SIMULATION OF THIRD ORDER SYSTEMS WITH ONE OPERATIONAL AMPLIFIER

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The third order linear system simulation by electronic analog computers is generally done with two or more operational amplifiers by techniques that are very well known. A method has been discussed which shows that third order linear systems can be simulated with the aid of only one operational amplifier and a few two-terminal impedances consisting of resistors and capacitors only. Three possible circuits capable of simulating a particular case of the general third order linear system have been presented and the resulting third and fourth degree non-linear algebraic equations have been solved for determining the validity conditions and the circuit component values. The method of simulation is simple. If the validity conditions are satisfied then a physically realizable network consisting of one operational amplifier, three capacitors, and five resistors exists and the component value may easily be calculated from the equations developed in the text.

INTRODUCTION

In analog computation need often arises for the simulation of third order linear systems and this is generally done with the aid of two or more operational amplifiers. The purpose of this paper is to indicate a method whereby third order linear systems can be simulated with the aid of only one operational amplifier and a few two-terminal impedances consisting of only resistors and capacitors.

Special features of this technique of simulation are that only one operational amplifier is required and this can be of significant advantage to establishments with small or overloaded computing capacity. The other features of this method are that the choice of standard value capacitors may be possible and that, in general, it requires a fewer number of resistors and capacitors.

THIRD ORDER SYSTEM SIMULATION

A network for the simulation of third order linear systems is shown in Fig. 1, and its transfer function as shown in Appendix I is

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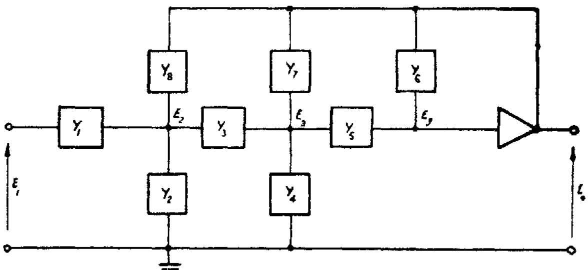
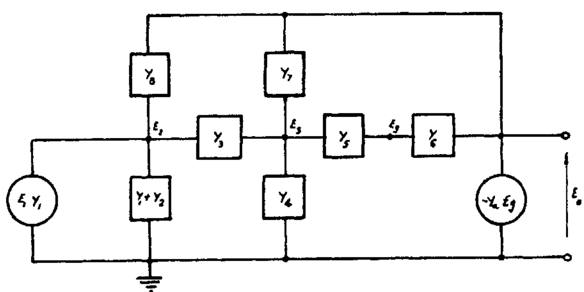
$$\frac{E_0}{E_1} = -\frac{Y_1Y_3Y_5}{Y_6(Y_1+Y_2+Y_8)(Y_3+Y_4+Y_5+Y_7)+Y_3Y_6(Y_4+Y_5+Y_7)+ Y_5Y_7(Y_1+Y_2+Y_3+Y_8)+Y_3Y_5Y_8} \quad .. \quad (1)$$

FIG. 1. Block diagram of a network for simulating third order systems.



(*a*) Block diagram of an operational amplifier and its associated input-feedback networks.



(*b*) Equivalent circuit of Fig. 1 (*a*).

A third order linear system is characterized by a transfer function of the form

$$F(S) = -\frac{b_2S^2+b_1S+b_0}{a_3S^3+a_2S^2+a_1S+1}, \quad .. \quad .. \quad .. \quad (2)$$

where a's and b's are real positive constants.

An inspection of (1) will indicate that simulation of the system as characterized by (2) is possible provided the admittances (Y's) are properly chosen.

The discussion in this paper will be confined to the type of systems for which

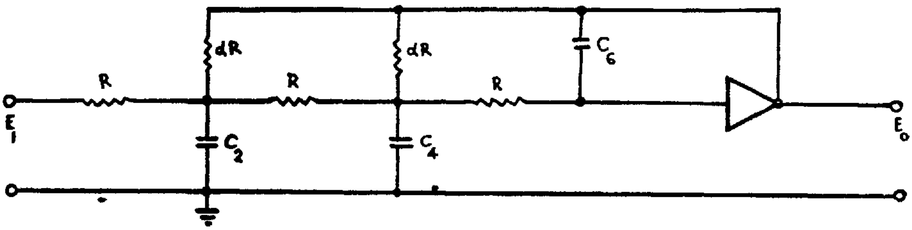
$$b_1 = b_2 = 0 \text{ and } b_0 > 0,$$

that is to the simulation of systems characterized by the transfer function

$$F(S) = -\frac{b_0}{a_3S^3+a_2S^2+a_1S+1}. \quad .. \quad .. \quad .. \quad (2a)$$

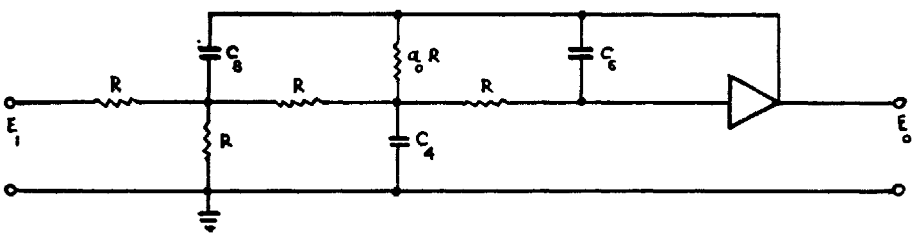
Simulation of the system characterized by (2*a*) with the network of Fig. 1 would require at least three capacitors and the three possible circuits, employing three capacitors each, are shown in Figs. 2(*a*), (*b*) and (*c*).

FIG. 2. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0}{a_3S^3+a_2S^2+a_1S+1}$.



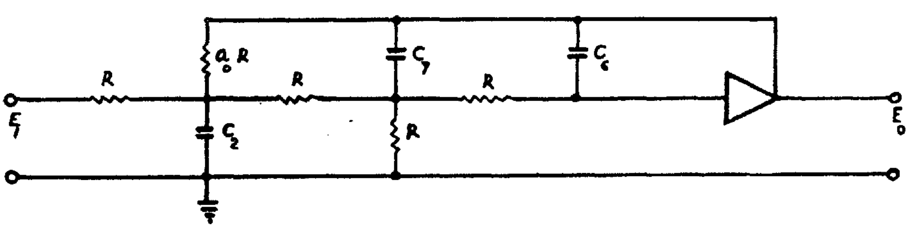
(a) $b_0 = \frac{\alpha^2}{(1+3\alpha)}$, $a_1 = \frac{\alpha}{(1+3\alpha)}RC_2+(1+\alpha)RC_6$, $a_2 = \frac{\alpha(1+2\alpha)}{(1+3\alpha)}R^2C_6(C_2+C_4)$,

$$a_3 = \frac{\alpha^2}{(1+3\alpha)}R^3C_2C_4C_6, \quad a_1 > \frac{a_3(1+2\alpha)}{a_2(1+3\alpha)}.$$



(b) $b_0 = \frac{a_0}{3}$, $a_1 = \frac{1}{3}[(5a_0+3)RC_6+(a_0+1)RC_8]$, $a_2 = \frac{1}{3}RC_6[3a_0RC_4+(2a_0+1)RC_8]$,

$$a_3 = \frac{a_0}{3}R^3C_4C_6C_8, \qquad a_1 > \frac{a_3}{a_2}(a_0+1).$$



(c) $b_0 = a_0$, $a_1 = [(5a_0+3)RC_6+(2a_0+1)RC_7]$,

$$a_2 = 3a_0R^2C_2C_6+a_0R^2C_2C_7+(2a_0+1)R^2C_6C_7, \quad a_3 = a_0R^3C_2C_6C_7.$$

(i) $a_1a_2 > a_0a_3$, (ii) $\Delta < 0$,

(iii) $\frac{2}{3}\frac{a_1}{(5a_0+3)} > \sqrt[3]{A}+\sqrt[3]{B}$ for $\Delta' > 0$ or $\frac{1}{3}\frac{a_1}{(5a_0+3)} > \begin{cases} +\sqrt{\frac{-p}{3}}\cos\frac{\phi}{3} \\ -\sqrt{\frac{-p}{3}}\cos\left(\frac{\pi}{3}-\frac{\phi}{3}\right) \\ -\sqrt{\frac{-p}{3}}\cos\left(\frac{\pi}{3}+\frac{\phi}{3}\right) \end{cases}$ for $\Delta' < 0$.

(a) Y_2, Y_4 and Y_6 capacitative

A network for simulating the system of (2*a*) with

$$\left.\begin{aligned} Y_2 &= SC_2 \\ Y_4 &= SC_4 \\ Y_6 &= SC_6 \\ Y_1 = Y_3 = Y_5 &= \frac{1}{R} \\ \text{and} \qquad Y_7 = Y_8 &= \frac{1}{\alpha R} \end{aligned}\right\} \quad \ldots \quad (3)$$

is shown in Fig. 2(*a*).

Substituting (3) into (1) and simplifying

$$\frac{E_0}{E_1} = -\frac{\dfrac{\alpha^2}{(1+3\alpha)}}{\dfrac{\alpha^2}{(1+3\alpha)} R^3 C_2 C_4 C_6 S^3 + \left[\dfrac{\alpha(1+2\alpha)}{(1+3\alpha)} R^2 C_6 (C_2+C_4)\right] S^2 + \left[\dfrac{\alpha}{(1+3\alpha)} RC_2 + (1+\alpha) RC_6\right] S + 1} \quad \ldots \quad (4)$$

Equations (2*a*) and (4) will be identical if

$$b_0 = \frac{\alpha^2}{(1+3\alpha)} \quad \ldots \quad (5)$$

$$a_1 = \frac{\alpha}{(1+3\alpha)} T_1 + (1+\alpha) T_3 \quad \ldots \quad (6)$$

$$a_2 = \frac{a(1+2\alpha)}{(1+3\alpha)} T_3 (T_1+T_2) \quad \ldots \quad (7)$$

$$a_3 = \frac{\alpha^2}{(1+3\alpha)} T_1 T_2 T_3, \quad \ldots \quad (8)$$

where

$$\left.\begin{aligned} T_1 &= RC_2 \\ T_2 &= RC_4 \\ T_3 &= RC_6 \end{aligned}\right\} \quad \ldots \quad (9)$$

Simulation of the system of (2*a*) with the network of Fig. 2(*a*) is possible only if the values of α, T_1, T_2 and T_3 obtained as the solution of (5) through (8) are real and positive. It is, therefore, required to determine the values of α, T_1, T_2 and T_3 in terms of the known real and positive constants b_0, a_1, a_2 and a_3, and find the conditions, if any, under which α, T_1, T_2, T_3 are real and positive.

The solution of (5) gives

$$\alpha = \frac{3b_0 + \sqrt{9b_0^2 + 4b_0}}{2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

as the negative root is inadmissible.

Eliminating T_2 and T_3 from (6) through (8) gives a cubic

$$T_1^3 - \frac{a_1(1+3\alpha)}{\alpha} T_1^2 + \frac{a_2(1+\alpha)(1+3\alpha)^2}{\alpha^2(1+2\alpha)} T_1 - \frac{a_3(1+\alpha)(1+3\alpha)^2}{\alpha^3} = 0, \quad \ldots \quad (11)$$

which will have either one real and two complex roots or all the three real roots, and these can be determined by methods that are very well known. It is obvious that (11) can have no negative real roots and therefore its real roots will be all positive. The corresponding values of T_2 and T_3 also will be real and positive, if, as shown in Appendix II,

$$a_1 > \frac{a_3(1+2\alpha)}{a_2(1+3\alpha)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Therefore, if the condition of expression (12) is satisfied, then it is possible to simulate the systems of (2*a*) with the network of Fig. 2(*a*). The circuit component values can be obtained by solving (11) for T_1 and substituting this value of T_1 and that of α as obtained from (10) in (6) and (8) to determine T_3 and T_2 respectively. Having thus determined T_1, T_2 and T_3 and choosing a convenient value for any one of the three capacitors—(say $C2$)—it is easy to calculate the values of R, C_4 and C_6 with the aid of (9).

(*b*) Y_4, Y_6 *and* Y_8 *capacitative*

A second possible choice, as shown in Fig. 2(*b*), is to make

$$\left.\begin{aligned} Y_4 &= SC_4 \\ Y_6 &= SC_6 \\ Y_8 &= SC_8 \\ Y_1 = Y_2 = Y_3 = Y_5 &= \frac{1}{R} \\ Y_7 &= \frac{1}{a_0R} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Substituting (13) into (1) and simplifying

$$\frac{E_0}{E_1} = -\frac{\frac{a_0}{3}}{\frac{a_0}{3}R^3C_4C_6C_8S^3 + \frac{1}{3}[3a_0R^2C_4C_6 + (2a_0+1)R^2C_6C_8]S^2 + \frac{1}{3}[(5a_0+3)RC_6 + (a_0+1)RC_8]S + 1} \quad \ldots \quad (14)$$

Equations (14) and (2*a*) will be identical if

$$b_0 = \frac{a_0}{3} \quad \ldots \quad (15)$$

$$a_1 = \tfrac{1}{3}[(5a_0+3)T_2+(a_0+1)T_3] \quad \ldots \quad (16)$$

$$a_2 = \frac{T_2}{3}[3a_0T_1+(2a_0+1)T_3] \quad \ldots \quad (17)$$

$$a_3 = \frac{a_0}{3}T_1T_2T_3, \quad \ldots \quad (18)$$

where

$$\left.\begin{aligned} T_1 &= RC_4 \\ T_2 &= RC_6 \\ T_3 &= RC_8 \end{aligned}\right\} \quad \ldots \quad (19)$$

Eliminating T_1 and T_2 from (16) through (18) gives a cubic

$$T_3^3 - \frac{3a_1}{(a_0+1)}T_3^2 + \frac{3a_2(5a_0+3)}{(a_0+1)(2a_0+1)}T_3 - \frac{9a_3(5a_0+3)}{(a_0+1)(2a_0+1)} = 0 \quad \ldots \quad (20)$$

which will have either one or all the three real and positive roots. It can be shown by the process of reasoning similar to that explained in Appendix II, that the corresponding values of T_1 and T_2 will be also real and positive if

$$a_1 > \frac{a_3}{a_2}(a_0+1). \quad \ldots \quad (21)$$

Therefore, if the condition of expression (21) is satisfied then it is possible to simulate the system of (2*a*) with the network of Fig. 2(*b*).

(*c*) *Y_2, Y_6 and Y_7 capacitative*

A third possible choice, as shown in Fig. 2(*c*), is to make

$$\left.\begin{aligned} Y_2 &= SC_2 \\ Y_6 &= SC_6 \\ Y_7 &= SC_7 \\ Y_1 = Y_3 = Y_4 = Y_5 &= \frac{1}{R} \\ Y_8 &= \frac{1}{a_0R} \end{aligned}\right\} \quad \ldots \quad (22)$$

Substituting (22) into (1) and simplifying

$$\frac{E_0}{E_1} = -\frac{a_0}{a_0R^3C_2C_6C_7S^3+[3a_0R^2C_2C_6+a_0R^2C_2C_7+(2a_0+1)R^2C_6C_7]S^2+[(5a_0+3)RC_6+(2a_0+1)RC_7]S+1} \quad \ldots \quad (23)$$

Equations (23) and (2a) will be identical if

$$b_0 = a_0 \qquad (24)$$

$$a_1 = (5a_0+3)T_2+(2a_0+1)T_3 \qquad (25)$$

$$a_2 = 3a_0T_1T_2+a_0T_1T_3+(2a_0+1)T_2T_3 \qquad (26)$$

$$a_3 = a_0T_1T_2T_3, \qquad (27)$$

where

$$\left.\begin{aligned} T_1 &= RC_2 \\ T_2 &= RC_6 \\ T_3 &= RC_7 \end{aligned}\right\} \qquad (28)$$

Eliminating T_1 and T_3 from (25) through (27) and simplifying

$$(5a_0+3)^2T_2^4-2a_1(5a_0+3)T_2^3+[a_1^2+a_2(5a_0+3)]T_2^2$$
$$-(a_1a_2-a_0a_3)T_2+a_1a_3 = 0. \qquad (29)$$

Now, (29) has four variations of signs if

$$a_1a_2 > a_0a_3 \qquad (30)$$

and according to Descartes' rule of signs, the number of roots that will be real and positive can be either four, two or zero. It is obvious, by inspection, that (29) can have no negative real roots. Therefore, (29) will have two real roots which will be positive if its discriminant is negative and the corresponding values of T_3 and T_1, as shown in Appendix III, will be real and positive if

$$\left.\begin{aligned} &\frac{2}{3}\frac{a_1}{(5a_0+3)} > \sqrt[3]{A}+\sqrt[3]{B} \text{ for } \Delta' > 0 \\ &\text{or} \\ &\frac{1}{3}\frac{a_1}{(5a_0+3)} > \begin{cases} +\sqrt{\frac{-p}{3}}\cos\frac{\phi}{3} \\ -\sqrt{\frac{-p}{3}}\cos\left(\frac{\pi}{3}-\frac{\phi}{3}\right) & \text{for } \Delta' < 0 \\ -\sqrt{\frac{-p}{3}}\cos\left(\frac{\pi}{3}+\frac{\phi}{3}\right) \end{cases} \end{aligned}\right\} \qquad (31)$$

where

$$\cos\phi = \frac{\sqrt{27q}}{2p\sqrt{-p}}$$

The roots of (29) can be easily determined by techniques that are very well known and discussed in textbooks on higher algebra. Having thus determined T_2 by solving (29) and knowing a_0 directly from (24) the corresponding values of T_3 and T_1 can be very conveniently determined from (25) and (27).

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Appendix I

Analysis of an Operational Amplifier and its Associated Networks

Block diagrammatic representation of an operational amplifier and its associated input feedback networks is shown in Fig. 1(*a*) and its equivalent circuit is shown in Fig. 1(*b*). The nodal equations of the network, by inspection, may be written as

$$E_1Y_1 = (Y_1+Y_2+Y_3+Y_8)E_2 - Y_3E_3 - O\,.\,E_g - Y_8E_0 \quad \ldots \quad (1.1)$$

$$O = -Y_3E_2 + (Y_3+Y_4+Y_5+Y_7)E_3 - Y_5E_g - Y_7E_0 \quad \ldots \quad (1.2)$$

$$O = O\,.\,E_2 - Y_5E_3 + (Y_5+Y_6)E_g - Y_6E_0 \quad \ldots \quad (1.3)$$

$$O = -Y_8E_2 - Y_7E_3 - (Y_6-Y_a)E_g + (Y_6+Y_7+Y_8)E_0 \quad \ldots \quad (1.4)$$

and the characteristic determinant is therefore

$$\Delta = \begin{vmatrix} (Y_1+Y_2+Y_3+Y_8), & -Y_3\,, & 0\,, & -Y_8 \\ -Y_3\,, & (Y_3+Y_4+Y_5+Y_7), & -Y_5\,, & -Y_7 \\ 0\,, & -Y_5\,, & (Y_5+Y_6), & -Y_6 \\ -Y_8\,, & -Y_7\,, & -(Y_6-Y_a), & (Y_6+Y_7+Y_8) \end{vmatrix} \quad (1.5)$$

whence

$$E_0 = \begin{vmatrix} (Y_1+Y_2+Y_3+Y_8), & -Y_3\,, & 0\,, & E_1Y_1 \\ -Y_3\,, & (Y_3+Y_4+Y_5+Y_7), & -Y_5\,, & 0 \\ 0\,, & -Y_5\,, & (Y_5+Y_6), & 0 \\ -Y_8\,, & -Y_7\,, & -(Y_6-Y_a), & 0 \end{vmatrix} \div \Delta \quad \ldots \quad (1.6)$$

assuming the gain of the amplifier to be very large, i.e. $Y_a \to \infty$, then

$$E_0 = -Y_a \begin{vmatrix} (Y_1+Y_2+Y_3+Y_8), & -Y_3\,, & E_1Y_1 \\ -Y_3\,, & (Y_3+Y_4+Y_5+Y_7), & 0 \\ 0\,, & -Y_5\,, & 0 \end{vmatrix}$$

$$\div -Y_a \begin{vmatrix} (Y_1+Y_2+Y_3+Y_8), & -Y_3\,, & -Y_8 \\ -Y_3\,, & (Y_3+Y_4+Y_5+Y_6), & -Y_7 \\ 0\,, & -Y_5\,, & -Y_6 \end{vmatrix} \quad \ldots \quad (1.7)$$

Simplifying and rearranging equation (1.7) gives the transfer function

$$\frac{E_0}{E_1} = -\frac{Y_1Y_3Y_5}{Y_6(Y_1+Y_2+Y_8)(Y_3+Y_4+Y_5+Y_7)+Y_3Y_6(Y_4+Y_5+Y_7)+Y_5Y_7(Y_1+Y_2+Y_3+Y_8)+Y_3Y_5Y_8} \quad (1.8)$$

It should be borne in mind that equation (1.8) is obtained on the assumptions that the amplifier has infinite input impedance, very wide band-width, very large gain and very low output impedance.

APPENDIX II

CONDITIONS FOR POSITIVE REAL ROOTS

Simulation of the third order system as characterized by (2*a*) by the circuit of Fig. 2(*a*) is possible only if α, T_1, T_2 and T_3 obtained as the solution of the equations

$$b_0 = \frac{\alpha^2}{(1+3\alpha)} \qquad \text{(2.1)}$$

$$a_1 = \frac{\alpha}{(1+3\alpha)} T_1 + (1+\alpha) T_3 \qquad \text{(2.2)}$$

$$a_2 = \frac{\alpha(1+2\alpha)}{(1+3\alpha)} T_3(T_1+T_2) \qquad \text{(2.3)}$$

$$a_3 = \frac{\alpha^2}{(1+3\alpha)} T_1 T_2 T_3 \qquad \text{(2.4)}$$

are real and positive; where b_0, a_1 a_2 and a_3 are real and positive constants.

Eliminating T_2 from equations (2.3) and (2.4)

$$T_3 = \frac{a_2(1+3\alpha)}{\alpha(1+2\alpha)} \cdot \frac{1}{T_1} - \frac{a_3(1+3\alpha)}{\alpha^2} \cdot \frac{1}{T_1^2} \qquad \text{(2.5)}$$

and plotting (2.2) and (2.5) in the T_1, T_3 plane will give respectively a straight line and a curve of the form shown in Fig. 2.1. The intersection of the straight line with the curve in the first quadrant will yield real and positive values for T_1 and T_3 and from (2.4) T_2 will also be real and positive.

Now, the curve of (2.5) and the straight line of (2.2) cross the $T_3 = 0$-axis (i.e. T_1-axis) at

$$T_1' = \frac{a_3(1+2\alpha)}{a_2 \cdot \alpha}$$

and

$$T_1'' = \frac{a_1(1+3\alpha)}{\alpha}$$

respectively. Therefore, if

$$a_1 > \frac{a_3(1+2\alpha)}{a_2(1+3\alpha)}, \qquad \text{(2.6)}$$

then the intersection of the straight line with the curve will give T_1 and T_3 both positive and real.

Now, on examining Fig. 2.1, it will be evident that the straight line and the curve, under certain conditions, can intersect at three points giving three sets of real and positive values of T_1 and T_3 and the corresponding values of T_2 will also be real and positive. Eliminating T_3 from (2.2) and (2.5) gives a cubic

$$T_1^3 - \frac{a_1(1+3\alpha)}{\alpha} T_1^2 + \frac{a_2(1+\alpha)(1+3\alpha)^2}{\alpha^2(1+2\alpha)} T_1 - \frac{a_3(1+\alpha)(1+3\alpha)^2}{\alpha^3} = 0 \quad .. \quad (2.7)$$

whose all the three roots will be real and positive if its discriminant is negative.

The roots of the cubic of (2.7) can be easily determined by techniques that are very well known and discussed at length in texts on higher algebra.

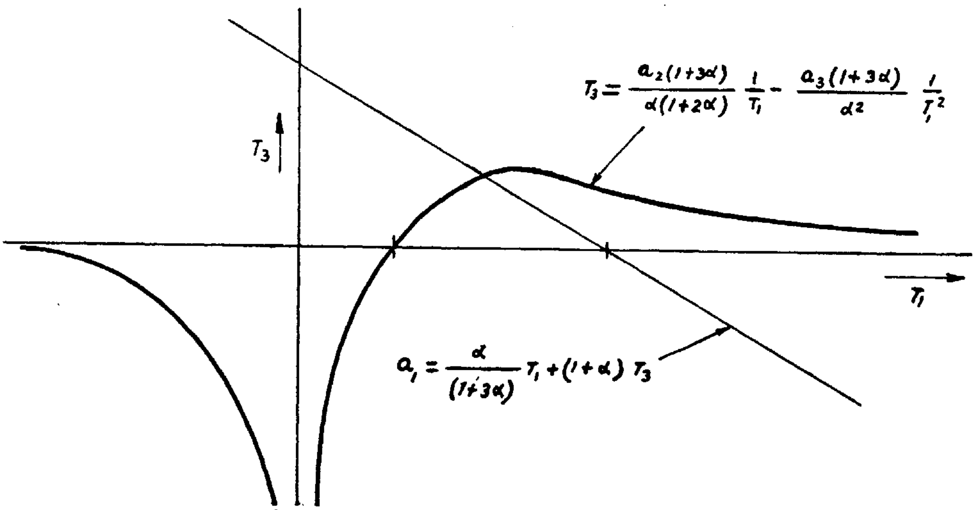

FIG. 2.1. Determination of conditions for real positive roots.

APPENDIX III

CONDITIONS FOR POSITIVE REAL ROOTS OF A QUARTIC

Simulation of the third order system of equation 2(a) with the circuit of Fig. 2(c) is possible only if a_0, T_1, T_2 and T_3 obtained as the solution of equations

$$b_0 = a_0 \quad .. \quad (3.1)$$

$$a_1 = (5a_0+3)T_2+(2a_0+1)T_3 \quad .. \quad (3.2)$$

$$a_2 = 3a_0T_1T_2+a_0T_1T_3+(2a_0+1)T_2T_3 \quad .. \quad (3.3)$$

$$a_3 = a_0T_1T_2T_3 \quad .. \quad (3.4)$$

are real and positive; a_0, a_1, a_2 and a_3 being real and positive constants.

Eliminating T_1 from equations (3.3) and (3.4)

$$a_2 = \frac{3a_3}{T_3}+\frac{a_3}{T_2}+(2a_0+1)T_2T_3 \quad \ldots \quad (3.5)$$

and from equations (3.2) and (3.5) after simplification and re-arrangement

$$\frac{3a_3}{T_3} = a_2-\frac{a_3}{T_2}-a_1T_2+(5a_0+3)T_2^2 \quad \ldots \quad (3.6)$$

and

$$(5a_0+3)^2T_2^4-2a_1(5a_0+3)T_2^3+\left[a_1^2+a_2(5a_0+3)\right]T_2^2-(a_1a_2-a_0a_3)T_2+a_1a_3 = 0 \quad (3.7)$$

The conditions for positive real T_1, T_2 and T_3 can be conveniently obtained graphically by plotting the curve of (3.6) and the straight line of (3.2) in the T_2, T_3 plane. The shape of the curve will depend upon the values of a_0, a_1, a_2 and a_3 and will assume any one of the three forms sketched in Fig. 3.1. It will assume the form of either Fig. 3.1(a) or 3.1(b) if the discriminant Δ' of the cubic obtained by equating (3.6) to zero is positive. That is, if

$$\Delta' = 4p^3+27q^2 > 0 \quad \ldots \quad (3.8)$$

then the only real root which will be positive is

$$T_2' = \sqrt[3]{A}+\sqrt[3]{B}+\frac{a_1}{3(5a_0+3)}, \quad \ldots \quad (3.9)$$

where

$$A = -\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$$

$$B = -\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$$

$$p = +\frac{a_2}{(5a_0+3)}-\frac{a_1^2}{3(5a_0+3)^2}$$

$$q = -\frac{a_3}{(5a_0+3)}+\frac{a_1a_2}{3(5a_0+3)^2}-\frac{2}{27}\frac{a_1^3}{(5a_0+3)^3}$$

and is represented in the sketch as point M. And if

$$\Delta' = 4p^3+27q^2 < 0 \quad \ldots \quad (3.10)$$

then the cubic has three real roots all of which will be positive and the largest of the three roots is represented as point N in Fig. 3.1(c). The largest of the roots can be determined by solving the cubic.

Now, if the discriminant of (3.7) is less than zero then it has two real roots, which means that the straight line of (3.2) and the curve of (3.6) intersect at two real points.

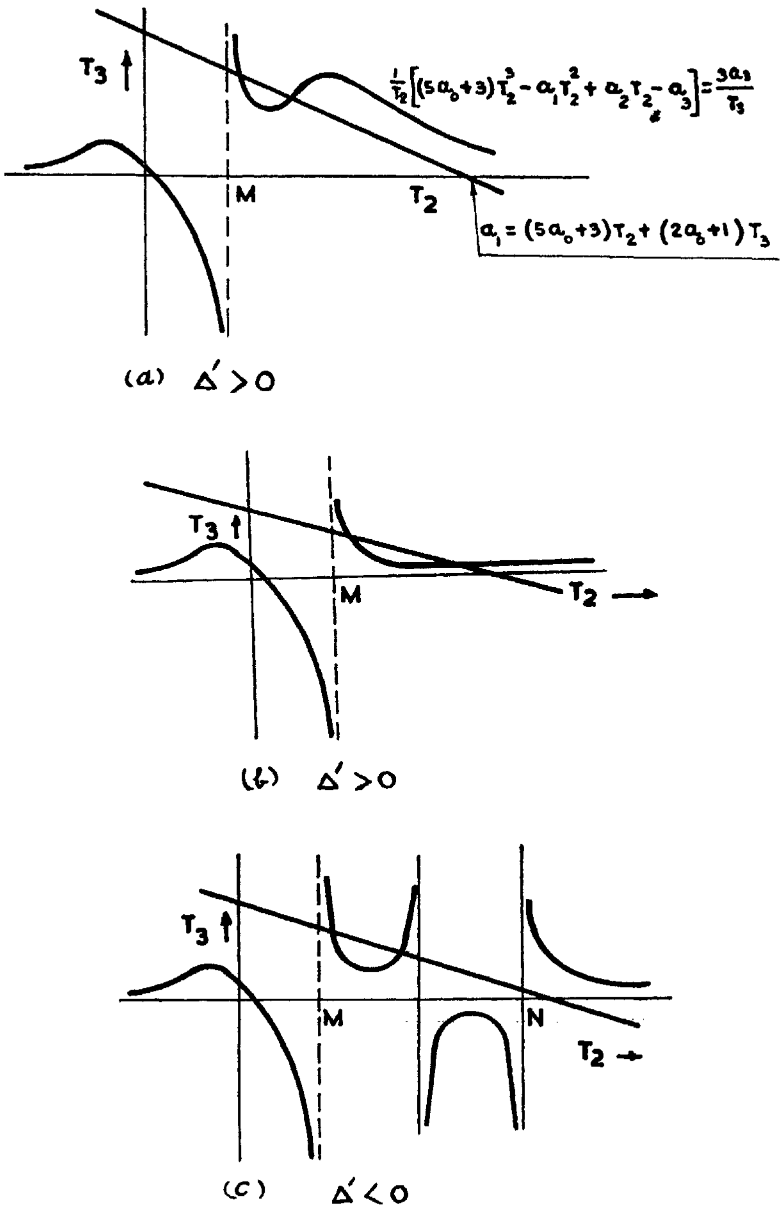

FIG. 3.1. Determination of two real and positive roots of a quartic.

Since it is obvious, by inspection, that (3.7) can have no negative real roots, therefore the two real roots will be positive.

The corresponding values of T_3 will be real and positive if the distance from the origin of the point of intersection of the straight line with the T_2-axis is greater than that of point M for the case $\Delta' > 0$; that is

$$\frac{a_1}{(5a_0+3)} > \sqrt[3]{A}+\sqrt[3]{B}+\frac{a_1}{3(5a_0+3)} \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

or for the case $\Delta' < 0$

$$\frac{a_1}{(5a_0+3)} > \begin{cases} 2\sqrt{\dfrac{-p}{3}}\,.\,\cos\dfrac{\phi}{3}+\dfrac{a_1}{3(5a_0+3)} \\ -2\sqrt{\dfrac{-p}{3}}\,.\,\cos\left(\dfrac{\pi}{3}-\dfrac{\phi}{3}\right)+\dfrac{a_1}{3(5a_0+3)} \\ -2\sqrt{\dfrac{-p}{3}}\,.\,\cos\left(\dfrac{\pi}{3}+\dfrac{\phi}{3}\right)+\dfrac{a_1}{3(5a_0+3)} \end{cases} \quad \ldots \quad \ldots \quad (3.12)$$

where

$$\cos\phi = \frac{\sqrt{27}q}{2p\sqrt{-p}}.$$

If T_2 and T_3 are real and positive then from (3.4) it is obvious that T_1 is also real and positive.