2^2-2C_2D_2E^2-4(CD_2+C_2D)EE_2],$$

$$R_{33} = \frac{1}{2}e^{-2\lambda}\Big[C_{11}+C_{22}-\frac{1}{2(CD+E^2)}\{C_1(C_1D-CD_1)+2E_1(C_1E-CE_1)$$
$$+C_2(C_2D-CD_2)+2E_2(C_2E-CE_2)\}\Big]$$

$$R_{44} = -\frac{1}{2}e^{-2\lambda}\Big[D_{11}+D_{22}-\frac{1}{2(CD+E^2)}\{D_1(CD_1-C_1D)+2E_1(D_1E-DE_1)$$
$$+D_2(CD_2-C_2D)+2E_2(D_2E-DE_2)\}\Big],$$

$$R_{12} = \frac{1}{2(CD+E^2)}[C_{12}D+CD_{12}+2EE_{12}-\lambda_1(C_2D+CD_2+2EE_2)$$
$$-\lambda_2(C_1D+CD_1+2EE_1)]$$
$$-\frac{1}{4(CD+E^2)^2}[C^2D_1D_2+D^2C_1C_2+2E^2E_1E_2-2CDE_1E_2-C_1D_2E^2-C_2D_1E^2$$
$$+2C_1DEE_2+2CD_1EE_2+2C_2DEE_1+2CD_2EE_1],$$

$$R_{34} = -\tfrac{1}{2}e^{-2\lambda}\Big[E_{11}+E_{22}-\frac{1}{2(CD+E^2)}\{E_1(C_1D+CD_1)-2EC_1D_1$$
$$+E_2(C_2D+CD_2)-2EC_2D_2\}\Big].$$