OPTIMUM TRAJECTORIES OF A ROCKET FOR MAXIMUM RANGE AND MINIMUM TIME

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ABSTRACT

Optimum trajectories of a rocket for maximum range and minimum time of flight have been obtained by using the method of calculus of variation. The equations determining the optimum trajectories have been established for the general case and exact solutions of Euler-Lagrange equations have been obtained for a few simpler cases for time minimization.

1. INTRODUCTION

In recent years, variational techniques have been extensively applied to optimization problems in rocket ballistics. The approach adopted in most of the literature on such types of problems is based largely on Mayer-type formulation. This approach can, however, be modified to lead to a simple numerical scheming by coupling the dynamical equations of motion along and normal to the path and thereby effecting reduction in the number of Lagrange multipliers. Recently this simpler formulation has been successfully employed by Theodorson (1959) to determine the minimum-time path to climb for an aeroplane. In this paper, optimum trajectories of a rocket for maximum range and minimum time have been found basing the analysis on this modified approach. The equations determining the optimum trajectory have been established for the general case and some exact solutions of Euler-Lagrange equations have been obtained for a few simpler cases for time minimization.

2. DYNAMICAL EQUATIONS OF MOTION AND KINEMATIC RELATIONS

The equations of motion of a rocket taking into account gravity variation and aerodynamic forces are

$$m\frac{dv}{dt} = T\cos\alpha - mg\left(\frac{R}{R+h}\right)^2 \sin\theta - D(h, v) \qquad \text{(1)}$$

$$mv\frac{d\theta}{dt} = T\sin\alpha - mg\left(\frac{R}{R+h}\right)^2 \cos\theta + L(h, v) + \frac{mv^2}{R+h}\cos\theta \qquad \text{(2)}$$

and the kinematic relations are

$$\frac{dx}{dt} = \frac{R}{R+h} v \cos \theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\frac{dh}{dt} = v \sin \theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where

T = rocket thrust.
v = velocity of the rocket.
m = instantaneous mass of the rocket.
α = angle between thrust vector and velocity vector.
θ = angle between velocity vector and the local horizontal.
R = earth's radius.
g = value of gravity at the surface of earth.
h = altitude from the surface of earth.
x = curvilinear co-ordinate measured along the surface of earth.

In this paper, the quantities with subscript 0 and f refer to their values at the initial and terminal points of the path. The dot and dash denote differentiation with respect to time t and altitude h respectively.

3. Range Maximization

In this section we derive a set of simultaneous differential equations of the trajectory along which the range measured on the surface of the earth is maximum. From equation (3), the integral to be extremized subject to (1), (2) and (4) as constraints is

$$x_f - x_0 = \int_{t_0}^{t_f} \frac{R}{R+h} v \cos \theta \,.\, dt. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

The dynamical equations (1) and (2) can be coupled by eliminating α thereby giving a relation in v and θ and their time derivatives. Introducing the Lagrange multipliers λ_1 and λ_2 (unspecified functions of time), the problem reduces to extremizing the integral

$$\int_{t_0}^{t_f} F(v, \dot{v}, \theta, \dot{\theta}, h, \dot{h}) dt \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where

$$F \equiv \frac{R}{R+h} v \cos \theta + \lambda_1 (\dot{h} - v \sin \theta) + \lambda_2 \left[m v \dot{\theta} - T \sin \alpha + mg \left(\frac{R}{R+h} \right)^2 \cos \theta - \frac{m v^2 \cos \theta}{R+h} - L \right] \ldots \quad \ldots \quad (7)$$

in which α is to be regarded as a function of v, $\dot{v}$, θ and h as given by equation (1).

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right) - \frac{\partial F}{\partial v} = 0. \qquad \text{.. .. .. ..} \quad (8)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\theta}}\right) - \frac{\partial F}{\partial \theta} = 0. \qquad \text{.. .. .. ..} \quad (9)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{h}}\right) - \frac{\partial F}{\partial h} = 0. \qquad \text{.. .. .. ..} \quad (10)$$

Carrying out the partial differentiation of F with respect to v, $\dot{v}$, θ, $\dot{\theta}$, h, $\dot{h}$ and using the fact that α as given by equation (1) is a function of v, $\dot{v}$, θ and h the above equations become

$$\alpha_1\dot{\lambda}_2+\beta_1\lambda_2+\gamma_1 = 0 \qquad \text{.. .. .. ..} \quad (11)$$

$$\alpha_2\dot{\lambda}_2+\beta_2\lambda_2+\gamma_2 = 0 \qquad \text{.. .. .. ..} \quad (12)$$

$$\dot{\lambda}_1+\beta_3\lambda_2+\gamma_3 = 0 \qquad \text{.. .. .. ..} \quad (13)$$

where

$$\alpha_1 = m\cot\alpha, \quad \alpha_2 = mv.$$

$$\beta_1 = \dot{m}\cot\alpha - m\dot{\alpha}\operatorname{cosec}^2\alpha - m\dot{\theta} - \frac{\partial D}{\partial v}\cot\alpha + \frac{2mv}{R+h}\cos\theta + \frac{\partial L}{\partial v}.$$

$$\beta_2 = \dot{m}v + m\dot{v} - \frac{mv^2}{R+h}\sin\theta - mg\left(\frac{R}{R+h}\right)^2(\cot\alpha\cos\theta - \sin\theta).$$

$$\beta_3 = 2mg\frac{R^2}{(R+h)^3}(\cot\alpha\sin\theta + \cos\theta) - \frac{mv^2}{(R+h)^2}\cos\theta - \frac{\partial D}{\partial h}\cot\alpha.$$

$$\gamma_1 = \lambda_1\sin\theta - \frac{R}{R+h}\sin\theta = \gamma_{10} + \lambda_1\sin\theta,\ \gamma_{10} = -\frac{R}{R+h}\sin\theta.$$

$$\gamma_2 = \lambda_1 v\cos\theta - \frac{R}{R+h}v\sin\theta = \gamma_{20} + \lambda_1 v\cos\theta,\ \gamma_{20} = -\frac{R}{R+h}v\sin\theta.$$

$$\gamma_3 = \frac{R}{(R+h)^2}v\cos\theta.$$

Eliminating λ_2 between equations (11) and (12) and also between equations (12) and (13), we get

$$b_1\dot{\lambda}_1 + b_2\lambda_1 + b_3 = 0 \qquad \text{.. .. .. ..} \quad (14)$$

and

$$a_1\ddot{\lambda}_1 + a_2\dot{\lambda}_1 + a_3\lambda_1 + a_4 = 0 \qquad \text{.. .. .. ..} \quad (15)$$

in which the coefficients a's and b's are given by

$$b_1 = (\alpha_2\sin\theta - \alpha_1 v\cos\theta)/(\alpha_1\beta_2 - \alpha_2\beta_1).$$

$$b_2 = \frac{d}{dt}\left(\frac{\alpha_2\sin\theta - \alpha_1 v\cos\theta}{\alpha_1\beta_2 - \alpha_2\beta_1}\right) - (\beta_1 v\cos\theta - \beta_2\sin\theta)/(\alpha_1\beta_2 - \alpha_2\beta_1).$$

$$b_3 = \frac{d}{dt}\left(\frac{\alpha_2\gamma_{10}-\alpha_1\gamma_{20}}{\alpha_1\beta_2-\alpha_2\beta_1}\right) - (\beta_1\gamma_{20}-\beta_2\gamma_{10})/(\alpha_1\beta_2-\alpha_2\beta_1).$$

$$a_1 = \alpha_2, \quad a_2 = \beta_2 - \alpha_2\frac{\dot{\beta}_3}{\beta_3}.$$

$$a_3 = -\beta_3 v \cos\theta, \quad a_4 = \alpha_2\dot{\gamma}_3 + \beta_2\gamma_3 - \beta_3\gamma_{20} - \alpha_2\gamma_3\frac{\dot{\beta}_3}{\beta_3}.$$

Eliminating λ_1 between equations (14) and (15), we get

$$\frac{(a_2b_2-a_3b_1)(b_2\dot{b}_3-\dot{b}_2b_3)-(a_4b_2-a_3b_3)(\dot{b}_1b_2+b_2^2-b_1\dot{b}_2)}{a_1(\dot{b}_1b_2+b_2^2-b_1\dot{b}_2)-b_1(a_2b_2-a_3b_1)}$$

$$= \frac{d}{dt}\left[\frac{b_1(a_4b_2-a_3b_3)-a_1(b_2\dot{b}_3-\dot{b}_2b_3)}{a_1(\dot{b}_1b_2+b_2^2-b_1\dot{b}_2)-b_1(a_2b_2-a_3b_1)}\right]. \quad \ldots \quad (16)$$

The set of simultaneous non-linear differential equations (1), (2), (4) and (16) can be integrated numerically with the aid of initial conditions on v, h and θ and their time derivatives to yield v, h and θ as functions of time for the maximum-range trajectory.

4. Time Minimization

In this section, the analysis has been carried out to determine the optimum path along which the time of transfer is minimum. In this case we have to extremize the integral

$$t_f - t_0 = \int_{h_0}^{h_f} \frac{dh}{v\sin\theta} \quad \ldots \quad (17)$$

subject to the dynamical relations (1) and (2) as constraints. Introducing the Lagrange multiplier λ the problem reduces to minimizing the integral

$$\int_{h_0}^{h_f} F(v, v', \theta, \theta')dh \quad \ldots \quad (18)$$

where

$$F \equiv \frac{1}{v\sin\theta} + \lambda\left[mvv'\sin\theta - T\cos\alpha + mg\left(\frac{R}{R+h}\right)^2\sin\theta + D(h, v)\right] \quad \ldots \quad (19)$$

in which α is to be regarded as function of v, h, θ and θ' as given by equation (2).

Carrying out the partial differentiation of F with respect to v, v', θ and θ' and using the fact that α is regarded as function of v, h, θ and θ' as given by equation (2), the corresponding Euler-Lagrange equations

$$\frac{d}{dh}\left(\frac{\partial F}{\partial v'}\right) - \frac{\partial F}{\partial v} = 0 \quad \ldots \quad (20)$$

$$\frac{d}{dh}\left(\frac{\partial F}{\partial \theta'}\right)-\frac{\partial F}{\partial \theta}=0 \quad \ldots \quad (21)$$

become

$$l_1\lambda'+m_1\lambda+n_1=0 \quad \ldots \quad (22)$$

$$l_2\lambda'+m_2\lambda+n_2=0 \quad \ldots \quad (23)$$

where

$$l_1=mv\sin\theta,\; l_2=mv^2\tan\alpha\sin\theta.$$

$$m_1=\left[(mv\sin\theta)'-mv'\sin\theta-\frac{\partial D}{\partial v}-\tan\alpha\left(2\,mv\theta'\sin\theta-\frac{2mv}{R+h}\cos\theta-\frac{\partial L}{\partial v}\right)\right].$$

$$m_2=\left[(mv^2\tan\alpha\sin\theta)'-mvv'\cos\theta-mg\left(\frac{R}{R+h}\right)^2\cos\theta\right.$$

$$\left.-\tan\alpha\left\{mv^2\,\theta'\cos\theta-mg\left(\frac{R}{R+h}\right)^2\sin\theta+\frac{mv^2}{R+h}\sin\theta\right\}\right].$$

$$n_1=\frac{1}{v^2\sin\theta},\; n_2=\frac{\cos\theta}{v\sin^2\theta}.$$

Eliminating λ between the equations (22) and (23), we get

$$\frac{m_1n_2-m_2n_1}{l_1m_2-l_2m_1}=\frac{d}{dh}\left(\frac{n_1l_2-n_2l_1}{l_1m_2-l_2m_1}\right). \quad \ldots \quad (24)$$

With the aid of prescribed initial conditions on v, v', θ and θ' and the set of non-linear differential equations (1), (2) and (24) can be integrated numerically to yield the minimum-time trajectory.

The approach adopted in this paper simplifies the programming as the differential equations to be integrated numerically are fewer than in Mayer type formulation.

5. A Few Simple Cases for Time Minimization

Case 1. Constant gravity, flat earth and no aerodynamic forces.

The kinematic relations and the dynamical equations for rocket flight, in this case, become

$$\frac{dx}{dt}=v\cos\theta,\; \frac{dh}{dt}=v\sin\theta. \quad \ldots \quad (25)$$

$$m\frac{dv}{dt}=T\cos\alpha-mg\sin\theta. \quad \ldots \quad (26)$$

$$mv\frac{d\theta}{dt}=T\sin\alpha-mg\cos\theta. \quad \ldots \quad (27)$$

Introducing h as the independent variable and transforming the dependent variables v and θ by

$$r=v^2,\; s=\cos\theta \quad \ldots \quad (28)$$

the dynamical equations reduce to

$$\frac{m}{2}r' = \frac{T\cos\alpha}{\sqrt{1-s^2}} - mg \qquad \ldots \quad (29)$$

$$mrs' = mgs - T\sin\alpha \qquad \ldots \quad (30)$$

and, accordingly, the integral to be minimized is

$$\int_{h_0}^{h_f} F(r, r', s, s')dh$$

where

$$F \equiv \frac{1}{\sqrt{r(1-s^2)}} + \lambda\left[\frac{m}{2}r' - \frac{T\cos\alpha}{\sqrt{1-s^2}} + mg\right]. \qquad \ldots \quad (31)$$

Carrying out the partial differentiation of F with respect to r, s and their derivatives with respect to and treating α as a function of r, s and s' as given by (30) the Euler-Lagrange equations become

$$\lambda' + \left(m' + \frac{2ms'\tan\alpha}{\sqrt{1-s^2}}\right)\lambda + \frac{1}{r\sqrt{r(1-s^2)}} = 0. \qquad \ldots \quad (32)$$

$$\frac{mr\tan\alpha}{\sqrt{1-s^2}}\lambda' + \left[\frac{d}{dh}\left(\frac{mr\tan\alpha}{\sqrt{1-s^2}}\right) - \frac{Ts\cos\alpha}{(1-s^2)^{\frac{3}{2}}} + \frac{mg\tan\alpha}{\sqrt{1-s^2}}\right]\lambda + \frac{s}{\sqrt{r}(1-s^2)^{\frac{3}{2}}} = 0. \qquad \ldots \quad (33)$$

Eliminating λ between these two differential equations, we get

$$\frac{s}{\sqrt{r}(1-s^2)^{\frac{3}{2}}}\left(m' + \frac{2ms'}{\sqrt{1-s^2}}\tan\alpha\right) - \frac{1}{r\sqrt{r(1-s^2)}}\left\{\frac{d}{dh}\left(\frac{mr\tan\alpha}{\sqrt{1-s^2}}\right) - \frac{Ts\cos\alpha}{(1-s^2)^{\frac{3}{2}}}\right.$$
$$\left. + \frac{mg\tan\alpha}{\sqrt{1-s^2}}\right\} = \Delta\frac{d}{dh}\left[\frac{1}{\Delta}\left\{\frac{m\tan\alpha}{\sqrt{r}(1-s^2)} - \frac{ms}{\sqrt{r}(1-s^2)^{\frac{3}{2}}}\right\}\right] \qquad \ldots \quad (34)$$

where

$$\Delta \equiv m\left\{\frac{d}{dh}\left(\frac{mr\tan\alpha}{\sqrt{1-s^2}}\right) - \frac{Ts\cos\alpha}{(1-s^2)^{\frac{3}{2}}} + \frac{mg\tan\alpha}{\sqrt{1-s^2}}\right\} - \frac{mr\tan\alpha}{\sqrt{1-s^2}}\left(m' + \frac{2ms'\tan\alpha}{\sqrt{1-s^2}}\right)$$

in which α, as given by (30), is to be replaced by

$$\tan^{-1}\left[\frac{m(gs-rs')}{\sqrt{T^2-m^2(gs-rs')^2}}\right].$$

Elimination of α between the dynamical equations (29) and (30) yields

$$(1-s^2)\left(\frac{r'}{2}+g\right)^2 + (gs-rs')^2 = \frac{T^2}{m^2}. \qquad \ldots \quad (35)$$

The simultaneous, non-linear differential equations (34) and (35) can be solved with the aid of initial conditions to obtain v and θ as functions of h for the optimum trajectory.

Case 2. $\alpha = 0$ and acceleration due to thrust is constant $\left(\frac{T}{m} = k\right)$.

In this case the dynamical equations reduce to

$$\frac{r'}{2} = \frac{k}{\sqrt{1-s^2}} - g \quad \ldots \quad (36)$$

$$rs' = gs \quad \ldots \quad (37)$$

and the corresponding Euler-Lagrange equations are

$$\lambda' + \frac{1}{r\sqrt{r(1-s^2)}} = 0. \quad \ldots \quad (38)$$

$$\lambda k \sqrt{r} = 1. \quad \ldots \quad (39)$$

Eliminating λ between (38) and (39), we get

$$r' = \frac{2k}{\sqrt{1-s^2}}. \quad \ldots \quad (40)$$

From equations (37) and (40), the equation determining s is found out to be

$$ss'' = s'^2 \left(1 - \frac{2k}{g\sqrt{1-s^2}}\right). \quad \ldots \quad (41)$$

With the substitution

$$s' = u, \text{ so that } s'' = u\frac{du}{ds}$$

the above differential equation becomes

$$\frac{du}{ds} = u\left(\frac{1}{s} - \frac{2k}{g}\frac{1}{s\sqrt{1-s^2}}\right). \quad \ldots \quad (42)$$

The solution of (42) subject to the initial conditions

$$\theta = \theta_0, u = u_0 = -\sin\theta_0 \left(\frac{d\theta}{dh}\right)_{\theta = \theta_0}$$

is

$$u = u_0 \frac{\cos\theta}{\cos\theta_0}\left(\frac{\tan\theta + \sec\theta}{\tan\theta_0 + \sec\theta_0}\right)^{2k/g} \quad \ldots \quad (43)$$

whence

$$\frac{d\theta}{dh} = -\frac{u_0}{\cos\theta_0}\cot\theta\left(\frac{\tan\theta + \sec\theta}{\tan\theta_0 + \sec\theta_0}\right)^{2k/g} \quad \ldots \quad (44)$$

and hence

$$h - h_0 = \frac{g}{2k}\frac{\cos\theta_0}{u_0}\left[1 - \left(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\right)^{k/g}\right] - \frac{\cos\theta_0}{u_0}\int_{\theta_0}^{\theta} \tan^{\frac{2k}{g}-1}\left(\frac{\pi}{4} + \frac{\theta}{2}\right) d\theta \quad (45)$$

using (37) and (44), the expression for v is given by

$$v = \sqrt{\frac{g\cos\theta_0}{u_0}}\left(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\right)^{k/g}. \quad \ldots \quad (46)$$

From the kinematic relations, we have

$$\frac{dx}{dh} = \cot\theta$$

so that

$$x - x_0 = -\frac{\cos\theta_0}{u_0}\left(\tan\theta_0 + \sec\theta_0\right)^{2k/g}\int_{\theta_0}^{\theta}\tan^{2k/g}\left(\frac{\pi}{4}+\frac{\theta}{2}\right)d\theta. \quad .. \quad (47)$$

Also from the kinematic relations, we have

$$\frac{d\theta}{dt} = v\sin\theta\,\frac{d\theta}{dh}$$

and hence, on using (44) and (46),

$$t - t_0 = \frac{1}{k}\sqrt{\frac{g\cos\theta_0}{u_0}}\left\{\left(\frac{\sec\theta_0 + \tan\theta_0}{\sec\theta + \tan\theta}\right)^{k/g} - 1\right\}. \quad .. \quad .. \quad (48)$$

Case 3. Drag form $= cv^2$ and acceleration due to thrust is constant.

The dynamical equations are

$$m\frac{dv}{dt} = T - mg\sin\theta - \frac{k_1}{2}v^2 \quad .. \quad .. \quad .. \quad (49)$$

$$mv\frac{d\theta}{dt} = -mg\cos\theta \quad .. \quad .. \quad .. \quad .. \quad .. \quad (50)$$

and the kinematic relations are

$$\frac{dx}{dt} = v\cos\theta,\ \frac{dh}{dt} = v\sin\theta.$$

Introducing the deceleration coefficient $c = \frac{k_1}{2m}$ and transforming the variables as in case (2), the dynamical equations become

$$\frac{r'}{2} = \frac{k - cr}{\sqrt{1-s^2}} - g \quad .. \quad .. \quad .. \quad .. \quad (51)$$

$$rs' = gs. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (52)$$

The integral to be minimized is

$$\int_{h_0}^{h_f} F\,dh$$

where F is now given by

$$F \equiv \frac{1}{\sqrt{r(1-s^2)}} + \lambda\left[\frac{r'}{2} - \frac{k-cr}{\sqrt{1-s^2}} + g\right] \quad .. \quad .. \quad .. \quad (53)$$

The Euler-Lagrange equations are

$$\lambda' - \frac{2c}{\sqrt{1-s^2}}\lambda + \frac{1}{r\sqrt{r(1-s^2)}} = 0. \quad .. \quad .. \quad .. \quad (54)$$

$$\lambda(k - cr)\sqrt{r} = 1. \quad .. \quad .. \quad .. \quad (55)$$

Eliminating λ between the above two equations

$$\frac{3cr-k}{2r(k-cr)^2}r' - \frac{2c}{\sqrt{1-s^2}(k-cr)} + \frac{1}{r\sqrt{1-s^2}} = 0. \quad \ldots \quad \ldots \quad (56)$$

Elimination of r between (52) and (56) yields

$$g(3cgs-ks')(s'^2-ss'')\sqrt{1-s^2}-4cgss'(ks'-cgs)+2s'(ks'-cgs)^2 = 0 \ldots \quad (57)$$

Applying the transformation

$$s' = u \text{ so that } s'' = u\frac{du}{ds}$$

the above equation reduces to the form

$$(u+f_1)u' = f_2u^2+f_3u+f_4 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

where f's are functions of the new independent variables alone and are given by

$$f_1 = -3cgs/k, \qquad f_2 = \frac{1}{s} - \frac{2k}{gs\sqrt{1-s^2}},$$

$$f_3 = \frac{2c}{\sqrt{1-s^2}} - \frac{3cg}{k}, \quad f_4 = -\frac{6gsc^2}{k\sqrt{1-s^2}}.$$

The equation (58) is a first order non-linear equation of Abel's type.

In the absence of drag ($c = 0$), the above Abel's equations reduces to equation (41) obtained in case (2).

Case 4.

$$\frac{T-D}{mg} = \text{constant}, \mu_1 \text{ and } \frac{L}{mg} = \text{constant}, \mu_2.$$

The dynamical equations of rocket flight, under the above assumptions, can be written as

$$\frac{dv}{dt} = g(\mu_1 - \sin\theta). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (59)$$

$$v\frac{d\theta}{dt} = g(\mu_2 - \cos\theta). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

With the transformation used earlier, these become

$$\frac{r'}{2} = g\left(\frac{\mu_1}{\sqrt{1-s^2}} - 1\right) \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

$$rs' = g(s-\mu_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

and the integral to be minimized is

$$\int_{h_0}^{h_f} F dh$$

where

$$F \equiv \frac{1}{\sqrt{r(1-s^2)}} + \lambda \left[\frac{r'}{2} - g \left(\frac{\mu_1}{\sqrt{1-s^2}} - 1 \right) \right]. \quad \ldots \quad \ldots (63)$$

In this case, the Euler-Lagrange equations are

$$\lambda' + \frac{1}{r\sqrt{r(1-s^2)}} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (64)$$

$$\mu_1 \lambda g \sqrt{r} = 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (65)$$

Eliminating λ between the above two equations and using the dynamical equation (62), we get

$$(s-\mu_2)s'' = s'^2 \left(1 - \frac{2\mu_1}{\sqrt{1-s^2}} \right) \ldots \quad \ldots \quad \ldots \quad \ldots (66)$$

Using the substitution $s' = u$ and the initial condition $u = u_0$, $\theta = \theta_0$ the first integral of the above differential equation is

$$u = -\sin\theta \frac{d\theta}{dh} = u_0 \frac{\cos\theta - \mu_2}{\cos\theta_0 - \mu_2} e^{-2\mu_1(\phi - \phi_0)} \quad \ldots \quad \ldots (67)$$

where

$$\phi = \frac{2}{\sqrt{\mu_2^2 - 1}} \tan^{-1} \left(\sqrt{\frac{\mu_2 + 1}{\mu_2 - 1}} \tan \frac{\theta}{2} \right).$$

From (67), the altitude is given by

$$h - h_0 = \frac{\cos\theta_0 - \mu_2}{u_0} \int_{\theta_0}^{\theta} \frac{\sin\theta}{\mu_2 - \cos\theta} e^{2\mu_1(\phi - \phi_0)} d\theta. \quad \ldots \quad \ldots (68)$$

From the dynamical equation (62) and (67), the expression for velocity v is

$$v = \sqrt{\frac{g(\cos\theta_0 - \mu_2)}{u_0}} e^{\mu_1(\phi - \phi_0)}. \quad \ldots \quad \ldots \quad \ldots (69)$$

Using the kinematic relations along with (67) and (69) the range and time corresponding to a point 'θ' on the minimum-time trajectory are given, respectively, by the integrals

$$x - x_0 = \frac{\cos\theta_0 - \mu_2}{u_0} \int_{\theta_0}^{\theta} \frac{\cos\theta}{\mu_2 - \cos\theta} e^{2\mu_1(\phi - \phi_0)} d\theta. \quad \ldots \quad \ldots (70)$$

$$t - t_0 = \sqrt{\frac{\cos\theta_0 - \mu_2}{g u_0}} \int_{\theta_0}^{\theta} \frac{e^{\mu_1(\phi - \phi_0)}}{\mu_2 - \cos\theta} d\theta.$$

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