SIMULATION OF THIRD ORDER SYSTEMS WITH DOUBLE LEAD USING ONE OPERATIONAL AMPLIFIER

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A method for the simulation of third order linear systems with only one operational amplifier is outlined. A particular case of the general third order systems, that is systems with double lead, is considered. Three networks each consisting of one operational amplifier, three capacitors and five resistors are presented. The circuits are analysed and the conditions discussed and obtained under which each circuit for the simulation of the given system is physically realizable. The design formulae and procedure are also given.

INTRODUCTION

The author (Wadhwa 1962*a*) has outlined a method for the simulation of third order linear systems with only one operational amplifier. The circuits capable of simulating, under certain conditions, the systems of the type

$$F(s) = -\frac{b_0}{a_3s^3+a_2s^2+a_1s+1}$$

and

$$F(s) = -\frac{b_1s}{a_3s^3+a_2s^2+a_1s+1}$$

have been discussed and the design formulae and the conditions of physical realizability given in earlier communications (Wadhwa 1961, 1962*b*, *c*, 1963*a*) on this subject.

The purpose of this paper is to discuss in detail the simulation of systems whose transfer function is of the form

$$F(s) = -\frac{b_2s^2}{a_3s^3+a_2s^2+a_1s+1} \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where b_2, a_1, a_2, a_3 are positive and real constants.

A circuit and the conditions under which it can simulate the system of (1) has been presented and discussed elsewhere (Wadhwa 1963*b*). The other three circuits, each employing three capacitors and five resistors, are presented and discussed in this paper.

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SIMULATION OF THIRD ORDER SYSTEMS

A network for the simulation of third order linear systems is shown in Fig. 1 and its transfer function has been shown (Wadhwa 1963*a*) to be

$$\frac{E_0}{E_1} = -\frac{y_1 y_3 y_5}{y_6(y_1+y_2+y_8)(y_3+y_4+y_5+y_7)+y_3 y_6(y_4+y_5+y_7)+y_5 y_7(y_1+y_2+y_3+y_8)+y_3 y_5 y_8} \quad (2)$$

Simulation of the system of (1) with the network of Fig. 1 is possible, if the admittances (y's) are properly chosen; and furthermore, it should be obvious from (2) that at least three of the appropriate admittances will be required to be purely capacitative.

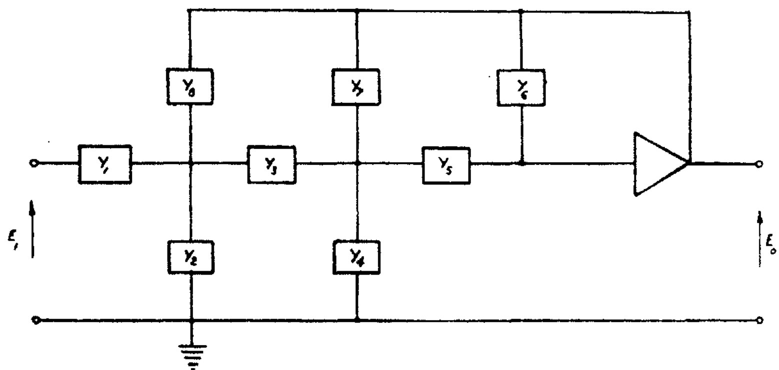

FIG. 1. Network for the simulation of third order systems.

Four circuits, each employing three capacitors and five resistors and capable of simulating, under certain conditions, the system of (1), are possible. One of these circuits has been discussed in detail elsewhere (Wadhwa 1963*b*) and the other three will be discussed here.

(*a*) y_1, y_3 *and* y_6 *capacitative*

A network for simulating the system of (1) with

$$\left.\begin{aligned} y_1 &= sc_1 \\ y_3 &= sc_3 \\ y_6 &= sc_6 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} y_2 = y_4 = y_7 = y_8 &= \frac{1}{R} \\ y_5 &= \frac{1}{aR} \end{aligned}\right\} \quad \ldots \quad (3)$$

is shown in Fig. 2(*a*).

Substituting (3) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\frac{1}{2}R^2c_1c_3s^2}{\frac{a}{2}R^3c_1c_3c_6s^3+\left[\frac{(2a+1)}{2}R^2c_1c_6+\frac{(4a+1)}{2}R^2c_3c_6\right]s^2+[\frac{1}{2}Rc_1+Rc_3+(2a+1)Rc_6]s+1} \quad (4)$$

Equations (1) and (4) will be identical if

$$b_2 = \tfrac{1}{2}T_1T_3 \quad (5)$$

$$a_1 = \tfrac{1}{2}T_1+T_3+(2a+1)T_6 \quad (6)$$

$$a_2 = \frac{(2a+1)}{2}T_1T_6+\frac{(4a+1)}{2}T_3T_6 \quad (7)$$

$$a_3 = \frac{a}{2}T_1T_3T_6 \quad (8)$$

where

$$T_n = Rc_n. \quad (9)$$

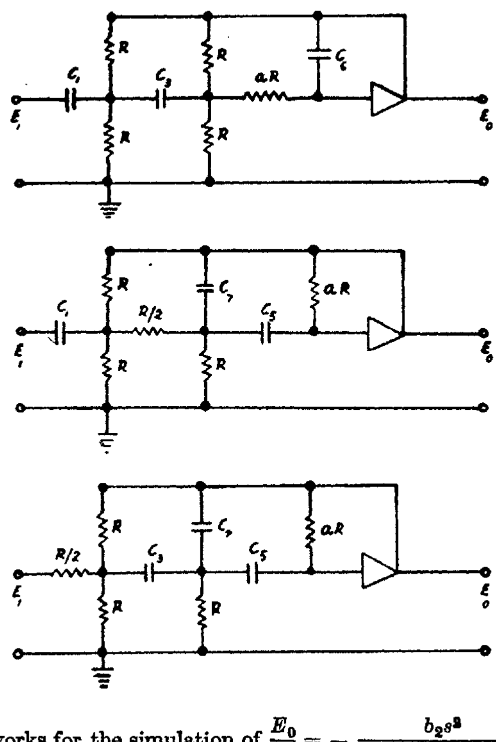

FIG. 2. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_2s^2}{a_3s^3+a_2s^2+a_1s+1}$.

Now, simulation of the system of (1) with the network of Fig. 2(*a*) is possible only if the values of a, T_1, T_3, T_6 obtained as the solution of (5) through (8) are real and positive. It is required to determine, therefore, a, T_1, T_3, T_6 in terms of the known real and positive constants b_2, a_1, a_2, a_3;

and find the conditions, if any, under which a, T_1, T_3, T_6 can be real and positive.

Elimination of a, T_1 and T_6 from (5) through (8) gives a quartic

$$T_3^4-(a_1b_2+2a_3)T_3^3+(2a_2b_2+2b_2+b_2^2)T_3^2-2a_1b_2^2T_3+2b_2^3 = 0 \quad .. \quad (10)$$

which, as is obvious, can have no negative real roots. Now, if its discriminant Δ is negative then (10) will have two positive real roots, and as shown in Appendix I, if

$$\left.\begin{aligned} a_1 &> \frac{2a_3}{b_2} \\ (a_1b_2-2a_3)^2 &> 4b_2^3 \\ b_2 &> \left(\frac{4a_3}{a_2}\right)^2 \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (11)$$

and, either

$$OQ > OB > OP > OA \quad .. \quad .. \quad .. \quad .. \quad (12)$$

or

$$OB > OQ > OA > OP \quad .. \quad .. \quad .. \quad .. \quad (13)$$

where

$$OA = \frac{(a_1b_2-2a_3)-\sqrt{(a_1b_2-2a_3)^2-4b_2^3}}{2b_2}$$

$$OB = \frac{(a_1b_2-2a_3)+\sqrt{(a_1b_2-2a_3)^2-4b_2^3}}{2b_2}$$

$$OP = \frac{a_2b_2-\sqrt{a_2^2b_2^2-16a_3^2b_2}}{4a_3}$$

$$OQ = \frac{a_2b_2+\sqrt{a_2^2b_2^2-16a_3^2b_2}}{4a_3}$$

then one set of positive real a, T_1, T_3 and T_6 exists. But if Δ is negative and (11) is satisfied then two sets of positive real a, T_1, T_3 and T_6 exist for either one of which the circuit of Fig. 2(a) is physically realizable provided that

$$OQ > OB > OA > OP \quad .. \quad .. \quad .. \quad .. \quad (14)$$

But, if its Δ is positive then (10) will have either no real roots or four positive real roots. Now, if Δ is positive and (11) is satisfied, then either one or three sets of positive real values exist, if

$$OB > OQ > OA > OP \quad .. \quad .. \quad .. \quad .. \quad (15)$$

and three sets of values exist if

$$OQ > OB > OP > OA \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

But if Δ is positive and (11) is satisfied then two sets of positive real values exist for either one of which the circuit of Fig. 2(a) is possible provided that

$$OB > OQ > OP > OA \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Combining and summarizing the restrictions on the inequalities of expressions (11) through (17); if

$$\left.\begin{aligned} a_1 &> \frac{2a_3}{b_2} \\ (a_1b_2-2a_3)^2 &> 4b_2^3 \\ b_2 &> \left(\frac{4a_3}{a_2}\right)^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

and, either $\quad OQ > OB > OP > OA$

or $\quad OB > OQ > OA > OP \quad \ldots \quad \ldots \quad \ldots \quad (18)$

then, irrespective of whether the discriminant of (10) is positive or negative, at least one positive real set of a, T_1, T_3 and T_6 exists. But, if

$\Delta < 0$

and $\quad OQ > OB > OA > OP \quad \ldots \quad \ldots \quad \ldots \quad (14)$

or, if $\quad \Delta > 0$

and $\quad OB > OQ > OP > OA \quad \ldots \quad \ldots \quad \ldots \quad (17)$

then two sets of positive real a, T_1, T_3 and T_6 exist for either one of which the circuit of Fig. 2(a) is possible.

For the design of the network, circuit component values are required to be determined. The proper procedure for design would be first to check and see if the conditions of expression (11) and any one of the three expressions (14), (17) or (18) are satisfied. If these are satisfied then the circuit of Fig. 2(a) for simulating the given system of (1) is physically realizable. The next step then would be to solve for T_3, the quartic of (10), by methods that are well known and discussed at length in textbooks (Uspensky 1948) on higher algebra. Substitution of the positive real value(s) of T_3 so obtained into (5) gives the corresponding value(s) of T_1; and T_6 may then be obtained by solving (6) and (9). Having thus determined a, T_1, T_3 and T_6 and choosing arbitrarily a convenient value for any one of the capacitors the remaining component values can be conveniently obtained with the aid of (3) and (9).

(b) y_1, y_5 and y_7 capacitative

Another possible circuit for the simulation of the system of (1) is shown in Fig. 2(*b*), in which

$$\left.\begin{aligned} y_1 &= sc_1 \\ y_5 &= sc_5 \\ y_7 &= sc_7 \\ y_2 = \frac{y_3}{2} = y_4 = y_8 &= \frac{1}{R} \\ y_6 &= \frac{1}{aR} \end{aligned}\right\} \qquad (19)$$

Substituting (19) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\frac{a}{4}R^2c_1c_5s^2}{\frac{a}{8}R^3c_1c_5c_7s^3+\left[\frac{1}{8}R^2c_1c_5+\frac{1}{8}R^2c_1c_7+\frac{a}{2}R^2c_5c_7\right]s^2+\left[\frac{3}{8}Rc_1+\frac{(a+2)}{4}Rc_5+\frac{1}{2}Rc_7\right]s+1} \qquad (20)$$

Equations (1) and (20) will be identical if

$$b_2 = \frac{a}{4}T_1T_5 \qquad (21)$$

$$a_1 = \frac{3}{8}T_1+\frac{(a+2)}{4}T_5+\frac{1}{2}T_7 \qquad (22)$$

$$a_2 = \frac{1}{8}T_1T_5+\frac{1}{8}T_1T_7+\frac{a}{2}T_5T_7 \qquad (23)$$

$$a_3 = \frac{a}{8}T_1T_5T_7 \qquad (24)$$

where

$$T_n = Rc_n \qquad (25)$$

Elimination of a, T_5 and T_7 from (21) through (24) gives a cubic

$$3T_1^3-8a_1T_1^2+8(4a_2+b_2)T_1-128a_3 = 0 \qquad (26)$$

which can have no negative real roots and will have either one or three real positive roots depending on whether its discriminant Δ is, respectively, positive or negative. If Δ is positive and

$$\left.\begin{aligned} a_1 &> \frac{a_3}{b_2} \\ (a_1b_2-a_3)^2 &> \tfrac{3}{2}b_2^3 \\ b_2 &> \left(\frac{2a_3}{a_2}\right)^2 \end{aligned}\right\} \qquad (27)$$

and

$$OQ > OB > OP > OA \qquad (28)$$

where

$$OA = \frac{4(a_1b_2-a_3)-\sqrt{16(a_1b_2-a_3)^2-24b_2^3}}{3b_2}$$

$$OB = \frac{4(a_1b_2-a_3)+\sqrt{16(a_1b_2-a_3)^2-24b_2^3}}{3b_2}$$

$$OP = \frac{2a_2b_2-\sqrt{4a_2^2b_2^2-16a_3^2b_2}}{a_3}$$

$$OQ = \frac{2a_2b_2+\sqrt{4a_2^2b_2^2-16a_3^2b_2}}{a_3}$$

then, as shown in Appendix II, one set of positive real a, T_1, T_5 and T_7 exists. But if Δ is negative; that is

$$\Delta = 4p^3+27q^2 < 0 \quad \ldots \quad (29)$$

where

$$p = \frac{8}{3}\left(4a_2+b_2\right) - \frac{1}{3}\left(\frac{8a_1}{3}\right)^2$$

$$q = -\frac{128}{3}a_3+\frac{64}{27}a_1\left(4a_2+b_2\right) - \frac{2}{27}\left(\frac{8a_1}{3}\right)^3$$

and (27) is satisfied then one and possibly two or three positive real sets can exist.

To summarize, if (27) and (28) are satisfied then either one or three sets of positive real a, T_1, T_5 and T_7 exist, depending respectively on whether Δ is positive or negative. But if (27) and (29) are satisfied then at least one and possibly two or three sets of positive real values can exist.

(c) y_3, y_5 and y_7 capacitative

Another possible circuit is shown in Fig. 2(c), in which

$$\left.\begin{aligned} y_3 &= sc_3 \\ y_5 &= sc_5 \\ y_7 &= sc_7 \\ \frac{y_1}{2} = y_2 = y_4 = y_8 &= \frac{1}{R} \\ y_6 &= \frac{1}{aR} \end{aligned}\right\} \quad \ldots \quad (30)$$

Substituting (30) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\frac{a}{2}R^2c_3c_5s^2}{\frac{a}{4}R^3c_3c_5c_7s^3+\left[\frac{(a+1)}{4}R^2c_3c_5+\frac{1}{4}R^2c_3c_7+aR^2c_5c_7\right]s^2+\left[5Rc_3+4Rc_5+4Rc_7\right]s+1} \quad (31)$$

Equations (1) and (31) will be identical if

$$b_2 = \frac{a}{2} T_3 T_5 \qquad (32)$$

$$a_1 = 5T_3 + 4T_5 + 4T_7 \qquad (33)$$

$$a_2 = \frac{(a+1)}{4} T_3 T_5 + \frac{1}{4} T_3 T_7 + a T_5 T_7 \qquad (34)$$

$$a_3 = \frac{a}{4} T_3 T_5 T_7 \qquad (35)$$

where

$$T_n = Rc_n \qquad (36)$$

Elimination of a, T_5 and T_7 from (32) through (35) gives a cubic

$$5T_3^3 - a_1 T_3^2 + 8(2a_2 - b_2) T_3 - 64 a_3 = 0 \qquad (37)$$

which will have either one or three real roots depending on whether its discriminant is positive or negative. If, as shown in Appendix III,

$$\left. \begin{aligned} a_1 &> \frac{8a_3}{b_2} \\ 2a_2 > b_2 &> 32 \left(\frac{a_3}{2a_2 - b_2} \right)^2 \end{aligned} \right\} \qquad (38)$$

then the real root(s) of (37) will be positive, and one corresponding set of real positive a, T_5, T_7 will exist, if

$$\frac{b_2(2a_2-b_2) + \sqrt{b_2^2(2a_2-b_2)^2 - 32a_3^2 b_2}}{2a_3} > \left(\frac{a_1 b_2 - 8a_3}{5b_2} \right) > \frac{b_2(2a_2-b_2) - \sqrt{b_2^2(2a_2-b_2)^2 - 32a_3^2 b_2}}{2a_3} \qquad (39)$$

But, if

$$\left(\frac{a_1 b_2 - 8a_3}{5b_2} \right) > \frac{b_2(2a_2-b_2) - \sqrt{b_2^2(2a_2-b_2)^2 - 32a_3^2 b_2}}{2a_3} \qquad (40)$$

and

$$4p^3 + 27q^2 < 0$$

where

$$p = \frac{8}{5}(2a_2 - b_2) - \frac{1}{3}\left(\frac{a_1}{5}\right)^2$$

$$q = -\frac{64}{5} a_3 + \frac{8}{75} a_1 (2a_2 - b_2) - \frac{2}{27}\left(\frac{a_1}{5}\right)^3$$

then either one or possibly two sets of positive real a, T_3, T_5 and T_7 can exist.

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Appendix I

Conditions under which the Circuit of Figure 2(*a*) is Physically Realizable

Simulation of the system as characterized by (1) with the network of Fig. 2(*a*) is possible only if a, T_1, T_3 and T_6 obtained as the solution of equations

$$b_2 = \frac{1}{2} T_1 T_3 \qquad (1.1)$$

$$a_1 = \frac{1}{2} T_1 + T_3 + (2a+1) T_6 \qquad (1.2)$$

$$a_2 = \frac{(2a+1)}{2} T_1 T_6 + \frac{(4a+1)}{2} T_3 T_6 \qquad (1.3)$$

$$a_3 = \frac{a}{2} T_1 T_3 T_6 \qquad (1.4)$$

are real and positive, where b_2, a_1, a_2, a_3 are real and positive constants.

It is evident that (1.1) through (1.4) are non-linear; and graphical methods may perhaps be a convenient means of determining the conditions under which their solutions can yield positive real a, T_1, T_3 and T_6.

Elimination of a and T_5 from (1.1), (1.2), (1.4) and (1.1), (1.3), (1.4) give the following two equations

$$T_6 = \left(a_1 - \frac{2a_3}{b_2}\right) - \frac{b_2}{T_3} - T_3 \qquad (1.5)$$

$$T_6 = -2a_3 + \frac{a_2 b_2 T_3 + 2a_3 b_2}{2b_2 + T_3^2} \qquad (1.6)$$

The intersection of the curves of (1.5) and (1.6) in the first quadrant of the T_3-T_6 plane will give both T_3 and T_6 as positive and real. It is obvious from (1.1) and (1.4) that the corresponding T_1 and a will be also real and positive. It should be clear, therefore, that only the portion of the curves lying on the right of the T_6-axis are of interest.

The branch of the curves of (1.5) lying on the right of the T_6-axis will cut the T_3-axis (i.e. $T_6 = 0$) at two real points A and B whose T_3 co-ordinates may be obtained by equating to zero the right-hand side of (1.5) and solving the resulting quadratic

$$b_2 T_3^2 - (a_1 b_2 - 2a_3) T_3 + b_2^2 = 0 \qquad (1.7)$$

The roots of (1.7) are

$$T_{3(A,B)} = \frac{(a_1b_2-2a_3) \pm \sqrt{(a_1b_2-2a_3)^2-4b_2^3}}{2b_2} \quad \ldots \quad (1.8)$$

which will be real, if

$$(a_1b_2-2a_3)^2 > 4b_2^3 \quad \ldots \quad (1.9)$$

and also positive, if

$$a_1 > \frac{2a_3}{b_2} \quad \ldots \quad (1.10)$$

Similarly, (1.6) will cut the T_3-axis at two points P and Q whose T_3 co-ordinates are

$$T_{3(P,Q)} = \frac{a_2b_2 \pm \sqrt{a_2^2b_2^2-16a_3^2b_2}}{4a_3} \quad \ldots \quad (1.11)$$

Now, P and Q will be real and positive, if

$$b_2 > \left(\frac{4a_3}{a_2}\right)^2 \quad \ldots \quad (1.12)$$

Therefore, if the conditions as expressed in (1.9), (1.10) and (1.12) are satisfied then it is possible for a portion of each curve of (1.5) and (1.6) to exist in the first quadrant, and it may be possible, under certain conditions, for these to intersect each other at one or more points in that region.

Elimination of T_6 from (1.5) and (1.6) gives a quartic

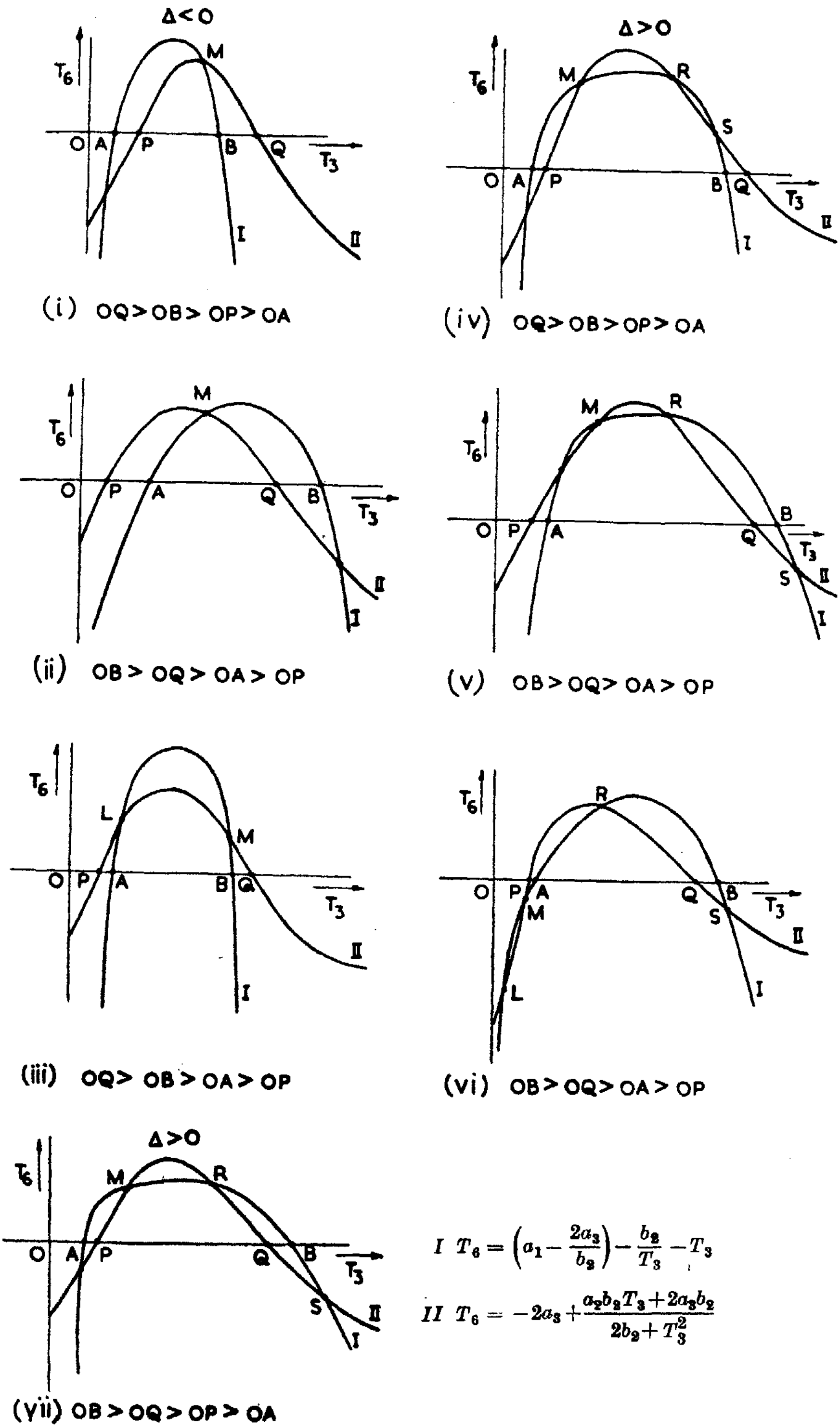
$$T_3^4-(a_1b_2+2a_3)T_3^3+(2a_2b_2+2b_2+b_2^2)T_3^2-2a_1b_2^2T_3+2b_2^3 = 0. \quad \ldots \quad (1.13)$$

The real roots of (1.13) give the real points of intersection of the curves of (1.5) and (1.6). It is obvious that (1.13) can have no negative real roots; therefore, the curves do not intersect at real points on the left of the T_6-axis. If the discriminant Δ is negative then (1.13) will have two real positive roots signifying that the curves intersect each other at two real points on the right of the T_6-axis; and if Δ is positive then the curves can intersect each other at four points or cannot intersect at all. The manner in which the curves of (1.5) and (1.6) can possibly intersect to yield one or more points in the first quadrant of the T_3-T_6 plane is shown in the sketches of Fig. 1.1.

Now, as is evident from Fig. 1.1 if the points A and B interlace with the points P and Q in the manner shown in the sketches, then irrespective of whether Δ of (1.13) is positive or negative the curves intersect each other at one or more points in the first quadrant; both the T_3 and T_6 co-ordinates of which are positive and the corresponding T_1 and a for which are also real and positive. That is, if either

$$\left.\begin{array}{l} OQ > OB > OP > OA \\ OB > OQ > OA > OP \end{array}\right\} \quad \ldots \quad (1.14)$$

or



$$I \quad T_6 = \left(a_1 - \frac{2a_3}{b_2}\right) - \frac{b_2}{T_3} - T_3$$

$$II \quad T_6 = -2a_3 + \frac{a_2 b_2 T_3 + 2a_3 b_2}{2b_2 + T_3^2}$$

FIG. 1.1. Conditions under which the curves can intersect each other and have at least one common point in the first quadrant.

where

$$OA = \frac{(a_1b_2-2a_3)-\sqrt{(a_1b_2-2a_3)^2-4b_2^3}}{2b_2}$$

$$OB = \frac{(a_1b_2-2a_3)+\sqrt{(a_1b_2-2a_3)^2-4b_2^3}}{2b_2}$$

$$OP = \frac{a_2b_2-\sqrt{a_2^2b_2^2-16a_3^2b_2}}{4a_3}$$

$$OQ = \frac{a_2b_2+\sqrt{a_2^2b_2^2-16a_3^2b_2}}{4a_3}$$

then one or more positive real sets of a, T_1, T_3 and T_6 exist.

But, if the points A and B lie in between the points P and Q and Δ of (1.13) is negative, or if the points P and Q lie in between the points A and B and Δ is positive then the curves intersect each other at two points in the first quadrant. That is, if either

$$\left.\begin{array}{l}\Delta < 0 \\ \text{and} \quad OQ > OB > OA > OP\end{array}\right\} \qquad (1.15)$$

or

$$\left.\begin{array}{l}\Delta > 0 \\ \text{and} \quad OB > OQ > OP > OA\end{array}\right\} \qquad (1.16)$$

then two positive real sets of a, T_1, T_3 and T_6 exist for either one of which the circuit of Fig. 2(a) is physically realizable.

To summarize, therefore, if

$$\left.\begin{array}{l}OQ > OB > OP > OA \\ \text{or} \quad OB > OQ > OA > OP\end{array}\right\} \qquad (1.17)$$

$$\left.\begin{array}{l}\text{or} \quad \Delta < 0 \\ \text{and} \quad OQ > OB > OA > OP\end{array}\right\} \qquad (1.17a)$$

$$\left.\begin{array}{l}\text{or} \quad \Delta > 0 \\ \text{and} \quad OB > OQ > OP > OA\end{array}\right\} \qquad (1.17b)$$

and also, if

$$\left.\begin{array}{c}a_1 > \dfrac{2a_3}{b_2} \\ (a_1b_2-2a_3)^2 > 4b_2^3 \\ b_2 > \left(\dfrac{4a_3}{a_2}\right)^2\end{array}\right\} \qquad (1.18)$$

then it is possible to simulate the system of (1) with the circuit of Fig. 2(a).

Appendix II

Conditions under which the Circuit of Figure 2(*b*) is Physically Realizable

If the values of a, T_1, T_5 and T_7 obtained as the solution of equations

$$b_2 = \frac{a}{4} T_1 T_5 \quad \ldots \quad (2.1)$$

$$a_1 = \frac{3}{8} T_1 + \frac{(a+2)}{4} T_5 + \frac{1}{2} T_7 \quad \ldots \quad (2.2)$$

$$a_2 = \frac{1}{8} T_1 T_5 + \frac{1}{8} T_1 T_7 + \frac{a}{2} T_5 T_7 \quad \ldots \quad (2.3)$$

$$a_3 = \frac{a}{8} T_1 T_5 T_7 \quad \ldots \quad (2.4)$$

are real and positive, then it is possible to simulate the system of (1) with the circuit of Fig. 2(*b*).

Elimination of a and T_7 from (2.1), (2.2), (2.4) and (2.1), (2.3), (2.4) gives the following two equations

$$T_5 = 2\left[\frac{a_1 b_2 - a_3}{b_2} - \frac{b_2}{T_1} - \frac{3}{8} T_1\right] \quad \ldots \quad (2.5)$$

$$T_5 = -\frac{2a_3}{b_2} + \frac{8}{T_1}\left(a_2 - \frac{4a_3}{T_1}\right) \quad \ldots \quad (2.6)$$

The intersection of the curves of (2.5) and (2.6) in the first quadrant of the T_1–T_5 plane will give both T_1 and T_5 as real and positive. It is evident from (2.1) and (2.4) that the corresponding a and T_7 are also real and positive.

The curve of (2.5) will cut the T_1-axis (i.e. $T_5 = 0$) at two points A and B whose T_1 co-ordinates may be obtained by equating to zero the right-hand side of (2.5) and solving the resulting quadratic

$$3b_2 T_1^2 - 8(a_1 b_2 - a_3) T_1 + 8b_2^2 = 0 \quad \ldots \quad (2.7)$$

whose roots are given by

$$T_{1(A,\,B)} = \frac{4(a_1 b_2 - a_3) \pm \sqrt{16(a_1 b_2 - a_3)^2 - 24 b_2^3}}{3b_2} \quad \ldots \quad (2.8)$$

Now A and B will be real, if

$$(a_1 b_2 - a_3)^2 > \tfrac{3}{2} b_2^3 \quad \ldots \quad (2.9)$$

and their co-ordinates will be positive, if

$$a_1 > \frac{a_3}{b_2} \quad \ldots \quad (2.10)$$

Similarly, (2.6) will cut the T_1-axis at two points P and Q whose T_1 co-ordinates are

$$T_{1(P,Q)} = \frac{2a_2b_2 \pm \sqrt{4a_2^2b_2^2 - 16a_3^2b_2}}{a_3} \quad \ldots \quad (2.11)$$

and which will be real, if

$$b_2 > \left(\frac{2a_3}{a_2}\right)^2 \quad \ldots \quad (2.12)$$

Elimination of T_5 from (2.5) and (2.6) gives a cubic

$$3T_1^3 - 8a_1T_1^2 + 8(4a_2+b_2)T_1 - 128a_3 = 0 \quad \ldots \quad (2.13)$$

which will have either one or three real and positive roots depending on whether its discriminant Δ is positive or negative.

The branches of the curves of (2.5) and (2.6) lying on the right of the T_5-axis in the T_1–T_5 plane and the manner in which these can possibly intersect at one or three points in that region with one or more common points in the first quadrant are sketched in Fig. 2.1.

Now, as is evident from Fig. 2.1, if the points A and B interlace with the points P and Q in the manner shown in the sketches 2.1(i) and 2.1(i*a*) then irrespective of whether Δ of (2.13) is positive or negative, at least one set of positive real a, T_1, T_5 and T_7 exists. That is, if

$$OQ > OB > OP > OA \quad \ldots \quad (2.14)$$

where

$$OA = \frac{4(a_1b_2-a_3) - \sqrt{16(a_1b_2-a_3)^2 - 24b_2^3}}{3b_2}$$

$$OB = \frac{4(a_1b_2-a_3) + \sqrt{16(a_1b_2-a_3)^2 - 24b_2^3}}{3b_2}$$

$$OP = \frac{2a_2b_2 - \sqrt{4a_2^2b_2^2 - 16a_3^2b_2}}{a_3}$$

$$OQ = \frac{2a_2b_2 + \sqrt{4a_2^2b_2^2 - 16a_3^2b_2}}{a_3}$$

then either one or three sets of positive real a, T_1, T_5 and T_7 exist depending respectively on whether Δ is positive or negative.

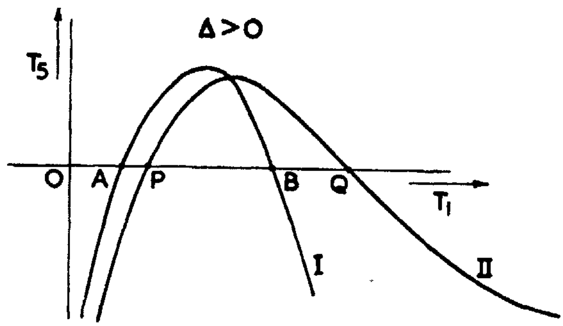
But if

$$\Delta = 4p^3 + 27q^2 < 0 \quad \ldots \quad (2.15)$$

where

$$p = \frac{8}{3}(4a_2+b_2) - \frac{1}{3}\left(\frac{8a_1}{3}\right)^2$$

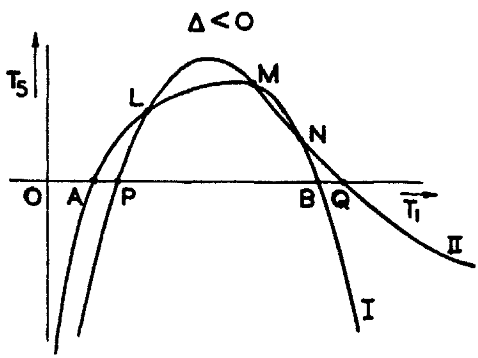
$$q = -\frac{128}{3}a_3 + \frac{64}{27}a_1(4a_2+b_2) - \frac{2}{27}\left(\frac{8a_1}{3}\right)^3$$



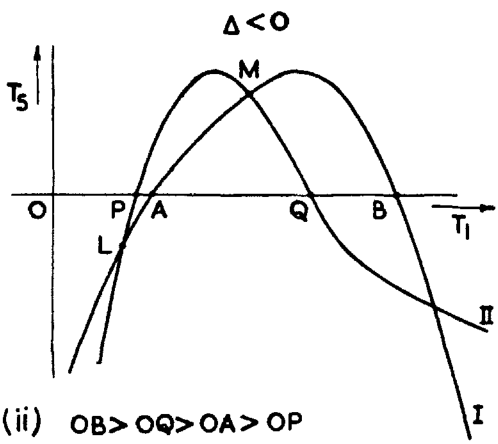
(ia) OQ > OB > OP > OA

$$I\ T_5 = 2\left[\frac{a_1b_2-a_3}{b_2} - \frac{b_2}{T_1} - \frac{3}{8}T_1\right]$$

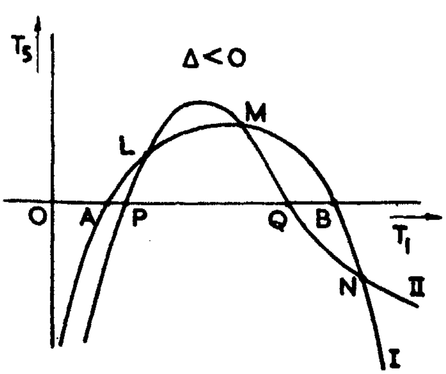
$$II\ T_5 = -\frac{2a_3}{2b_2}+\frac{8}{T_1}\left(a_2-\frac{4a_3}{T_1}\right)$$

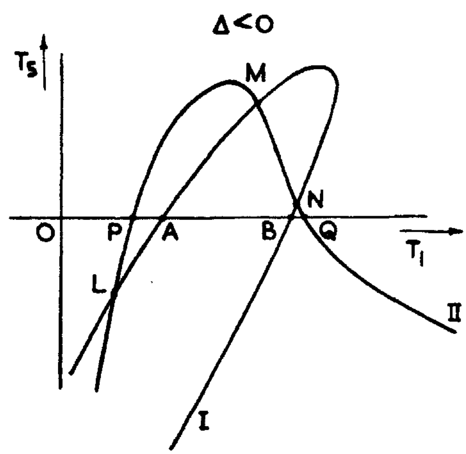


(i) OQ>OB>OP>OA



(ii) OB>OQ>OA>OP





(iii) OB > OQ > OP > OA

(iv) OQ > OB > OA > OP

FIG. 2.1. Conditions under which the curves can intersect each other and have at least one common point in the first quadrant.

then one, two, or three sets of positive real values are possible depending on the manner in which the curves of (2.5) and (2.6) intersect each other.

To summarize, if

$$\left.\begin{aligned} a_1 &> \frac{a_3}{b_2} \\ (a_1b_2-a_3)^2 &> \frac{3}{2}b_2^3 \\ b_2 &> \left(\frac{2a_3}{a_2}\right)^2 \end{aligned}\right\} \qquad \text{(2.16)}$$

and either

$$OQ > OB > OP > OA \qquad \text{(2.14)}$$

or

$$\Delta = 4p^3+27q^2 < 0 \qquad \text{(2.15)}$$

then it is possible to simulate the system of (1) with the circuit of Fig. 2(b).

Appendix III

Conditions under which the Circuit of Figure 2(c) is Physically Realizable

Simulation of the system of (1) with the network of Fig. 2(c) is possible only if a, T_3, T_5 and T_7 obtained as the solution of equations

$$b_2 = \frac{a}{2}T_3T_5 \qquad \text{(3.1)}$$

$$a_1 = 5T_3+4T_5+4T_7 \qquad \text{(3.2)}$$

$$a_2 = \frac{(a+1)}{4}T_3T_5+\tfrac{1}{4}T_3T_7+aT_5T_7 \qquad \text{(3.3)}$$

$$a_3 = \frac{a}{4}T_3T_5T_7 \qquad \text{(3.4)}$$

are real and positive.

Elimination of a and T_7 from (3.1), (3.2), (3.4) and (3.1), (3.3) and (3.4) gives the following two equations:

$$5T_3+4T_5 = \left(a_1-\frac{8a_3}{b_2}\right) \qquad \text{(3.5)}$$

and

$$T_5 = -\frac{2a_3}{b_2}+\frac{4}{T_3}\left(a_2-\frac{b_2}{2}-\frac{4a_3}{T_3}\right) \qquad \text{(3.6)}$$

The sketches of the straight line of (3.5) and the branch of the curve of

(3.6) lying on the right of the T_5-axis are shown in Fig. 3.1. The straight line will make positive and real intercepts with the T_3-axis, if

$$a_1 > \frac{8a_3}{b_2} \quad \ldots \quad (3.7)$$

and the curve will cut the T_3-axis at two points P and Q whose T_3 co-ordinates can be obtained by equating to zero the right-hand side of (3.6) and solving the resulting quadratic

$$a_3T_3^2 - b_2(2a_2 - b_2)T_3 + 8a_3b_2 = 0 \quad \ldots \quad (3.8)$$

The roots of (3.3) are given by

$$T_{3(P,\,Q)} = \frac{b_2(2a_2 - b_2) \pm \sqrt{b_2^2(2a_2 - b_2)^2 - 32a_3^2b_2}}{2a_3} \quad \ldots \quad (3.9)$$

which will be real, if

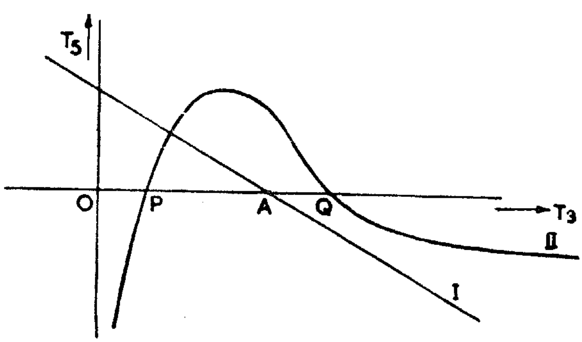
$$b_2 > 32\left(\frac{a_3}{2a_2 - b_2}\right)^2 \quad \ldots \quad (3.10)$$

and also positive, if

$$a_2 > \frac{b_2}{2} \quad \ldots \quad (3.11)$$

Elimination of T_5 from (3.5) and (3.6) gives a cubic

$$5T_3^3 - a_1T_3^2 + 8(2a_2 - b_2)T_3 - 64a_3 = 0 \quad \ldots \quad (3.12)$$



OA > OP

$$I \quad \left(a_1 - \frac{8a_3}{b_2}\right) = 5T_3 + 4T_5$$

$$II \quad T_5 = \frac{4}{T_3}\left(a_2 - \frac{b_2}{2} - \frac{4a_3}{T_3}\right) - \frac{2a_3}{b_2}$$

FIG. 3.1. Conditions under which the straight line and the curve can intersect each other and have at least one common point in the first quadrant.

which, in view of (3.11), can have no negative real roots. Therefore (3.12) will have either one or three positive real roots depending on whether its discriminant Δ is positive or negative.

Now, the straight line will cut the T_3-axis at a point A and if the point A lies in between the points P and Q then irrespective of whether Δ is positive or negative one set of positive real a, T_3, T_5 and T_7 exists. That is, if

$$\frac{b_2(2a_2-b_2)+\sqrt{b_2^2(2a_2-b_2)^2-32a_3^2b_2}}{2a_3} > \left(\frac{a_1b_2-8a_3}{5b_2}\right) > \frac{b_2(2a_2-b_2)-\sqrt{b_2^2(2a_2-b_2)^2-32a_3^2b_2}}{2a_3} \quad \ldots \quad (3.13)$$

then it is possible to simulate the system of (1) with the circuit of Fig. 2(*c*).

But if the point A lies on the right of Q then for a positive real set of values to exist the Δ must be negative. That is, if

$$\left(\frac{a_1b_2-8a_3}{5b_2}\right) > \frac{b_2(2a_2-b_2)+\sqrt{b_2^2(2a_2-b_2)^2-32a_3^2b_2}}{2a_3} \quad \ldots \quad (3.14)$$

then

$$\Delta = 4p^3+27q^2 < 0 \quad \ldots \quad (3.15)$$

where

$$p = \frac{8}{5}(2a_2-b_2)-\frac{1}{3}\left(\frac{a_1}{5}\right)^2$$

$$q = -\frac{64}{5}a_3+\frac{8}{75}a_1(2a_2-b_2)-\frac{2}{27}\left(\frac{a_1}{5}\right)^3$$

To summarize, if

$$\left.\begin{aligned} a_1 &> \frac{8a_3}{b_2} \\ 2a_2 > b_2 &> 32\left(\frac{a_3}{2a_2-b_2}\right)^2 \end{aligned}\right\} \quad \ldots \quad (3.16)$$

and either

$$\frac{b_2(2a_2-b_2)+\sqrt{b_2^2(2a_2-b_2)^2-32a_3^2b_2}}{2a_3} > \left(\frac{a_1b_2-8a_3}{5b_2}\right) > \frac{b_2(2a_2-b_2)-\sqrt{b_2^2(2a_2-b_2)^2-32a_3^2b_2}}{2a_3} \quad \ldots \quad (3.13)$$

or

$$\left.\begin{aligned} \left(\frac{a_1b_2-8a_3}{5b_2}\right) &> \frac{b_2(2a_2-b_2)-\sqrt{b_2^2(2a_2-b_2)^2-32a_3^2b_2}}{2a_3} \\ \text{and} \quad \Delta &= 4p^3+27q^2 < 0 \end{aligned}\right\} \quad \ldots \quad (3.17)$$

then it is possible to simulate the system of (1) with the circuit of Fig. 2(*c*).