CURVES AND INVARIANTS ASSOCIATED WITH A VECTOR FIELD OF A RIEMANNIAN V_m IN RELATION TO A CURVE C IN A SUBSPACE V_n

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R. S. Mishra and Shri Krishna (1956) have defined the absolute curvature and normal curvature of a congruence of curves in a Riemannian V_m w.r.t. a curve C in a subspace V_n by considering the derived vector of λ along C, and its normal component in V_m. In this paper, we deal with the tangential component of this derived vector, and define the geodesic curvature of the congruence λ w.r.t. C and study some of its properties. We also introduce lines of curvature and asymptotic lines of two different types associated with the congruence λ, and study a number of relations between the corresponding curvatures. We next obtain a general expression for the tendency of λ w.r.t. a curve C in V_n. Our formula reduces to a theorem of C. E. Weatherburn as a particular case when λ is taken tangential to V_n. We also obtain a number of results on tendency.

1. In 1952, Pan introduced the idea of the absolute curvature vector of a vector field v on a surface in ordinary space w.r.t. a curve C on it, by defining it as the derived vector (intrinsic derivative) of v along C. In the same paper, he also extended it to a hypersurface V_n of a Riemannian V_{n+1} confining himself to a vector field in V_n. Mishra and Shri Krishna (1956) extended the notion by considering a congruence* of curves in V_m and defined absolute curvature of the congruence w.r.t. a curve C in a subspace V_n. They further introduced the notions of principal directions, principal curvatures, lines of curvature, etc., along V_n w.r.t. the congruence. The notion of lines of curvature had been earlier given by Mishra (1952) in a different way.

2. Following the work of Mishra and Shri Krishna, we have developed this subject further in this paper and obtained a number of new results. We have also felt the need to systematize this work, because in the papers of Mishra, and Mishra and Shri Krishna, identical names have been proposed for notions which are not equivalent.

3. Let V_m be a Riemannian space of m dimensions with y^α as the coordinates of any point, and $a_{\alpha\beta}\, dy^\alpha\, dy^\beta$ as the fundamental metric (α, $\beta = 1$,

* Actually Mishra and Shri Krishna refer to $m-n$ congruences of curves. But their work is almost exclusively confined to a single congruence, and so this is equivalent to considering a vector field of V_m tangential to this congruence at points on V_n.

$2, \ldots, m$). Let V_n be a subspace with coordinates x^i and fundamental metric $g_{ij}dx^i dx^j$ $(i, j = 1, 2, \ldots, n)$.
Therefore

$$g_{ij} = a_{\alpha\beta}\, y^{\alpha}_{;i}\, y^{\beta}_{;j}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let

$$N^{\alpha}_{\nu|},\ (\nu = 1, 2, \ldots, m-n)$$

be the contravariant components of a system of $m-n$ linearly independent orthogonal unit normal vectors to V_n, so that

$$a_{\alpha\beta} N^{\alpha}_{\nu|} N^{\beta}_{\mu|} = \delta^{\nu}_{\mu} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$a_{\alpha\beta}\, y^{\alpha}_{;i} N^{\beta}_{\nu|} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where the semicolon indicates tensor differentiation w.r.t. the x's.

We have also

$$y^{\alpha}_{;i} = y^{\alpha}_{,i}.$$

Let us consider a congruence of curves in V_m such that one curve of the congruence passes through each point of V_m and hence through each point of V_n also. Let λ^{α} be the contravariant components in V_m of the unit vector in the direction of a curve of the congruence (we shall call it λ), passing through a point P of V_n. The congruence is of general position, but we may later consider as particular cases a normal congruence and a congruence tangential to V_n.

Resolving λ^{α} tangentially to V_n and along the chosen set of normals to V_n, we have

$$\lambda^{\alpha} = t^i y^{\alpha}_{;i} + \sum_{\nu} c_{\nu|} N^{\alpha}_{\nu|}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where t^i form the components of a vector in V_n and $c_{\nu|}$ has the value $\cos\,\theta_{\nu|}$, $\theta_{\nu|}$ being the angle between the vectors λ and $N_{\nu|}$.

The tensor derivative of (4) gives

$$\lambda^{\alpha}_{;i} = t^{l}_{;i}\, y^{\alpha}_{;l} + t^{l} y^{\alpha}_{;li} + \sum_{\nu} c_{\nu|;i} N^{\alpha}_{\nu|} + \sum_{\nu} c_{\nu|} N^{\alpha}_{\nu|;i}. \qquad \ldots \quad \ldots \quad (5)$$

Using the known formulae (Weatherburn 1957, p. 163, § 90, p. 170, § 95),

$$y^{\alpha}_{;ij} = \sum_{\nu} \Omega_{\nu|ij} N^{\alpha}_{\nu|}$$

$$N^{\alpha}_{\nu|;i} = -\Omega_{\nu|ik}\, g^{kl} y^{\alpha}_{;l} + \sum_{\mu} \theta_{\mu\nu|i} N^{\alpha}_{\mu|}, \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

where $\theta_{\mu\nu|i}$ are certain constants, skew-symmetric in μ and ν, and defined by $\theta_{\mu\nu|i} = a_{\alpha\beta} N^{\alpha}_{\nu|;i} N^{\beta}_{\mu|}$, the above simplifies to

$$\lambda^{\alpha}_{;i} = \sum_{\nu} \left(t^l \Omega_{\nu|li} + c_{\nu|;i} + \sum_{\mu} c_{\mu|} \theta_{\mu\nu|i} \right) N^{\alpha}_{\nu|} + \left(t^l_{;i} - \sum_{\nu} c_{\nu|} \Omega_{\nu|ik} g^{kl} \right) y^{\alpha}_{;l}. \qquad (7)$$

Let $x^i = x^i(s)$ define a curve C in V_n passing through the point P. The intrinsic derivative or the derived vector of λ in the direction of C will be denoted by $\frac{\delta\lambda}{\delta s}$ and is given by $\frac{\delta\lambda^{\alpha}}{\delta s} = \lambda^{\alpha}_{;i} \frac{dx^i}{ds}$. Hence from (7)

$$\frac{\delta\lambda^{\alpha}}{\delta s} = \sum_{\nu} \left(t^l \Omega_{\nu|li} \frac{dx^i}{ds} + \frac{d}{ds}(c_{\nu|}) + \sum_{\mu} c_{\mu|} \theta_{\mu\nu|i} \frac{dx^i}{ds} \right) N^{\alpha}_{\nu|} + \left(\frac{\delta t^l}{\delta s} - \sum_{\nu} c_{\nu|} \Omega_{\nu|ik} g^{kl} \frac{dx^i}{ds} \right) y^{\alpha}_{;l}. \qquad (8)$$

We shall denote the magnitude of this vector by $\kappa_{\lambda|}$ and the components of the unit vector along this direction by $\omega^{\alpha}_{\lambda|}$, so that

$$\frac{\delta\lambda^{\alpha}}{\delta s} = \kappa_{\lambda|} \omega^{\alpha}_{\lambda|}. \qquad (9)$$

$\kappa_{\lambda|}$ is defined by Mishra and Shri Krishna as *the absolute curvature of the congruence* λ *w.r.t. the curve* C. They then define *the normal curvature of the congruence* λ *w.r.t.* C *as the magnitude of the normal component of the derived vector of* λ *divided by the magnitude of the tangential component* t *of* λ. We shall write this as $\kappa_{\lambda|n}$.
Hence

$$\kappa^2_{\lambda|n} = \sum_{\nu} \left(t^j \Omega_{\nu|ij} \frac{dx^i}{ds} + \frac{d}{ds}(c_{\nu|}) + \sum_{\mu} c_{\mu|} \theta_{\mu\nu|i} \frac{dx^i}{ds} \right) \times \left(t^l \Omega_{\nu|kl} \frac{dx^k}{ds} + \frac{d}{ds}(c_{\nu|}) + \sum_{\sigma} c_{\sigma|} \theta_{\sigma\nu|k} \frac{dx^k}{ds} \right) \div \left(1 - \sum_{\nu} c^2_{\nu|} \right), \qquad (10)$$

the square of the magnitude of t being $1 - \sum_{\nu} c^2_{\nu|}$.

4. We next consider the magnitude of the tangential component of the derived vector of λ in (8) divided by the magnitude of the tangential component t of λ. This is the tangential component of the absolute curvature divided by the magnitude of t relative to V_n. Hence, we shall call this as the *geodesic curvature of the congruence* λ *w.r.t.* C, following the lines of the definition of geodesic curvature of a curve as the curvature relative to V_n. Pan (1952) calls this as the associate curvature of the vector field w.r.t. the curve.

Denoting this by $\kappa_{\lambda|g}$ we have

$$\kappa^2_{\lambda|g} = g_{lm}\left(t^m_{;i} - \sum_\nu c_{\nu|}\Omega_{\nu|ik}g^{km}\right)$$

$$\left(t^l_{;j} - \sum_\nu c_{\nu|}\Omega_{\nu|jk}g^{kl}\right)\frac{dx^i}{ds}\frac{dx^j}{ds} \div g_{kl}t^k t^l. \quad \ldots \quad \ldots \quad (11)$$

When λ is tangential to V_n at P, $t = 1$
and

$$\kappa^2_{\lambda|g} = g_{lm}t^l_{;i}t^m_{;j}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

$$= g_{lm}\frac{\delta t^l}{\delta s}\frac{\delta t^m}{\delta s}.$$

In this case, $\kappa_{\lambda|g}$ = magnitude of the derived vector of t along C.

When λ defines a curve tangential to C, the expressions $\kappa_{\lambda|n}$ and $\kappa_{\lambda|g}$ reduce to the normal and geodesic curvature respectively of C in V_n. This offers the justification for the nomenclatures adopted by Mishra and Shri Krishna and in the present paper.

(8) can be written

$$\frac{\delta\lambda^\alpha}{\delta s} = \kappa_{\lambda|}\omega^\alpha_{\lambda|} = (\kappa_{\lambda|n}N^\alpha + \kappa_{\lambda|g}a^\alpha)t, \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where a and N are the unit vectors along the tangential and normal components of $\frac{\delta\lambda}{\delta s}$ expressed in the coordinates of V_m, and where t is the magnitude of the vector t^i. Squaring (12), we get

$$\kappa^2_{\lambda|} = t^2(\kappa^2_{\lambda|n} + K^2_{\lambda|g}). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Also $\kappa_{\lambda|}\cos\theta = \kappa_{\lambda|n}t$ where θ is the angle between the direction of $\frac{\delta\lambda}{\delta s}$ and N. *This is the analogue of Meunier's Theorem for the congruence under consideration.*

5. A direction in V_n for which $\kappa_{\lambda|n}$ is an extremum has been called by Mishra and Shri Krishna (1956) as a principal direction of the congruence λ and the corresponding value of $\kappa_{\lambda|n}$ as a principal curvature of the congruence λ. A curve in V_n whose direction at each point is a principal direction of the congruence is called a line of curvature of the congruence λ. The principal directions of the congruence λ are given by

$$(\kappa^2_{\lambda|n}g_{ij} - \psi_{ij})dx^j = 0 \quad (i = 1, 2, \ldots, n) \quad \ldots \quad \ldots \quad (14)$$

where $\kappa^2_{\lambda|n}$ are the roots of the equation

$$\left|\kappa^2_{\lambda|n}g_{ij} - \psi_{ij}\right| = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

and ψ_{ij} is the symmetric covariant tensor of the second order, given by

$$t^2\psi_{ij}=\sum_{\nu}\left(t^k\Omega_{\nu\,|\,ik}+c_{\nu\,|\,;\,i}+\sum_{\mu}c_{\mu\,|}\,\theta_{\mu\nu\,|\,i}\right)$$

$$\times\left(t^l\Omega_{\nu\,|\,jl}+c_{\nu\,|\,;\,j}+\sum_{\sigma}c_{\sigma\,|}\,\theta_{\sigma\nu\,|\,j}\right). \qquad \ldots \qquad \ldots \quad (16)$$

It is a known result that principal directions so defined are mutually orthogonal for any two different values of $\kappa_{\lambda\,|\,n}$. These values are all real, if $g_{ij}dx^i dx^j$ is positive definite.

In his earlier paper, Mishra (1952), however, defines lines of curvature corresponding to λ by the property that the component in V_n of $\frac{\delta\lambda}{\delta s}$ is co-directional with the curve C, in other words

$$\left(\frac{\delta t^k}{\delta s}-\sum_{\nu}c_{\nu\,|}\,\Omega_{\nu\,|\,il}\,g^{lk}\frac{dx^i}{ds}\right)y^{\alpha}_{;\,k}=cy^{\alpha}_{;\,i}\frac{dx^i}{ds}, \qquad \ldots \qquad \ldots \quad (17)$$

where c is a scalar. The directions tangential to these lines of curvature are not mutually orthogonal.

It is necessary to point out that the two definitions are not equivalent. According to the first definition, a line of curvature corresponding to λ is tangential to the principal direction, i.e. the direction for which $\kappa_{\lambda\,|\,n}$ is a maximum or minimum. The component in V_n of the vector $\frac{\delta\lambda}{\delta s}$ which figures in the second definition is not a 'principal direction' in this sense, except when λ happens to be normal to V_n in a V_{n+1}.

In the general case, there is justification to introduce the concept of lines of curvature associated with λ by either method. But the system of curves obtained are different. We shall accordingly call the two systems as *lines of curvature of the first and second types* respectively associated with λ. It is, however, found that the matrix $\|\psi_{ij}\|$ is of rank 1. Hence there is only one non-zero root of (15), the others being all zero. The non-zero value gives a unique principal direction. Hence there is a unique line of curvature of the first type through each point of V_n and in general n lines of curvature of the second type. The normal curvatures corresponding to the directions of these curves will be called *principal curvature and quasi-principal curvatures* respectively.

Now choose the curve C to be a line of curvature of the second type associated with λ. Using (17), we have

$$\lambda^{\alpha}_{;i}\,e^i=t\kappa_{\lambda\,|\,n}N+cy^{\alpha}_{;i}\,e^i,$$

where c is the quasi-principal curvature associated with λ. On squaring the above equation, we have

$$\kappa_{\lambda|}^2 = t^2\kappa_{\lambda|n}^2 + c^2 ,$$

a relation connecting the absolute and normal curvatures of the congruence λ w.r.t. C and the quasi-principal curvature.

6. Let us next define the tensor ϕ_{ij} by the relation

$$t^2\phi_{ij} = g_{lm}\left(t^m_{;i} - \sum_\nu c_{\nu|}\Omega_{\nu|ik}g^{km}\right)\left(t^l_{;j} - \sum_\nu c_{\nu|}\Omega_{jh}g^{hl}\right) \quad .. \quad (18)$$

so that

$$\kappa_{\lambda|g}^2 = \phi_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

The directions for which the geodesic curvature $\kappa_{\lambda|g}$ is an extremum are given by

$$\left(\kappa_{\lambda|g}^2 g_{ij} - \phi_{ij}\right) dx^j = 0 \quad (i = 1, 2, \ldots, n)$$

where $\kappa_{\lambda|g}^2$ are the roots of the equation

$$|\kappa^2 g_{ij} - \phi_{ij}| = 0. \quad .. \quad .. \quad .. \quad (18')$$

As before, $\|\phi_{ij}\|$ is of rank 1, and there is a unique non-zero root in κ of (18′) giving a unique direction which we call the *geodesic principal direction* of the congruence λ, and the corresponding value of $\kappa_{\lambda|g}$ as the *geodesic principal curvature* of the congruence.

A curve in V_n whose direction at each point is a geodesic principal direction of λ at the point will be called *a geodesic line of curvature** of λ. Also if the geodesic curvature of the congruence λ w.r.t. C vanishes at every point of a curve, the curve will be called *a λ-geodesic.* Hence *the absolute curvature of the congruence w.r.t. a λ-geodesic curve is equal to the normal curvature of the congruence w.r.t. that curve.* The differential equation of the λ-geodesics on V_n is given by

$$\phi_{ij}dx^i dx^j = 0.$$

7. If $\kappa_{\lambda|} = 0$, then for a real curve C, $\kappa_{\lambda|n} = 0$, $\kappa_{\lambda|g} = 0$, then C will be called an *absolute geodesic* w.r.t. the congruence λ. When λ is tangential to the curve C, we have $\kappa_{\lambda|g} = \kappa_g$. If, therefore, C is an absolute geodesic relative to its own tangent field, C is a geodesic as well as an asymptotic line of V_n in the usual sense. Hence by a well-known result, C is a geodesic of V_m.

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* The nomenclature is by no means happy, but no better names can be suggested. There will, however, be no occasion for confusing this curve as a combination of a geodesic and a line of curvature, since we are dealing with the congruence λ.

8. If the curve C be such that at each point the normal curvature of the congruence λ, i.e. the normal component of the intrinsic derivative $\frac{\delta\lambda}{\delta s}$, is zero, then Mishra and Shri Krishna (1956) call C as an asymptotic line of the congruence λ. We shall call this as *an asymptotic line of the first type.* The differential equation of an asymptotic line of the first type is therefore

$$\psi_{ij}dx^i dx^j = 0.$$

The absolute curvature of the congruence λ w.r.t. an asymptotic line of the first type is equal to its geodesic curvature w.r.t. that curve.

It is possible to introduce the concept of an asymptotic line in a different way as a curve in V_n such that the derived vector of λ relative to C is orthogonal to C. We shall call such a curve as *an asymptotic line of the second type relative to* λ.

The condition that the derived vector of λ w.r.t. C is orthogonal to C is that

$$a_{\alpha\beta}\lambda^{\alpha}_{;i}e^i y^{\beta}_{;j}e^j = 0,$$

i.e.
$$\left(t_{j;i} - \sum_{\nu} c_{\nu|}\Omega_{\nu|ij}\right)e^i e^j = 0.$$

The differential equation of the asymptotic lines of the second type is therefore

$$\left(t_{j;i} - \sum_{\nu} c_{\nu|}\Omega_{\nu|ij}\right)dx^i dx^j = 0.$$

Singal and Behari (1955) have defined 'a generalized asymptotic line relative to the congruence' by this property, in other words a generalized asymptotic line according to them is an asymptotic line of the second type according to the present paper. They also define generalized normal curvature of the curve on the hypersurface relative to the congruence by the expression

$$(r\Omega_{ij} - t_{j;i})\frac{dx^i}{ds}\frac{dx^j}{ds} \text{ where } r = a_{\alpha\beta}\lambda^{\alpha}N^{\beta},$$

and hence the generalized asymptotic line relative to a congruence is a curve in V_n such that at every point of it the generalized normal curvature relative to the congruence as defined by them is zero.

Mishra (1952) suggests a third possible definition for asymptotic lines. He defines two directions dx^i and δx^i as conjugate directions w.r.t. a set of $m-n$ mutually orthogonal congruences $\lambda_{\Gamma|}$ ($\Gamma = 1, 2, \ldots, m-n$) if

$$\sum_{\Gamma}\left(t_{\Gamma|j;i} - \sum_{\nu} c_{\Gamma\nu|}\Omega_{\nu|ij}\right)\left(t_{\Gamma|k;l} - \sum_{\mu} c_{\Gamma\mu|}\Omega_{\mu|kl}\right)dx^i\delta x^j dx^k \delta x^l = 0.$$

A curve whose direction is self-conjugate is an asymptotic line w.r.t. this set of congruences. Here and only here Mishra uses all the congruences, and he uses only one congruence in the rest of his work. As we have also been doing likewise, we shall not refer again to this third type of asymptotic curves.

9. Let $e^i_{h|}$ $(h = 1, 2, \ldots, n)$ be the unit vectors of an orthogonal ennuple in V_n.
Now

$$\kappa^2_{\lambda|n} = \psi_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$$

and

$$\kappa^2_{\lambda|g} = \phi_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}.$$

Since

$$\sum_h \psi_{ij} e^i_{h|} e^j_{h|} = \psi_{ij} g^{ij},$$

and

$$\sum_h \phi_{ij} e^i_{h|} e^j_{h|} = \phi_{ij} g^{ij},$$

we obtain the following results.

The sum of the squares of the normal curvatures of the congruence λ w.r.t. curves forming an orthogonal ennuple on V_n at a point P is an invariant independent of the orthogonal ennuple selected. We may call this as the *mean curvature of the congruence λ w.r.t. V_n at P.* In particular consider the principal direction of the congruence and a set of $n-1$ mutually orthogonal asymptotic directions orthogonal to the principal direction. $\kappa_{\lambda|n} = 0$, for the asymptotic direction. Hence the mean curvature of the congruence λ $=$ square of the principal normal curvature. We get a *similar result for the sum of the squares of the geodesic curvatures of the congruence λ.*

From (12), we get the same result for the *sum of the squares of the absolute curvatures of the congruence w.r.t. an orthogonal ennuple of V_n.*

For a given λ, choose $e_{h|}$ $(h = 1, 2, \ldots, n)$ to be the unique geodesic principal direction and a set of directions tangential to mutually orthogonal λ-geodesics, which are also orthogonal to the geodesic principal direction.

The geodesic principal curvature $\bar{c}_{1|}$ is given by

$$\bar{c}^{\,2}_{1|} = \phi_{ij} e^i_{1|} e^j_{1|} \text{ while } \phi_{ij} e^i_{h|} e^j_{h|} = 0, \quad h = 2, \ldots, n.$$

Let u^i be the unit tangent vector to any curve C on V_n

$$\begin{aligned}\kappa^2_{\lambda|g} &= \phi_{ij} u^i u^j \\ &= \sum_h \phi_{ij} e^i_{h|} e^j_{h|} \cos^2 \theta_h,\end{aligned}$$

where θ_1 is the angle between the curve C and the geodesic principal direction $e_{1|}$

$$\kappa^2_{\lambda|g} = \bar{c}^2_{1|} \cos^2\theta_1.$$

This gives the geodesic curvature of the congruence λ *w.r.t. any curve in terms of the geodesic principal curvature, and resembles Euler's formula for normal curvature.*

We can obtain a similar result starting with

$$\kappa^2_{\lambda|n} = \psi_{ij} e^i_{h|} e^j_{h|}.$$

Euler's formula in fact can be generalized. Let a_{ij} be any symmetric covariant tensor and let $E^i_{h|}$ $(h = 1, 2, \ldots, n)$ be the associated principal directions (Weatherburn 1957, pp. 47–50, § 31). These are known to be orthogonal. Let $a_{ij}u^iu^j$ be called a curvature of some type or any invariant associated with the direction u and $a_{ij}E^i_{h|}E^j_{h|}$, $(h = 1, 2, \ldots, n)$ the 'principal curvatures' or invariants of the same type. Then the above proof can be repeated and we have the analogue of Euler's formula for this curvature or invariant.

10. Squaring $\frac{\delta\lambda}{\delta s}$ and using equations (16) and (18) we can write

$$\kappa^2_{\lambda|} = t^2(\psi_{ij}+\phi_{ij})e^ie^j.$$

Let us call $t^2(\psi_{ij}+\phi_{ij}) = \Phi_{ij}$.

Φ_{ij} is therefore a symmetric covariant tensor.

The differential equation of absolute geodesics is therefore given by

$$\Phi_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0.$$

The principal directions associated with Φ_{ij} may be called the *absolute principal directions* of the congruence λ. We can prove that $\|\Phi_{ij}\|$ is of rank 2 and hence two absolute lines of curvature which are mutually orthogonal pass through any point of V_n. The curves tangential to these are the *absolute lines of curvature.* Observing the remarks made at the end of § 9 we have the analogue of Euler's formula with regard to absolute curvatures.

A pair of directions in V_n, for which the derived vector of λ in one direction is orthogonal to the derived vector of λ in the other direction, is given by

$$a_{\alpha\beta}\lambda^{\alpha}_{;i}e^i\lambda^{\beta}_{;j}a^j = 0,$$

$$\text{i.e.} \qquad \Phi_{ij}e^ia^j = 0.$$

We may call these directions as *absolute conjugate directions* of the congruence λ. It follows that the absolute geodesics are absolute self-conjugate curves—otherwise obvious since a vector is orthogonal to itself if and only if it is a null vector.

11. Let E^{α}, e^{i} be the components of the unit tangent vector to C in V_m and in V_n respectively.

$$a_{\alpha\beta}\lambda^{\beta}_{;k}E^{\alpha} = a_{\alpha\beta}\left(t^{l}_{;k} - \sum_{\nu} c_{\nu|}\Omega_{\nu|kj}g^{jl}\right) y^{\beta}_{;l} y^{\alpha}_{;i} e^{i}$$

$$= g_{il}\left(t^{l}_{;k} - \sum_{\nu} c_{\nu|}\Omega_{\nu|kj}g^{jl}\right)e^{i}$$

$$= g_{il}t^{l}_{;k}e^{i} - \sum_{\nu} c_{\nu|}\Omega_{\nu|kj}e^{j}.$$

Multiply by e^{k} and sum w.r.t. k.

$$a_{\alpha\beta}\lambda^{\beta}_{;k}E^{\alpha}e^{k} = g_{il}t^{l}_{;k}e^{i}e^{k} - \sum_{\nu} c_{\nu|}\Omega_{\nu|kj}e^{j}e^{k} \quad \ldots \quad (19)$$

The left side is the tendency of λ in the direction of the curve C in V_m. Call it $D_{\lambda|}$. The first term on the right is the tendency of the vector t along the curve. Call it $d_{\lambda|}$.

Now

$$c_{\nu|} = \cos\theta_{\nu|} = a_{\alpha\beta}\lambda^{\alpha}N^{\beta}_{\nu|}.$$

$$\sum_{\nu} c_{\nu|}\Omega_{\nu|kl}e^{k}e^{l} = a_{\alpha\beta}\sum_{\nu}\Omega_{\nu|kl}e^{k}e^{l}N^{\alpha}_{\nu|}\lambda^{\beta}$$

$$= a_{\alpha\beta}\kappa_n N^{\alpha}\lambda^{\beta},$$

where $\kappa_n N^{\alpha}$ is the normal curvature vector (Weatherburn 1957, p. 164, § 92) of C,

$= \kappa_n \cos\Theta$, Θ being the angle made by λ with the normal curvature vector of C. We thus have the result

$$D_{\lambda|} = d_{\lambda|} - \kappa_n \cos\Theta \quad \ldots \quad (20)$$

We have the following particular cases:

(1) When λ is normal to V_n, we may take $\lambda = N_{\nu|}$. The right-hand side of (19) reduces to $\Omega_{\nu|kl}e^{k}e^{l}$ = resolved part of the normal curvature vector in the direction of the normal λ.

Hence we have *the tendency of any normal w.r.t.* C = *resolved part of the normal curvature vector in the direction of that normal.*

(2) When λ is tangential to V_n, the formula (20) reduces to $D_{\lambda|} = d_{\lambda|}$, i.e. the tendency of λ w.r.t. the curve C in V_m = the tendency of λ w.r.t. C in V_n. This is a result due to Weatherburn (1957, p. 94, § 53). *The result (20) is therefore the generalization of Weatherburn's result for any vector of* V_m *passing through a point of* V_n.

If V_n is a hypersurface of V_{n+1}, Θ becomes the angle between λ and the normal to the hypersurface.

(3) If C is an asymptotic line (in the usual sense) of V_n, $\kappa_n = 0$, and hence the *tendency of any vector λ in the direction of an asymptotic line* = the *tendency of the tangential component of λ along V_n, in the same direction.*

12. The tendency of the vector λ in the direction of the unit vector $v_{h|}$ along V_n is $a_{\alpha\beta}\lambda^{\alpha}_{;i}v^i_{h|}y^{\beta}_{;l}v^l_{h|}$

$$= g_{lm}v^i_{h|}v^l_{h|}\left(t^m_{;i} - \sum_\nu c_{\nu|}\Omega_{\nu|ik}g^{km}\right),$$

using (5) and (6).

Hence, the sum of the squares of the tendencies of the vector λ along n mutually orthogonal directions in V_n

$$= \sum_h \left[g_{lm}v^i_{h|}v^l_{h|}\left(t^m_{;i} - \sum_\nu c_{\nu|}\Omega_{\nu|ik}g^{km}\right)\right.$$

$$\left. g_{pq}v^j_{h|}v^q_{h|}\left(t^p_{;j} - \sum_\nu c_{\nu|}\Omega_{\nu|jk}g^{kp}\right)\right]$$

$$= g_{lm}g_{pq}g^{ij}g^{lq}\left(t^m_{;i} - \sum_\nu c_{\nu|}\Omega_{\nu|ik}g^{km}\right)$$

$$\times \left(t^p_{;j} - \sum_\nu c_{\nu|}\Omega_{\nu|jk}g^{kp}\right)$$

$$= g_{pm}\left(t^m_{;i} - \sum_\nu c_{\nu|}\Omega_{\nu|ik}g^{km}\right)\left(t^p_{;j} - \sum_\nu c_{\nu|}\Omega_{\nu|jk}g^{kp}\right)g^{ij}$$

$$= t^2\phi_{ij}g^{ij}, \text{ from (18).}$$

Hence, the sum of the squares of the tendencies of the vector λ along n mutually orthogonal directions in V_n = the sum of the squares of the geodesic curvatures of the congruence λ w.r.t. these directions multiplied by the square of the magnitude of the tangential component of λ along V_n. The sum is therefore independent of the ennuple chosen.

13. We shall finally consider the congruence λ to be tangential to V_n. Then

$$\lambda^\alpha = t^i y^\alpha_{;i}$$

We have

$$a_{\alpha\beta}\lambda^\alpha N^\beta_{\nu|} = 0,$$

i.e. $\lambda^\alpha N_{\nu|\alpha} = 0$ for all values of ν. $(\nu = 1, 2, \ldots, m-n)$.

$$\lambda^\alpha_{,\beta}N_{\nu|\alpha} + \lambda^\alpha N_{\nu|\alpha,\beta} = 0.$$

Multiplying by λ^β,

$$\lambda^\alpha{}_{,\beta}\lambda^\beta N_{\nu|\alpha} = -N_{\nu|\alpha,\beta}\lambda^\alpha\lambda^\beta.$$

The right side is the negative of the tendency of the unit normal in the direction of a curve of the congruence λ.

$$\lambda^{\alpha}_{,\beta}\lambda^{\beta}N_{\nu|\alpha} = \lambda^{\alpha}_{,\beta}y^{\beta}_{;i}t^{i}N^{\alpha}_{\nu|}$$
$$= a_{\alpha\gamma}\lambda^{\alpha}_{;i}N^{\gamma}_{\nu|}\, t^{i}.$$

Since λ^{α} is tangential to V_n, equation (7) reduces to

$$\lambda^{\alpha}_{;i} = t^{l}_{;i}y^{\alpha}_{;l} + \sum_{\nu}\Omega_{\nu|li}t^{l}N^{\alpha}_{\nu||}.$$

Hence

$$\lambda^{\alpha}_{,\beta}\lambda^{\beta}N_{\nu|\alpha} = \Omega_{\nu|li}t^{l}t^{i}.$$

Choose $e_{h|}\,(h = 1, 2, \ldots, n)$ to be the unit vectors along the principal directions of V_n at the point P.

We have $t^{i} = \sum_{h} e^{i}_{h|} \cos\theta_h$, where θ_h is the angle between λ and the principal direction. Substituting for t,

$$\lambda^{\alpha}_{,\beta}\lambda^{\beta}N_{\nu|\alpha} = \Omega_{\nu|li}\sum_{h} e^{l}_{h|}\cos\theta_h \sum_{k} e^{i}_{k|}\cos\theta_k$$
$$= \sum_{h} \Omega_{\nu|li}e^{l}_{h|}\,e^{i}_{h|}\cos^2\theta_h$$
$$= \sum_{h} \kappa_h \cos^2\theta_h,$$

where κ_h is the principal curvature of V_n corresponding to the normal $N_{\nu|}$. Hence we have the following result: *The negative of the tendency of the unit normal in the direction of a curve of the congruence is equal to* $\sum_{h}\kappa_h \cos^2\theta_h$.

In particular, if λ is tangential to a line of curvature, only one term in the above summation remains. Therefore, *the tendency of the unit normal in the direction of a line of curvature is equal to the corresponding principal curvature.*

When λ is tangential to the curve C, we get the result that *the negative of the tendency of the unit normal in the direction of the curve C is equal to the resolved part of the normal curvature vector of the curve in the direction of the corresponding normal.* This result has been obtained otherwise by Mishra (1952).

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