SIMULATION OF THE GENERAL THIRD ORDER SYSTEMS BY A SINGLE OPERATIONAL AMPLIFIER—I

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(Communicated by V. R. Thiruvenkatachar, F.N.I.)

(*Received September 27, 1962*)

A method for the simulation of the general third order linear systems using only one operational amplifier is described. Two basic circuits each employing one operational amplifier, three capacitors and seven resistors are presented. The circuits are analysed and the conditions of physical realizability discussed and obtained. The design formulae are also given.

INTRODUCTION

In the earlier communications (Wadhwa 1962*a*, *b*, *c*, 1963*a*, *b*; Wadhwa and Rao 1963) on this subject certain particular types of the general third order systems were considered for simulation by a single operational amplifier and two-terminal impedances consisting of capacitors and resistors only. The purpose of this paper is to consider the simulation of the general third order systems, that is systems characterized by a transfer function of the form

$$F(S) = -\frac{b_0(b_2S^2+b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where a's and b's are non-zero positive real constants.

Of the various possible circuits, each employing three capacitors and seven resistors only two will be presented here, and their design formulae and the conditions of physical realizability will also be discussed and obtained.

THIRD ORDER SYSTEM SIMULATION

A network for the simulation of third order systems is shown in Fig. 1 and its transfer function has been shown (Wadhwa 1963*b*) to be

$$\frac{E_0}{E_1} = -\frac{Y_1Y_3Y_5}{Y_6(Y_1+Y_2+Y_8)(Y_3+Y_4+Y_5+Y_7)+Y_3Y_6(Y_4+Y_5+Y_7)+ Y_5Y_7(Y_1+Y_2+Y_3+Y_8)+Y_3Y_5Y_8} \quad \ldots \quad (2)$$

Simulation of the system of eqn. (1) with the network of Fig. 1 is possible if the admittances (Y's) are properly chosen; and, furthermore, it should be obvious from eqn. (2) that at least three of the appropriate admittances will be required to be capacitative. Six basic circuits each employing three capacitors and seven resistors are possible.

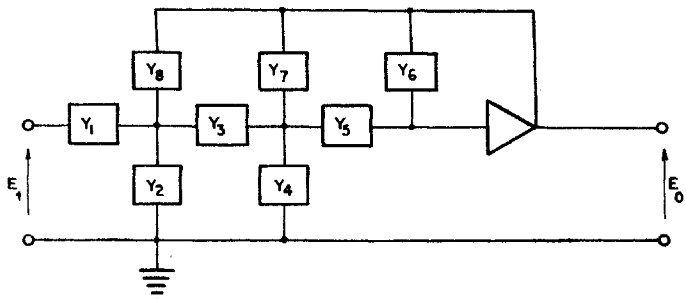

FIG. 1. Network for the simulation of third order systems.

Each basic circuit can have ten parameters and as the number of constants in the system represented by eqn. (1) is six, the choice of resistor values is required to be made so as to reduce the number of circuit parameters to the number of constants in eqn. (1). This requirement offers considerable latitude in the choice of the intended design values of the resistors as a result of which a number of circuits are possible which are essentially a variation of the basic circuit employing three capacitors and seven resistors. Each circuit, for its physical realizability, will require satisfaction of a set of conditions which will be different for every circuit.

For obvious reasons and other practical considerations two of the six basic circuits with a certain arbitrary choice of resistor values will be presented here. The other four basic circuits will be presented in Part II and Part III of the papers on this subject.

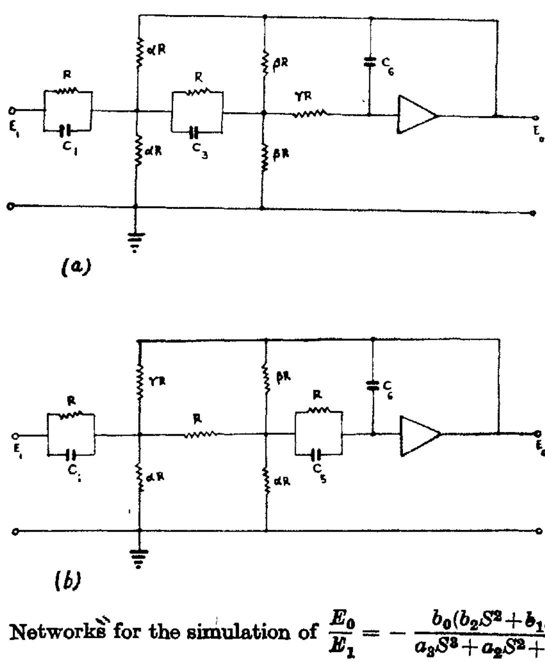

FIG. 2. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0(b_2S^2+b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}$.

(*a*) Y_1, Y_3 *and* Y_6 *capacitative*

A possible circuit for simulating the system of eqn. (1) is shown in Fig. 2 (*a*), in which

$$\left.\begin{aligned} Y_1 &= \left(SC_1+\frac{1}{R}\right); \quad Y_3 = \left(SC_3+\frac{1}{R}\right); \quad Y_6 = SC_6; \\ Y_2 &= Y_8 = \frac{1}{\alpha R}; \quad Y_4 = Y_7 = \frac{1}{\beta R}; \quad Y_5 = \frac{1}{\gamma R} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

Substituting eqn. (3) into eqn. (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\left(\frac{\alpha\beta}{2\alpha+\beta+2}\right)[R^2C_1C_3S^2+R(C_1+C_3)S+1]}{\frac{\alpha\beta\gamma}{(2\alpha+\beta+2)}R^3C_1C_3C_6S^3+\left[\frac{\alpha(\beta+\beta\gamma+2\gamma)}{(2\alpha+\beta+2)}R^2C_1C_6+\frac{(\alpha\beta+\alpha\beta\gamma+2\alpha\gamma+2\beta\gamma)}{(2\alpha+\beta+2)}R^2C_3C_6\right]S^2+\left[\frac{\alpha}{(2\alpha+\beta+2)}RC_1+\frac{(\alpha+\beta)}{(2\alpha+\beta+2)}RC_3+\frac{(\alpha+2)(\beta+\beta\gamma+2\gamma)+\alpha(\beta+2\gamma)}{(2\alpha+\beta+2)}RC_6\right]S+1} \quad \ldots \quad (4)$$

Eqns. (1) and (4) will be identical if

$$b_0 = \frac{\alpha\beta}{(2\alpha+\beta+2)} \quad \ldots \quad (5)$$

$$b_1 = T_1+T_3 \quad \ldots \quad (6)$$

$$b_2 = T_1T_3 \quad \ldots \quad (7)$$

$$a_1 = \frac{\alpha}{(2\alpha+\beta+2)}T_1+\frac{(\alpha+\beta)}{(2\alpha+\beta+2)}T_3+\frac{(\alpha+2)(\beta+\beta\gamma+2\gamma)+\alpha(\beta+2\gamma)}{(2\alpha+\beta+2)}T_6 \quad (8)$$

$$a_2 = \frac{\alpha(\beta+\beta\gamma+2\gamma)}{(2\alpha+\beta+2)}T_1T_6+\frac{(\alpha\beta+\alpha\beta\gamma+2\alpha\gamma+2\beta\gamma)}{(2\alpha+\beta+2)}T_3T_6 \quad \ldots \quad (9)$$

$$a_3 = \frac{\alpha\beta\gamma}{(2\alpha+\beta+2)}T_1T_3T_6 \quad \ldots \quad (10)$$

where

$$T_n = RC_n. \quad \ldots \quad (11)$$

Now, simulation of the system of eqn. (1) with the network of Fig. 2(*a*) is possible only if the values of α, β, γ, T_1, T_3, T_6 obtained as the solution of eqns. (5) through (10) are positive and real. It is required to determine therefore, in terms of the given positive real *a*'s and *b*'s, the values of α, β, γ, T_1, T_3, T_6 and find the conditions, if any, under which these can be positive and real.

It is obvious from eqn. (5), that

$$\beta = \frac{2b_0(\alpha+1)}{(\alpha-b_0)}, \quad \ldots \quad (12)$$

and β can be real positive, if α is real and

$$\alpha > b_0. \qquad (13)$$

From eqns. (6) and (7)

$$T_1 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \qquad (14a)$$

$$T_3 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \qquad (14b)$$

and these will be real positive, if

$$b_1^2 \geqslant 4b_2. \qquad (15)$$

Elimination of β, γ, T_1, T_3 and T_6 from eqns. (5) through (10) gives a cubic

$$P_0\alpha^3 - P_1\alpha^2 + P_2\alpha - 8a_3T_3 = 0, \qquad (16)$$

which has at least one real root that will be positive and corresponding to which, as shown in Appendix I, positive real T_6 and γ exist provided that a set of conditions listed under any one of the four cases mentioned below is satisfied.

(i) $A_1 > 0$; $A_2 > 0$.

Either

$$\left.\begin{aligned} \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) &> \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) \\ \frac{A_2}{b_1} &> \frac{A_1}{4} \end{aligned}\right\} \qquad (17)$$

or

$$\left.\begin{aligned} \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) &> \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) \\ \frac{A_1}{4} &> \frac{A_2}{b_1} \end{aligned}\right\} \qquad (18)$$

(ii) $A_1 > 0$; $A_2 < 0$, $B_2 > 0$.

$$\left.\begin{aligned} &\Delta_2 > 0 \\ \text{and} \quad &\left(\frac{-B_2-\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) \end{aligned}\right\} \qquad (19)$$

(iii) $A_1 < 0$, $B_1 > 0$; $A_2 > 0$.

$$\left.\begin{aligned} &\Delta_1 > 0 \\ \text{and} \quad &\left(\frac{-B_1-\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) \end{aligned}\right\} \qquad (20)$$

(iv) $A_1 < 0$, $B_1 > 0$; $A_2 < 0$, $B_2 > 0$.

$$\left.\begin{aligned} \Delta_1 > 0 \\ \Delta_2 > 0 \end{aligned}\right\} \qquad (21)$$

and

$$\text{either } \left(\frac{-B_2-\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1-\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) \quad .. \text{ (21a)}$$

$$\text{or } \left(\frac{-B_1-\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2-\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) \quad .. \text{ (21b)}$$

where

$$\left.\begin{aligned}
&P_0 = b_2(4a_2+b_1^2)-2b_1(a_1b_2+a_3)\,;\\
&P_1 = 2(a_1b_1b_2+a_3b_1+4a_3T_3)+b_0b_1b_2(T_1-T_3)-8a_2b_2\,;\\
&P_2 = 4(a_2b_2-4a_3T_3)\,;\ A_1 = 2a_1b_0b_2-(4a_3+2a_3b_0+b_0b_1b_2)\,;\\
&A_2 = a_2b_0b_2-a_3b_1(1+b_0)\,;\ B_1 = b_0b_2\{2a_1+b_0(T_1-T_3)\}-2a_3(2+b_0)\,;\\
&B_2 = b_0(a_2b_2-2a_3T_3)\,;\ \Delta_1 = (B_1^2+8A_1b_0^2b_2T_3)\,;\ \Delta_2 = (B_2^2+8A_2a_3b_0T_3)
\end{aligned}\right\} \quad (22)$$

For the design of the network, circuit component values are required to be determined. The proper procedure for design would be first to check and see if the inequality of expression (15) is satisfied and then solve (Uspensky 1948) the cubic of eqn. (16) to see if its real root(s) is/are greater than b_0—thus satisfying the inequality of expression (13). Thereafter we compute the values of A_1, A_2, B_1, B_2 with the aid of eqn. (22) and find if any one of the four cases mentioned above is applicable; and then check and see if the set of conditions listed under the relevant case is satisfied. The satisfaction of these conditions signifies that the circuit of Fig. 2(*a*) for the simulation of the system of eqn. (1) is physically realizable. The circuit component values may then be obtained by substituting the positive real and larger than b_0 value of α into eqn. (12) and obtain β; T_1 and T_3 may be obtained from eqns. (14*a*) and (14*b*). T_6 and γ may then be obtained from eqns. (1.11) and (7). Having thus determined T_1, T_3, T_6 and choosing arbitrarily a convenient value for any one of the capacitors, the values for the resistors and the remaining capacitors can then be determined with the aid of eqn. (11).

(*b*) Y_1, Y_5 *and* Y_6 *capacitative*

Another possible circuit for simulating the system of eqn. (1) with

$$\left.\begin{aligned}
&Y_1 = \left(SC_1+\frac{1}{R}\right);\quad Y_5 = \left(SC_5+\frac{1}{R}\right);\quad Y_6 = SC_6\,;\\
&Y_3 = \frac{1}{R};\quad Y_2 = Y_4 = \frac{1}{\alpha R};\quad Y_7 = \frac{1}{\beta R};\quad Y_8 = \frac{1}{\gamma R}
\end{aligned}\right\} \quad .. \ (23)$$

is shown in Fig. 2(*b*).

Substituting eqn. (23) into eqn. (2) and simplifying

$$\frac{E_0}{E_1} = - \frac{\frac{\alpha\beta\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}[R^2C_1C_5S^2+R(C_1+C_5)S+1]}{\frac{\alpha\beta\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}R^3C_1C_5C_6S^3+\left[\frac{\alpha\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}R^2C_1C_5+\frac{\gamma(\alpha+2\alpha\beta+\beta)}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}R^2C_1C_6+\frac{\beta(\alpha+2\alpha\gamma+\gamma)}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}R^2C_5C_6\right]S^2+\left[\frac{\alpha\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}RC_1+RC_5+\frac{(\alpha+\alpha\gamma+\gamma)(\alpha+2\alpha\beta+\beta)+\alpha\gamma(\alpha+\alpha\beta+\beta)}{\alpha(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}RC_6\right]S+1}. \quad (24)$$

Eqns. (1) and (24) will be identical if

$$b_0 = \frac{\alpha\beta\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)} \quad (25)$$

$$b_1 = T_1+T_5 \quad (26)$$

$$b_2 = T_1T_5 \quad (27)$$

$$a_1 = \frac{\alpha\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1+T_5+\frac{(\alpha+\alpha\gamma+\gamma)(\alpha+2\alpha\beta+\beta)+\alpha\gamma(\alpha+\alpha\beta+\beta)}{\alpha(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_6 \quad (28)$$

$$a_2 = \frac{\alpha\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1T_5+\frac{\gamma(\alpha+2\alpha\beta+\beta)}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1T_6+\frac{\beta(\alpha+2\alpha\gamma+\gamma)}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_5T_6 \quad (29)$$

$$a_3 = \frac{\alpha\beta\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1T_5T_6 \quad (30)$$

where

$$T_n = RC_n. \quad (31)$$

The solution of eqns. (26) and (27) gives

$$T_1 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2-4b_2}\right] \quad (32a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2-4b_2}\right] \quad (32b)$$

and these will be positive real, if

$$b_1^2 > 4b_2. \quad (33)$$

Elimination of α, γ, T_1, T_5 and T_6 from eqns. (25) through (30) gives a quartic

$$P_1\beta^4+P_2\beta^3+P_3\beta^2+P_4\beta+P_5 = 0, \quad (34)$$

where

$$\left.\begin{aligned}
P_1 &= a_3b_0b_1A_1+2a_3b_0T_5(a_2b_2-2a_3b_1)-a_3^2T_5^2-b_0^2(a_2b_2-2a_3b_1)^2\\
P_2 &= a_3b_0T_1A_1+a_3b_0b_1A_2+4a_3^2b_0T_5^2+2a_3b_0T_5(a_2b_2-2a_3b_1)(1-2b_0)\\
&\quad +2b_0^2(a_3T_1+b_0b_2^2)(a_2b_2-2a_3b_1)-2a_3b_0T_5(a_3T_1+b_0b_2^2)-2b_0^2(a_2b_2-2a_3b_1)^2\\
P_3 &= a_3b_0(T_1A_2+b_1A_3)+4b_0^2(a_3T_1+b_0b_2^2)(a_2b_2-2a_3b_1)\\
&\quad -b_0^2\left\{(a_2b_2-2a_3b_1)^2+(a_3T_1+b_0b_2^2)^2\right\}-4a_3^2b_0^2T_5^2\\
&\quad -2a_3b_0T_5(a_3T_1+b_0b_2^2)(1-2b_0)-4a_3b_0^2T_5(a_2b_2-2a_3b_1)\\
P_4 &= a_3b_0(T_1A_3+b_1A_4)+4a_3b_0^2T_5(a_3T_1+b_0b_2^2)\\
&\quad +2b_0^2(a_3T_1+b_0b_2^2)\left\{(a_2b_2-2a_3b_1)-(a_3T_1+b_0b_2^2)\right\}\\
P_5 &= a_3b_0T_1A_4-b_0^2(a_3T_1+b_0b_2^2)^2\\
A_1 &= b_0b_1b_2(a_1-T_5)-a_3b_1(2+3b_0)\\
A_2 &= a_3T_5+b_0b_2(a_1-T_5)(b_1+T_1)-a_3(2T_1+3b_0T_1+b_1+b_0b_1)\\
&\quad -b_0(a_2b_2-2a_3b_1)-b_0^2b_1b_2T_1\\
A_3 &= b_0b_2T_1(a_1-T_5)+b_0(a_3T_1+b_0b_2^2)-2a_3b_0T_5-a_3T_1(1+b_0)\\
&\quad -b_0(a_2b_2-2a_3b_1)-b_0^2b_2T_1(b_1+T_1)\\
A_4 &= b_0(a_3T_1+b_0b_2^2)-b_0^2b_2T_1^2.
\end{aligned}\right\} \quad (35)$$

Now, eqn. (34) will have at least one positive real root corresponding to which, as shown in Appendix II, a set of positive real α, γ, T_6 exists provided that

$$a_2^2 < 4a_3b_1\left(1+\frac{a_3T_1}{b_0b_2^2}\right) \quad \ldots \quad (36)$$

and a set of conditions listed under any one of the eight cases mentioned below is satisfied

(i) <u>$A_1 > 0,\ A_4 > 0;\ B_1 > 0.$</u>

Either

$$\left.\begin{aligned}
\Delta &> 0\\
B_1 &> \sqrt{a_3b_0b_1A_1}
\end{aligned}\right\} \quad \ldots \quad (37)$$

or

$$\left.\begin{aligned}
\Delta &< 0\\
A_2 &> 0\\
A_3 &> 0\\
B_1 &> \sqrt{a_3b_0b_1A_1}
\end{aligned}\right\} \quad \ldots \quad (37a)$$

or

$$\left.\begin{array}{rl} & \Delta < 0 \\ \text{and either} & A_3 < 0 \\ \text{or} & \left.\begin{array}{l} A_2 < 0 \\ A_3 > 0 \end{array}\right\} \\ \text{and either} & \lambda_2 > \mu_1 \\ \text{or} & \left.\begin{array}{l} \lambda_3 > \mu_1 > \lambda_2 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} \mu_1 > \lambda_3 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1} \end{array}\right\} \end{array}\right\} \qquad (37b)$$

(ii) $A_1 > 0,\ A_4 < 0;\ B_1 > 0.$

Either

$$\left.\begin{array}{rl} & \Delta > 0 \\ \text{and either} & \left.\begin{array}{l} \lambda_0 > \mu_1 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} \mu_1 > \lambda_0 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1} \end{array}\right\} \end{array}\right\} \qquad (38)$$

or

$$\left.\begin{array}{rl} & \Delta < 0 \\ \text{and either} & \left.\begin{array}{l} \lambda_3 > \mu_1 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1 \end{array}\right\},\ \text{or}\ \left.\begin{array}{l} \mu_1 > \lambda_3 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1} \end{array}\right\},\ \text{or} \\ & \left.\begin{array}{l} \lambda_1 > \mu_1 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1 \end{array}\right\},\ \text{or}\ \lambda_2 > \mu_1 > \lambda_1,\ \text{or} \\ & \left.\begin{array}{l} \lambda_3 > \mu_1 > \lambda_2 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1 \end{array}\right\},\ \text{or}\ \left.\begin{array}{l} \mu_1 > \lambda_3 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1} \end{array}\right\} \end{array}\right\} \qquad (38a)$$

(iii) $A_1 < 0,\ A_4 > 0;\ B_1 > 0.$

Either

$$\left.\begin{array}{l} \Delta > 0 \\ \lambda_0 > \mu_1 \end{array}\right\} \qquad (39)$$

or

$$\left.\begin{array}{rl} & \left.\begin{array}{l} \Delta < 0 \\ \lambda_3 > \mu_1 \end{array}\right\} \\ \text{and either} & A_3 > 0 \\ \text{or} & \left.\begin{array}{l} A_2 < 0 \\ A_3 < 0 \end{array}\right\} \end{array}\right\} \qquad (39a)$$

$$\left.\begin{array}{lr}
\text{or} & \Delta < 0 \\
 & A_2 > 0 \\
 & A_3 < 0 \\
\text{and either} & \lambda_1 > \mu_1 \\
\text{or} & \lambda_3 > \mu_1 > \lambda_2
\end{array}\right\} \quad \ldots \quad (39b)$$

(iv) $\underline{A_1 < 0,\ A_4 < 0;\ B_1 > 0.}$

$$\left.\begin{array}{lr}
 & \Delta < 0 \\
 & \lambda_3 > \mu_1 > \lambda_2 \\
\text{and either} & A_2 > 0 \\
\text{or} & \left.\begin{array}{r} A_2 < 0 \\ A_3 > 0 \end{array}\right\}
\end{array}\right\} \quad \ldots \quad (40)$$

(v) $\underline{A_1 > 0,\ A_4 > 0;\ B_1 < 0,\ B_2 > 0,\ \Delta_1 > 0.}$

$$\left.\begin{array}{lr}
 & \Delta < 0 \\
\text{and either} & A_3 < 0 \\
\text{or} & \left.\begin{array}{r} A_2 < 0 \\ A_3 > 0 \end{array}\right\} \\
\text{and either} & \mu_2 > \lambda_2 > \mu_1 \\
\text{or} & \mu_2 > \lambda_3 > \mu_1
\end{array}\right\} \quad \ldots \quad (41)$$

(vi) $\underline{A_1 > 0,\ A_4 < 0;\ B_1 < 0,\ B_2 > 0,\ \Delta_1 > 0.}$

$$\left.\begin{array}{lr}
\text{Either} & \Delta > 0 \\
 & \mu_2 > \lambda_0 > \mu_1
\end{array}\right\} \quad \ldots \quad (42)$$

$$\left.\begin{array}{lr}
\text{or} & \left.\begin{array}{r} \Delta < 0 \\ \mu_2 > \lambda_3 > \mu_1 \end{array}\right\} \\
\text{and either} & A_2 > 0 \\
\text{or} & \left.\begin{array}{r} A_2 < 0 \\ A_3 < 0 \end{array}\right\}
\end{array}\right\} \quad \ldots \quad (42a)$$

$$\left.\begin{array}{lr}
\text{or} & \Delta < 0 \\
 & A_2 < 0 \\
 & A_3 > 0 \\
\text{and either} & \lambda_2 > \mu_2 > \lambda_1 > \mu_1 \\
\text{or} & \mu_2 > \lambda_2 > \mu_1 > \lambda_1 \\
\text{or} & \mu_2 > \lambda_3 > \mu_1
\end{array}\right\} \quad \ldots \quad (42b)$$

(vii) $\underline{A_1 < 0,\ A_4 > 0;\ B_1 < 0,\ B_2 > 0,\ \Delta_1 > 0.}$

$$\left.\begin{array}{lr}
\text{Either} & \Delta > 0 \\
 & \mu_2 > \lambda_0 > \mu_1
\end{array}\right\} \quad \ldots \quad (43)$$

$$\left.\begin{array}{ll} \text{or} & \left.\begin{array}{r}\Delta < 0\\ \mu_2 > \lambda_3 > \mu_1\end{array}\right\} \\ \text{and either} & A_3 > 0 \\ \text{or} & \left.\begin{array}{l}A_2 < 0\\ A_3 < 0\end{array}\right\} \end{array}\right\} \quad \ldots \quad (43a)$$

$$\text{or}\quad \left.\begin{array}{ll} & \Delta < 0 \\ & A_2 > 0 \\ & A_3 < 0 \\ \text{and either} & \mu_2 > \lambda_1 > \mu_1 \\ \text{or} & \lambda_3 > \mu_2 > \lambda_2 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 > \lambda_2 \end{array}\right\} \quad \ldots \quad (43b)$$

(viii) $A_1 < 0,\ A_4 < 0;\ B_1 < 0,\ B_2 > 0,\ \Delta_1 > 0$.

$$\left.\begin{array}{ll} & \Delta < 0 \\ \text{and either} & A_2 > 0 \\ \text{or} & \left.\begin{array}{l}A_2 < 0\\ A_3 > 0\end{array}\right\} \\ \text{and either} & \lambda_3 > \mu_2 > \lambda_2 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 > \lambda_2 \end{array}\right\} \quad \ldots \quad (44)$$

where

$$\left.\begin{array}{l}
B_1 = a_2 b_0 b_2 - a_3(2b_0 b_1 + T_5);\ B_2 = b_0\left[a_2 b_2 - 3a_3 T_1 - b_0 b_2^2\right]; \\
\Delta_1 = B_2^2 + 4b_0 B_1\left(a_3 T_1 + b_0 b_2^2\right);\ \Delta = 4p^3 + 27q^2; \\
p = \dfrac{A_3}{A_1} - \dfrac{1}{3}\left(\dfrac{A_2}{A_1}\right)^2;\ q = \dfrac{A_4}{A_1} - \dfrac{A_2 A_3}{3A_1^2} + \dfrac{2}{27}\left(\dfrac{A_2}{A_1}\right)^3; \\
\lambda_0 = \sqrt[3]{A} + \sqrt[3]{B} - \dfrac{1}{3}\left(\dfrac{A_2}{A_1}\right);\ A = \dfrac{-q}{2} + \sqrt{\dfrac{\Delta}{108}}; \\
B = \dfrac{-q}{2} - \sqrt{\dfrac{\Delta}{108}}; \\
\begin{array}{l}(\lambda_3, \lambda_2, \lambda_1) = \\ \lambda_3 > \lambda_2 > \lambda_1\end{array} \left\{\begin{array}{l}
2\sqrt{\dfrac{-p}{3}}\cos\dfrac{\phi}{3} - \dfrac{1}{3}\left(\dfrac{A_2}{A_1}\right) \\
-2\sqrt{\dfrac{-p}{3}}\cos\left(\dfrac{\pi}{3} - \dfrac{\phi}{3}\right) - \dfrac{1}{3}\left(\dfrac{A_2}{A_1}\right) \\
-2\sqrt{\dfrac{-p}{3}}\cos\left(\dfrac{\pi}{3} + \dfrac{\phi}{3}\right) - \dfrac{1}{3}\left(\dfrac{A_2}{A_1}\right)
\end{array}\right. \\
\phi = \tan^{-1}\left[\dfrac{-\sqrt{-\Delta}}{q\sqrt{27}}\right];\ \mu_1 = \left(\dfrac{-B_2 + \sqrt{\Delta_1}}{2B_1}\right);\ \mu_2 = \left(\dfrac{-B_2 - \sqrt{\Delta_1}}{2B_1}\right)
\end{array}\right\} \quad (45)$$

To summarize, if the inequalities of expressions (33) and (36) together with any one relevant set of conditions listed under any one of the eight cases mentioned above is satisfied then at least one positive real set of α, β, γ, T_1, T_5 and T_6 exists for which the circuit of Fig. 2(*b*) is physically realizable. The circuit component values may be determined with the aid of eqns. (32*a*), (32*b*), (34), (2.10), (2.11), (30) and (31).

ACKNOWLEDGEMENTS

The author wishes to thank the Director of Armaments, Defence Research and Development Organization, for his permission to publish this paper and Dr. V. R. Thiruvenkatachar, F.N.I., for his valuable suggestions and criticism.

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APPENDIX I

CONDITIONS UNDER WHICH THE CIRCUIT OF FIG. 2(*a*) IS PHYSICALLY REALIZABLE

Simulation of the system represented by eqn. (1) with the network of Fig. 2(*a*) is possible only if the values of α, β, γ, T_1, T_3 and T_6 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta}{(2\alpha+\beta+2)} \qquad (1.1)$$

$$b_1 = T_1+T_3 \qquad (1.2)$$

$$b_2 = T_1T_3 \qquad (1.3)$$

$$a_1 = \frac{\alpha}{(2\alpha+\beta+2)}T_1+\frac{(\alpha+\beta)}{(2\alpha+\beta+2)}T_3+\frac{(\alpha+2)(\beta+\beta\gamma+2\gamma)+\alpha(\beta+2\gamma)}{(2\alpha+\beta+2)}T_6 \qquad (1.4)$$

$$a_2 = \frac{\alpha(\beta+\beta\gamma+2\gamma)}{(2\alpha+\beta+2)}T_1T_6+\frac{(\alpha\beta+\alpha\beta\gamma+2\alpha\gamma+2\beta\gamma)}{(2\alpha+\beta+2)}T_3T_6 \qquad (1.5)$$

$$a_3 = \frac{\alpha\beta\gamma}{(2\alpha+\beta+2)}T_1T_3T_6 \qquad (1.6)$$

are positive and real, where a's and b's are non-zero positive real constants.

It is therefore required to determine the conditions under which α, β, γ, T_1, T_3, T_6 can be positive real, and graphical methods may be perhaps a convenient means of obtaining these.

Simplification and rearrangement of eqn. (1.1) gives

$$\beta = \frac{2b_0(\alpha+1)}{(\alpha-b_0)} \quad \ldots \quad (1.7)$$

from which it is obvious that β can be real positive, if α is real and

$$\alpha > b_0. \quad \ldots \quad (1.8)$$

The solution of eqns. (1.2) and (1.3) gives

$$T_1 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (1.9a)$$

$$T_3 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (1.9b)$$

from which it is evident that T_1 and T_3 will be positive real, if

$$b_1^2 \geqslant 4b_2. \quad \ldots \quad (1.10)$$

Elimination of β, γ, T_1 and T_3 from eqns. (1.1), (1.2), (1.3) (1.4), (1.5) and (1.1), (1.2), (1.3), (1.5), (1.6) give the following two equations

$$T_6 = \frac{A_1\alpha^2 + B_1\alpha - 2b_0^2 b_2 T_3}{4b_0^2 b_2(\alpha+1)^2} \quad \ldots \quad (1.11)$$

$$T_6 = \frac{A_2\alpha^2 + B_2\alpha - 2a_3 b_0 T_3}{b_0^2 b_1 b_2 \alpha(\alpha+1)} \quad \ldots \quad (1.12)$$

The intersection of the curves of eqns. (1.11) and (1.12) in the first quadrant of the α-T_6 plane will give both α and T_6 positive real. Since β, T_1 and T_3 will be positive real subject to the restrictions of expressions (1.8) and (1.10), therefore the corresponding γ, as is obvious from eqn. (1.6), will also be positive real. It should be clear therefore that only the portion of the curves lying on the right of the T_6-axis are of interest.

Now the function of (1.11) has a double pole at $\alpha = -1$, a point on the left of the T_6-axis, and two zeros the values of which may be obtained by equating its numerator to zero and solving the resulting quadratic

$$A_1\alpha^2 + B_1\alpha - 2b_0^2 b_2 T_3 = 0. \quad \ldots \quad (1.13)$$

The roots of eqn. (1.13) are

$$\alpha_{(A,\,B)} = \left(\frac{-B_1 \pm \sqrt{\Delta_1}}{2A_1}\right) \quad \ldots \quad (1.14)$$

both of which will be real and one of which will be positive and the other negative, if

$$\left.\begin{array}{l} A_1 > 0 \\ B_1 \gtrless 0 \end{array}\right\} \quad \ldots \quad (1.15)$$

But, if

$$\left.\begin{array}{l} A_1 < 0 \\ B_1 > 0 \\ \Delta_1 \geqslant 0 \end{array}\right\} \quad \ldots \quad (1.16)$$

then both the roots of eqn. (1.13) will be positive real.

Similarly, the function of eqn. (1.12) has two simple poles one at $\alpha = 0$ and the other at $\alpha = -1$, and two zeros whose α-coordinates are

$$\alpha_{(P,\,Q)} = \left(\frac{-B_2 \pm \sqrt{\Delta_2}}{2A_2}\right) \quad \ldots \quad (1.17)$$

both of which will be real and one of which will be positive and the other negative, if

$$\left.\begin{matrix} A_2 > 0 \\ B_2 \gtrless 0 \end{matrix}\right\} \quad \ldots \quad (1.18)$$

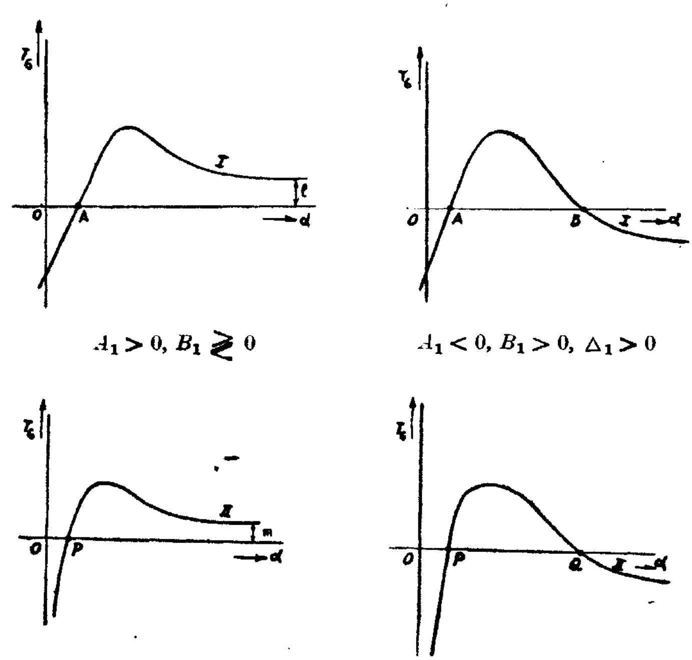
But, if

$$\left.\begin{matrix} A_2 < 0 \\ B_2 > 0 \\ \Delta_2 \geqslant 0 \end{matrix}\right\} \quad \ldots \quad (1.19)$$

then both roots of eqn. (1.17) will be positive real.

Therefore, if the inequalities of either expressions (1.15) or (1.16) and either (1.18) or (1.19) are satisfied then it is possible for a portion of each curve to exist in the first quadrant of the α-T_6 plane and sketches of the curves for positive α are shown in Fig. 1.1. Now, the curves can intersect each other

FIG. 1.1. Sketches of the curves for positive α.



$A_1 > 0,\ B_1 \gtrless 0$ $\qquad$ $A_1 < 0,\ B_1 > 0,\ \Delta_1 > 0$

$A_2 > 0,\ B_2 \gtrless 0$ $\qquad$ $A_2 < 0,\ B_2 > 0,\ \Delta_2 > 0$

$$\text{I}\quad T_6 = \frac{A_1\alpha^2 + B_1\alpha - 2b_0^2 b_2 T_3}{4b_0^2 b_2(\alpha+1)^2} \qquad \text{II}\quad T_6 = \frac{A_2\alpha^2 + B_2\alpha - 2a_3 b_0 T_3}{b_0^2 b_1 b_2 \alpha(\alpha+1)}$$

at at least one point in the first quadrant if a set of conditions listed under any one of the four cases mentioned below is satisfied.

(i) $A_1 > 0$; $A_2 > 0$.

For very large values of α (i.e. $\alpha \to \infty$), the values of T_6, as seen from eqns. (1.11) and (1.12), are respectively given by

$$T'_{6\infty} \to \frac{A_1}{4b_0^2 b_2}$$

$$T''_{6\infty} \to \frac{A_2}{b_0^2 b_1 b_2}$$

and therefore

Either

$$\left.\begin{aligned} &\left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) \\ &\text{i.e.}\quad OP > OA \\ \text{and}\quad &\frac{A_2}{b_1} > \frac{A_1}{4} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.20)$$

or

$$\left.\begin{aligned} &\left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) \\ \text{and}\quad &\frac{A_1}{4} > \frac{A_2}{b_1} \end{aligned}\right. \quad \ldots \quad \ldots \quad (1.21)$$

(ii) $A_1 > 0,\ A_2 < 0,\ B_2 > 0$.

$$\left.\begin{aligned} &\Delta_2 > 0 \\ \text{and}\quad &\left(\frac{-B_2-\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) \end{aligned}\right\} \quad \ldots \quad (1.22)$$

i.e. $OQ > OA > OP$

(iii) $A_1 > 0,\ B_1 > 0$; $A_2 > 0$.

$$\left.\begin{aligned} &\Delta_1 > 0 \\ \text{and}\quad &\left(\frac{-B_1-\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) \end{aligned}\right\} \quad \ldots \quad (1.23)$$

i.e. $OB > OP > OA$.

(iv) $A_1 < 0,\ B_1 > 0$; $A_2 < 0,\ B_2 > 0$.

$$\left.\begin{aligned} \Delta_1 > 0 \\ \Delta_2 > 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.24)$$

and

$$\text{either}\left(\frac{-B_2-\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1-\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) \quad (1.24a)$$

$$\text{or}\left(\frac{-B_1-\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2-\sqrt{\Delta_2}}{A_2}\right) > \left(\frac{-B_1+\sqrt{\Delta_1}}{A_1}\right) > \left(\frac{-B_2+\sqrt{\Delta_2}}{A_2}\right) \quad (1.24b)$$

where

$$\left.\begin{aligned} A_1 &= 2a_1b_0b_2-(4a_3+2a_3b_0+b_0b_1b_2);\; A_2 = a_2b_0b_2-a_3b_1(1+b_0);\\ B_1 &= b_0b_2\{2a_1+b_0(T_1-T_3)\}-2a_3(2+b_0);\; B_2 = b_0(a_2b_2-2a_3T_3);\\ \Delta_1 &= B_1^2+8A_1b_0^2b_2T_3;\; \Delta_2 = B_2^2+8A_2a_3b_0T_3 \end{aligned}\right\} \quad (1.25)$$

To summarize, if the inequalities of expressions (1.8) and (1.10) together with a set of conditions listed under any one of the four cases mentioned above are satisfied then a set of positive real α, β, γ, T_1, T_3 and T_6 exists; and it is possible to simulate the system of eqn. (1) with the circuit of Fig. 2(a).

Appendix II

Conditions under which the Circuit of Fig. 2(b) is Physically Realizable

It is possible to simulate the system represented by eqn. (1) with the circuit of Fig. 2(b) only if the values of α, β, γ, T_1, T_5 and T_6 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)} \quad \text{.. (2.1)}$$

$$b_1 = T_1+T_5 \quad \text{.. (2.2)}$$

$$b_2 = T_1T_5 \quad \text{.. (2.3)}$$

$$a_1 = \frac{\alpha\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1+T_5+\frac{(\alpha+\alpha\gamma+\gamma)(\alpha+2\alpha\beta+\beta)+\alpha\gamma(\alpha+\alpha\beta+\beta)}{\alpha(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_6 \quad (2.4)$$

$$a_2 = \frac{\alpha\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1T_5+\frac{\gamma(\alpha+2\alpha\beta+\beta)}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1T_6+\frac{\beta(\alpha+2\alpha\gamma+\gamma)}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_5T_6 \quad \text{.. (2.5)}$$

$$a_3 = \frac{\alpha\beta\gamma}{(\alpha+\alpha\beta+2\alpha\gamma+\gamma)}T_1T_5T_6 \quad \text{.. (2.6)}$$

are positive and real.

The solution of eqns. (2.2) and (2.3) gives

$$T_1 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2-4b_2}\right] \quad \text{.. (2.7a)}$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2-4b_2}\right] \quad \text{.. (2.7b)}$$

from which it is evident that T_1 and T_5 will be positive real, if

$$b_1^2 > 4b_2 \quad \ldots \quad (2.8)$$

Elimination of γ, T_5 and T_6 from eqns. (2.1), (2.2), (2.3), (2.4), (2.6) and (2.1), (2.2), (2.3), (2.5), (2.6) give the following two equations

$$\frac{1}{\alpha^2} = \frac{A_1\beta^3+A_2\beta^2+A_3\beta+A_4}{a_3b_0\beta^2(b_1\beta+T_1)} \quad \ldots \quad (2.9)$$

$$\frac{1}{\alpha} = \frac{B_1\beta^2+B_2\beta-b_0\left(a_3T_1+b_0b_2^2\right)}{a_3b_0\beta(b_1\beta+T_1)} \quad \ldots \quad (2.10)$$

The intersection of the curves of eqns. (2.9) and (2.10) in the first quadrant of the $\beta-\frac{1}{\alpha}$ plane will give both α and β positive real. In order to see if the corresponding γ and T_6 will also be positive real let us eliminate α from eqns. (2.1) and (2.10) and obtain the relation between β and γ in the form

$$\frac{1}{\gamma} = \frac{a_3b_1\beta^2-a_2b_0b_2\beta+b_0\left(a_3T_1+b_0b_2^2\right)}{a_3b_0\beta(b_1\beta+T_1)}. \quad \ldots \quad (2.11)$$

Now, the function of eqn. (2.11) has two simple poles one at $\beta = 0$ and the other at $\beta = -\frac{T_1}{b_1}$ and two zeros both of which may be positive real or complex depending on whether the discriminant Δ_3 of the quadratic obtained by equating to zero the numerator of eqn. (2.11) is positive or negative. Hence, if

$$\Delta_3 = a_2^2b_0^2b_2^2-4a_3b_0b_1\left(a_3T_1+b_0b_2^2\right) < 0,$$

$$\text{i.e.} \quad a_2^2 < 4a_3b_1\left(1+\frac{a_3T_1}{b_0b_2^2}\right), \quad \ldots \quad (2.12)$$

then for all positive real β the corresponding γ is positive real, and if T_1, T_5, α are also positive real then T_6, as evident from eqn. (2.6), is necessarily positive real.

It is evident from the function of eqn. (2.9) that it has three zeros which are given by

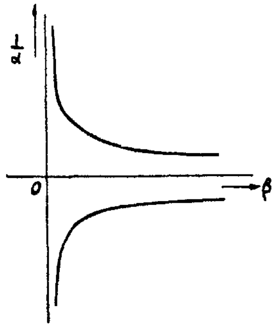
$$A_1\beta^3+A_2\beta^2+A_3\beta+A_4 = 0 \quad \ldots \quad (2.13)$$

one of which is real and the other two may be real or complex depending on whether its discriminant

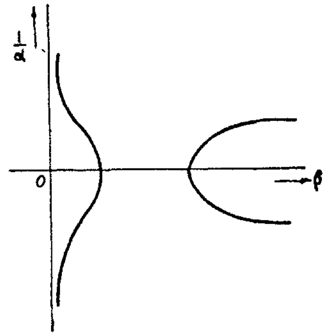
$$\Delta = 4p^3+27q^2 \quad \ldots \quad (2.14)$$

is negative or positive. The nature of the function of eqn. (2.9) under various possible conditions wherein a portion of the curve can exist in the first quadrant of the $\beta-\frac{1}{\alpha}$ plane is sketched in Fig. 2.1.

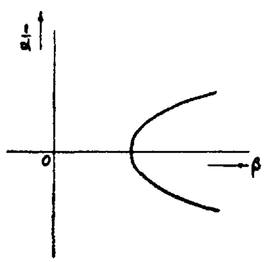
FIG. 2.1. Sketches of $\frac{1}{\alpha^2} = \frac{A_1\beta^3 + A_2\beta^2 + A_3\beta + A_4}{a_3 b_0 \beta^2 (b_1\beta + T_1)}$ for positive β.



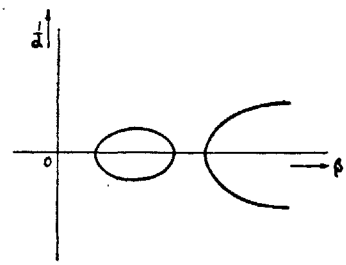
(*a*) Either 1 or 3 negative real roots, i.e. $A_1 > 0$, $A_4 > 0$, and either $\Delta > 0$ or $A_2 > 0$, $A_3 > 0$, $\Delta < 0$.



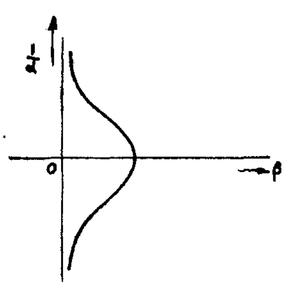
(*b*) 2 positive and 1 negative real roots, $A_1 > 0$, $A_4 > 0$, $\Delta < 0$, and either $A_2 \gtrless 0$, $A_3 < 0$ or $A_2 < 0$, $A_3 > 0$.



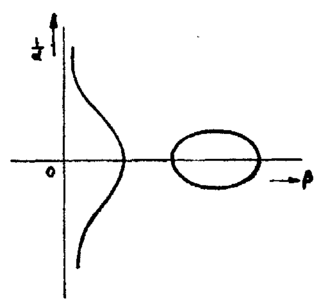
(*c*) 1 positive real and either 0 or 2 negative real roots, i.e. $A_1 > 0$, $A_4 < 0$, and either $\Delta > 0$ or $\Delta < 0$, and either $A_2 > 0$, $A_3 \gtrless 0$ or $A_2 < 0$, $A_3 < 0$.



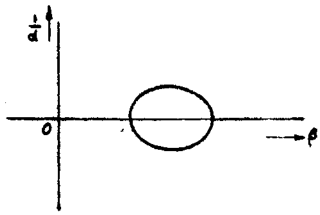
(*d*) 3 positive real roots, i.e. $A_1 > 0$, $A_4 < 0$, $A_2 < 0$, $A_3 > 0$, $\Delta < 0$.



(*e*) 1 positive real and either, 0 or 2 negative real roots, i.e. $A_1 < 0$, $A_4 > 0$, and either $\Delta > 0$ or $\Delta < 0$, and either $A_2 \gtrless 0$, $A_3 > 0$ or $A_2 < 0$, $A_3 < 0$.



(*f*) 3 positive real roots, i.e. $A_1 < 0$, $A_4 > 0$, $A_2 > 0$, $A_3 < 0$, $\Delta < 0$.



(g) 2 positive and 1 negative real roots, i.e. $A_1 < 0$, $A_4 < 0$; $\Delta < 0$, and either $A_2 > 0$, $A_3 \gtrless 0$ or $A_2 < 0$, $A_3 > 0$.

The function of eqn. (2.10) has two real zeros given by

$$B_1\beta^2 + B_2\beta - b_0(a_3T_1 + b_0b_2^2) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.15)$$

one of which will be positive and the other negative, if

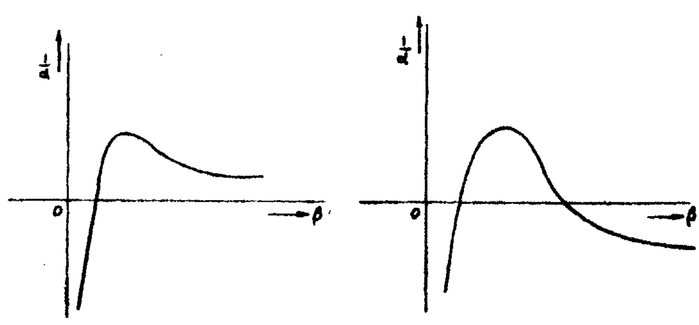
$$\left.\begin{aligned} B_1 &> 0 \\ B_2 &\gtrless 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.16)$$

or both of these will be positive, if

$$\left.\begin{aligned} B_1 &< 0 \\ B_2 &> 0 \\ \Delta_1 = [B_2^2 + 4b_0B_1(a_3T_1 + b_0b_2^2)] &> 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.17)$$

Sketches of the function of eqn. (2.10) for positive β and wherein a portion of it can exist in the first quadrant are shown in Fig. 2.2.

FIG. 2.2. Sketches of $\frac{1}{\alpha} = \frac{B_1\beta^2 + B_2\beta - b_0(a_3T_1 + b_0b_2^2)}{a_3b_0\beta(b_1\beta + T_1)}$ for positive β.



(a) $B_1 > 0$, $B_2 \gtrless 0$ (b) $B_1 < 0$, $B_2 > 0$, $\Delta_1 > 0$

Now, it is possible for the curves of eqns. (2.9) and (2.10) to intersect each other at at least one point in the first quadrant if a set of conditions listed under any one of the eight cases mentioned below is satisfied.

(i) $A_1 > 0,\ A_4 > 0;\ B_1 > 0.$

Under this case the function of eqn. (2.9) will have one negative real zero if Δ is positive. But if Δ is negative then all the three zeros will either be negative real or two positive and one negative real. When β is very large (i.e. $\beta \to \infty$) then the value of the function tends to

$$\frac{1}{\alpha_\infty^2} \to \frac{A_1}{a_3 b_0 b_1}.$$

Similarly, when $B_1 > 0$ and $B_2 \gtrless 0$ the function of eqn. (2.10) will have two real zeros one of which will be positive and the other negative. The value of the function for very large β tends to

$$\frac{1}{\alpha_\infty} \to \frac{B_1}{a_3 b_0 b_1}.$$

Now, if

$$\text{Either} \quad \Delta > 0 \qquad \ldots \qquad (2.18)$$

$$\text{or} \quad \left.\begin{array}{l} \Delta < 0 \\ A_2 > 0 \\ A_3 > 0 \end{array}\right\} \qquad \ldots \qquad (2.19)$$

then the real zeros of eqn. (2.9) will be negative and the sketch of the curve is shown in Fig. 2.1(a). The sketch of the curve of eqn. (2.10), when

$$\left.\begin{array}{l} B_1 > 0 \\ B_2 \gtrless 0 \end{array}\right\} \qquad \ldots \qquad (2.20)$$

is shown in Fig. 2.2(a). The curves of eqns. (2.9) and (2.10) can intersect each other in the first quadrant if the final values of the function (i.e. $\beta \to \infty$) of eqn. (2.10) is greater than that of eqn. (2.9).

That is, if

$$B_1 > \sqrt{a_3 b_0 b_1 A_1}. \qquad \ldots \qquad (2.21)$$

And if

$$\left.\begin{array}{ll} & \Delta < 0 \\ \text{and either} & \left.\begin{array}{l} A_2 \gtrless 0 \\ A_3 < 0 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} A_2 < 0 \\ A_3 > 0 \end{array}\right\} \end{array}\right\} \qquad \ldots \qquad (2.22)$$

then the function of eqn. (2.9) has two positive real zeros (λ_2, λ_3) and one negative real zero (λ_1); such that

$$\lambda_3 > \lambda_2 > 0$$

$$\lambda_1 < 0$$

and the sketch of the curve is shown in Fig. 2.1(b). It will be possible for the curves of Fig. 2.1(a) and Fig. 2.2(a) to intersect each other in the first quadrant, if

either $\lambda_2 > \mu_1$ .. .. (2.23a)

or $\left.\begin{array}{c}\lambda_3 > \mu_1 > \lambda_2 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1\end{array}\right\}$ .. .. (2.23b)

or $\left.\begin{array}{c}\mu_1 > \lambda_3 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1}\end{array}\right\}$ .. .. (2.23c)

where μ_1 is the positive real zero of eqn. (2.10).

(ii) $A_1 > 0,\ A_4 < 0;\ B_1 > 0.$

Under this case the function of eqn. (2.9) has either one positive real zero (λ_0), if

$$\Delta > 0$$

or one positive (λ_3) and two negative (λ_1, λ_2) real zeros, if

$$\Delta < 0$$

and either $\left.\begin{array}{c}A_2 > 0 \\ A_3 \gtrless 0\end{array}\right\}$

or $\left.\begin{array}{c}A_2 < 0 \\ A_3 < 0\end{array}\right\}$

and the sketch of the curve is shown in Fig. 2.1(c). It will be possible for the curves of Fig. 2.1(c) and Fig. 2.2(a) to intersect each other in the first quadrant, if

Either

$$\left.\begin{array}{ll} & \Delta > 0 \\ \text{and either} & \left.\begin{array}{c}\lambda_0 > \mu_1 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1\end{array}\right\} \\ \text{or} & \left.\begin{array}{c}\mu_1 > \lambda_0 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1}\end{array}\right\}\end{array}\right\} \quad \ldots \quad (2.24)$$

or

$$\left.\begin{array}{ll} & \Delta < 0 \\ \text{and either} & \left.\begin{array}{c}\lambda_3 > \mu_1 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1\end{array}\right\} \\ \text{or} & \left.\begin{array}{c}\mu_1 > \lambda_3 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1}\end{array}\right\}\end{array}\right\} \quad \ldots \quad (2.24a)$$

But, if

$$\left.\begin{array}{c}\Delta < 0 \\ A_2 < 0 \\ A_3 > 0\end{array}\right\}$$

then the function of eqn. (2.9) has three positive real zeros (λ_1, λ_2, λ_3) and its sketch is shown in Fig. 2.1(d). The curves of Fig. 2.1(d) and Fig. 2.2(a) can intersect each other in the first quadrant if

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{c}\lambda_1 > \mu_1 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1}\end{array}\right\} \\ \text{or} & \lambda_2 > \mu_1 > \lambda_1 \\ \text{or} & \left.\begin{array}{c}\lambda_3 > \mu_1 > \lambda_2 \\ \sqrt{a_3 b_0 b_1 A_1} > B_1\end{array}\right\} \\ \text{or} & \left.\begin{array}{c}\mu_1 > \lambda_3 \\ B_1 > \sqrt{a_3 b_0 b_1 A_1}\end{array}\right\} \end{array}\right\} \qquad (2.25)$$

(iii) $A_1 < 0, A_4 > 0;\ B_1 > 0.$

If either $\Delta > 0$

or

$\Delta < 0$

$$\text{and either} \quad \left.\begin{array}{c} A_2 \gtrless 0 \\ A_3 > 0 \end{array}\right\} \qquad \text{or} \quad \left.\begin{array}{c} A_2 < 0 \\ A_3 < 0 \end{array}\right\}$$

then the function of eqn. (2.9) has one positive and either none or two negative real zeros, and if

$$\left.\begin{array}{c} \Delta < 0 \\ A_2 > 0 \\ A_3 < 0 \end{array}\right\}$$

then the function has three real zeros all of which are positive and the sketches are shown in Figs. 2.1(e) and (f). The curve of Fig. 2.2(a) can intersect with the curves of either Fig. 2.1(e) or Fig. 2.1(f) in the first quadrant provided that

$$\left.\begin{array}{ll} \text{either} & \lambda_0 > \mu_1 \\ \text{or} & \lambda_3 > \mu_1 \\ \text{or} & \lambda_1 > \mu_1 \\ \text{or} & \lambda_3 > \mu_1 > \lambda_2 \end{array}\right\} \qquad (2.26)$$

(iv) $A_1 < 0, A_4 < 0;\ B_1 > 0.$

Under this case positive real zeros are possible only when

$$\Delta < 0$$

$$\text{and either} \quad \left.\begin{array}{c} A_2 > 0 \\ A_3 \gtrless 0 \end{array}\right\} \qquad \text{or} \quad \left.\begin{array}{c} A_2 < 0 \\ A_3 > 0 \end{array}\right\}$$

and the sketch of the function is shown in Fig. 2.1(g). The curves of Fig. 2.1(g) and Fig. 2.2(a) can intersect each other in the first quadrant, if

$$\lambda_3 > \mu_1 > \lambda_2.$$

(v) $A_1 > 0, A_4 > 0;\ B_1 < 0, B_2 > 0, \Delta_1 > 0.$

Under this case the function of eqn. (2.10) has two positive real zeros μ_1, and μ_2, as sketched in Fig. 2.2(b); and it is possible for this curve to intersect with that of Fig. 2.1(b) in the first quadrant provided that

$$\left.\begin{array}{ll} & \Delta < 0 \\ \text{and either} & \mu_2 > \lambda_2 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 \end{array}\right\} \quad (2.27)$$

(vi) $A_1 > 0, A_4 < 0;\ B_1 < 0, B_2 > 0, \Delta_1 > 0.$

The curve of Fig. 2.2(b) can intersect with that of either Figs. 2.1(c) or (d) in the first quadrant, if

$$\left.\begin{array}{ll} \text{either} & \mu_2 > \lambda_0 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 \\ \text{or} & \lambda_2 > \mu_2 > \lambda_1 > \mu_1 \\ \text{or} & \mu_2 > \lambda_2 > \mu_1 > \lambda_1 \end{array}\right\} \quad (2.28)$$

(vii) $A_1 < 0, A_4 > 0;\ B_1 < 0, B_2 > 0, \Delta_1 > 0.$

This relates to the case wherein the functions of eqns. (2.9) and (2.10) can be described by the sketches of Figs. 2.1(e), (f) and 2.2(b) respectively. The intersection of the curve of Fig. 2.2(b) with that of either Figs. 2.1(e) or (f) is possible in the first quadrant, if

$$\left.\begin{array}{ll} \text{either} & \mu_2 > \lambda_0 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 \\ \text{or} & \mu_2 > \lambda_1 > \mu_1 \\ \text{or} & \lambda_3 > \mu_2 > \lambda_2 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 > \lambda_2 \end{array}\right\} \quad (2.29)$$

(viii) $A_1 < 0, A_4 < 0;\ B_1 < 0, B_2 > 0, \Delta_1 > 0.$

This describes the sketches of Fig. 2.1(g) and Fig. 2.2(b) which can intersect each other in the first quadrant, if

$$\left.\begin{array}{ll} \text{either} & \lambda_3 > \mu_2 > \lambda_2 > \mu_1 \\ \text{or} & \mu_2 > \lambda_3 > \mu_1 > \lambda_2 \end{array}\right\} \quad (2.30)$$

To summarize, if the inequalities of expressions (2.8) and (2.12) together with a set of conditions listed under any one of the eight cases mentioned above is satisfied then at least one set of positive real α, β, γ, T_1, T_5 and T_6 exists for which the circuit of Fig. 2(b), simulating the system of eqn. (1), is physically realizable.