SIMULATION OF THE GENERAL THIRD ORDER SYSTEMS BY A SINGLE OPERATIONAL AMPLIFIER—II

by L. K. WADHWA,* *Defence Research and Development Organization, New Delhi*

(Communicated by V. R. Thiruvenkatachar, F.N.I.)

(*Received September 27, 1962*)

In the previous paper (Wadhwa 1963) on the same topic two basic circuits, each employing three capacitors and seven resistors and capable of simulating, under certain conditions, the general third order linear systems, were presented. In this paper two more basic circuits for simulating similar third order systems are presented. The circuits are analysed, the design formulae obtained and the conditions of their physical realizability discussed.

INTRODUCTION

In the previous communication (Wadhwa 1963) on this subject two basic circuits, each capable of simulating, under certain conditions, the general third order systems, were presented. The purpose of this paper is to present two more basic circuits, obtain the design formulae and discuss the conditions of their physical realizability.

THIRD ORDER SYSTEM SIMULATION

Two basic circuits, each employing three capacitors and seven resistors (with a certain arbitrary choice of resistor values) and capable of simulating, under certain conditions, the general third order systems of the type

$$F(S) = -\frac{b_0(b_2S^2+b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

are shown in Fig. 1.

The transfer function for the circuit of Fig. 1(a) can be shown to be

$$\frac{E_0}{E_1} = -\frac{\dfrac{\alpha\beta\gamma}{K}\left[R^2C_1C_5S^2+R(C_1+C_5)S+1\right]}{\dfrac{\alpha\beta\gamma}{K}R^3C_1C_5C_7S^3+\left[\dfrac{\alpha\beta}{K}R^2C_1C_5+\dfrac{\alpha\beta(\gamma+1)}{K}R^2C_1C_7+\dfrac{2\beta\gamma(\alpha+1)}{K}R^2C_5C_7\right]S^2+\left[\dfrac{\alpha(2\beta+1)}{K}RC_1+\dfrac{\beta(2\alpha+\gamma+2)}{K}RC_5+\dfrac{2\beta(\alpha+1)(\gamma+1)}{K}RC_7\right]S+1}. \quad \ldots \quad (2)$$

---

* *Present address :* Electronics and Radar Development Establishment, Bangalore 1.

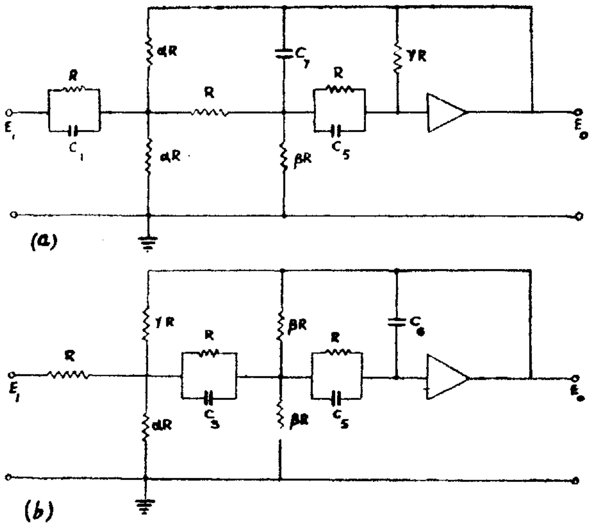


FIG. 1. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0(b_2S^2+b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}$.

Eqns. (1) and (2) will be identical if

$$b_0 = \frac{\alpha\beta\gamma}{K} \quad \text{(3)}$$

$$b_1 = T_1+T_5 \quad \text{(4)}$$

$$b_2 = T_1T_5 \quad \text{(5)}$$

$$a_1 = \frac{\alpha(2\beta+1)}{K}T_1+\frac{\beta(2\alpha+\gamma+2)}{K}T_5+\frac{2\beta(\alpha+1)(\gamma+1)}{K}T_7 \quad \text{(6)}$$

$$a_2 = \frac{\alpha\beta}{K}T_1T_5+\frac{\alpha\beta(\gamma+1)}{K}T_1T_7+\frac{2\beta\gamma(\alpha+1)}{K}T_5T_7 \quad \text{(7)}$$

$$a_3 = \frac{\alpha\beta\gamma}{K}T_1T_5T_7 \quad \text{(8)}$$

where

$$K = (2\alpha+3\alpha\beta+4\beta+\beta\gamma+2) \quad \text{(9)}$$

$$T_n = RC_n. \quad \text{(10)}$$

The solution of eqns. (4) and (5) gives

$$T_1 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2-4b_2}\right] \quad \text{(11a)}$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2-4b_2}\right] \quad \text{(11b)}$$

from which it is obvious that T_1 and T_5 will be positive real, provided that

$$b_1^2 > 4b_2. \quad \text{(12)}$$

Elimination of β, γ and T_7 from eqns. (3) through (8) gives a cubic

$$\frac{P_1}{\alpha^3}-\frac{P_2}{\alpha^2}+\frac{P_3}{\alpha}-P_4=0, \quad \ldots \quad (13)$$

where

$$\left.\begin{aligned}
&P_1 = 2a_3T_5A_1;\; P_2 = kb_2A_1-2a_3T_5A_2;\\
&P_3 = 2a_3T_5A_3-kb_2A_2-b_2m\left(\frac{a_3}{T_5}+b_0b_2\right);\\
&P_4 = kb_2A_3-b_2\left(\frac{a_3}{T_5}+b_0b_2\right)(l-m);\; A_1=\frac{4T_5}{b_2}\left(\frac{a_3}{T_5}+b_0b_2\right);\\
&A_2=\frac{T_5}{b_2}\left(\frac{a_3}{T_5}+b_0b_2\right)\left[8-\frac{b_2}{a_3T_5}\left(\frac{2a_3}{T_5}+b_0b_2\right)\right];\\
&A_3=b_0T_1+\frac{T_5}{b_2}\left(\frac{a_3}{T_5}+b_0b_2\right)\left[\frac{b_2^2}{2a_3T_5^2}(2a_1+b_0T_1)-\frac{b_2}{a_3T_5}\left(\frac{2a_3}{T_5}+b_0b_2\right)+4\right];\\
&k=\left(a_2-\frac{a_3b_1}{b_2}-\frac{a_3T_5}{b_2}\right);\; p=\frac{2a_3T_5}{b_2};\; q=\left(\frac{a_3}{T_5}+b_0b_2\right);\\
&l=\frac{2a_1b_2k+b_0T_1(a_2b_2-a_3b_1)-a_3b_2(2+b_0)}{2a_3T_5};\\
&m=\frac{a_2b_2(2a_3+b_0b_2T_5)+2a_3^2T_5-a_3(2a_3b_1+b_0b_1b_2T_5+b_0b_2T_5^2)}{a_3b_2T_5}
\end{aligned}\right\} \quad \ldots \quad (14)$$

Now, as shown in Appendix I, positive real α, γ exist if

$$k=\left(a_2-\frac{a_3b_1}{b_2}-\frac{a_3T_5}{b_2}\right)>0 \quad \ldots \quad (15)$$

and any one of the four cases mentioned below is applicable, and any one set of conditions listed under the relevant case is satisfied.

(i) $A_3>0$, $(l-m)>0$ and either $A_2>0$ or $A_2<0$, $\Delta_1<0$.

Either

$$\left.\begin{aligned}
&z_2>z_1\\
\text{i.e. } &OA>OP\\
\text{and } &\left(\frac{l-m}{A_3}\right)>\frac{k}{q}\\
\text{i.e. } &OQ>OB
\end{aligned}\right\} \quad \ldots \quad (16a)$$

or

$$\left.\begin{aligned}
&z_1>z_2\\
&\frac{k}{q}>\left(\frac{l-m}{A_3}\right)
\end{aligned}\right\} \quad \ldots \quad (16b)$$

(ii) $A_2 \gtrless 0, A_3 < 0$.

Either $\left.\begin{array}{l}(l-m) > 0 \\ z_2 > z_1\end{array}\right\}$ .. .. (17a)

or $\left.\begin{array}{l}(l-m) < 0 \\ \frac{k}{q} > \left(\frac{l-m}{A_3}\right)\end{array}\right\}$ .. .. (17b)

(iii) $A_2 < 0, A_3 > 0, \Delta_1 > 0, (l-m) > 0$.

Either $\lambda_2 > \lambda_1 > z_1$
and either $\left.\begin{array}{l}z_2 > z_1 \\ \left(\frac{l-m}{A_3}\right) > \frac{k}{q}\end{array}\right\}$
or $\left.\begin{array}{l}z_1 > z_2 \\ \frac{k}{q} > \left(\frac{l-m}{A_3}\right)\end{array}\right\}$ .. .. (18a)

or $\lambda_2 > z_1 > \lambda_1$
and either $z_2 > z_1$
or $\frac{k}{q} > \left(\frac{l-m}{A_3}\right)$ .. .. (18b)

or $z_1 > \lambda_2 > \lambda_1$
and either $z_2 > z_1$
or $\frac{k}{q} > \left(\frac{l-m}{A_3}\right)$ .. .. (18c)

(iv) $A_2 < 0, A_3 > 0, \Delta_1 > 0, (l-m) < 0$.

$$P_2 > 0, P_3 > 0, P_4 > 0, \Delta > 0. \quad \text{.. .. ..} \quad (19)$$

The corresponding γ and $\frown$ also T_7 will be positive real if the positive real root, $\frac{1}{\alpha_0}$, of eqn. (13) satisfies a set of conditions listed in any one of the three expressions given below:

Either $B_1 > 0, B_2 > 0, \Delta_2 > 0$
and either $\mu_2 > \mu_1 > z_2 > \frac{1}{\alpha_0}$, or $\mu_2 > z_2 > \mu_1 > \frac{1}{\alpha_0}$,
or $z_2 > \mu_2 > \mu_1 > \frac{1}{\alpha_0}$, or $z_2 > \frac{1}{\alpha_0} > \mu_2 > \mu_1$ .. (20)

or $B_1 < 0, B_2 > 0, z_2 > \frac{1}{\alpha_0}$ .. .. (21)

or $B_1 \gtrless 0, B_2 < 0, z_2 > \frac{1}{\alpha_0} > \mu_2$ .. .. .. (22)

where

$$\left.\begin{aligned}
&B_1 = \left(\frac{q+4k-3p}{4p}\right);\; B_2 = \left(\frac{q-3b_0k}{4b_0p}\right);\; \Delta_1 = A_2^2-4A_1A_3;\; \Delta_2 = B_1^2-4B_2;\\
&\Delta = 4p_1^3+27q_1^2;\; p_1 = \frac{P_3}{P_1}-\frac{1}{3}\left(\frac{P_2}{P_1}\right)^2;\; q_1 = -\frac{P_4}{P_1}+\frac{1}{3}\cdot\frac{P_2P_3}{P_1^2}-\frac{2}{27}\left(\frac{P_2}{P_1}\right)^3;\\
&\lambda_1 = \frac{-A_2-\sqrt{\Delta_1}}{2A_1};\; \lambda_2 = \left(\frac{-A_2+\sqrt{\Delta_1}}{2A_1}\right);\; \mu_1 = \frac{B_1-\sqrt{\Delta_2}}{2};\\
&\mu_2 = \frac{B_1+\sqrt{\Delta_2}}{2};\; z_1 = OP = \left(\frac{l-m}{m}\right);\; z_2 = OA = \frac{k}{p};\; OB = \frac{k}{q};\; OQ = \left(\frac{l-m}{A_3}\right)
\end{aligned}\right\} \quad \ldots (23)$$

For the design of the network circuit component values are required to be determined. The proper procedure for design would be first to check and see if the inequalities of expressions (12) and (15) are satisfied; and then compute the values of A's, B's, Δ's, $(l—m)$ with the aid of eqns. (14) and (23) and see if any one of the four cases listed above is applicable and a set of conditions mentioned therein is satisfied. Thereafter, solve (Uspensky 1948) the cubic of eqn. (13) and see if any one of its positive real roots, $\frac{1}{\alpha_0}$, satisfies any one set of conditions listed in any one of the expressions (20) through (22). The satisfaction of these conditions signifies that the circuit of Fig. 1(a) for the simulation of the system of eqn. (1) is physically realizable. The circuit component values may then be obtained by computing T_1 and T_5 from eqns. (11a) and (11b) and substituting $\frac{1}{\alpha_0}$ into eqns. (1.10), (1.16) to obtain γ and β respectively, T_7 may then be obtained with the aid of eqn. (9). Choosing arbitrarily a convenient value for any one of the capacitors, the values for the resistors and the remaining capacitors can then be obtained with the aid of eqn. (10).

Referring to Fig. 1(b), its transfer function can be shown to be

$$\frac{E_0}{E_1} = -\frac{\frac{\alpha\beta\gamma}{K}[R^2C_3C_5S^2+R(C_3+C_5)S+1]}{\frac{\alpha\beta\gamma}{K}R^3C_3C_5C_6S^3+\left[\frac{\alpha(\beta+\gamma)}{K}R^2C_3C_5+\frac{(\alpha\beta+2\alpha\beta\gamma+2\alpha\gamma+\beta\gamma)}{K}R^2C_3C_6+\frac{\beta(\alpha+2\alpha\gamma+\gamma)}{K}R^2C_5C_6\right]S^2+\left[\frac{\alpha(\beta+\gamma)}{K}RC_3+RC_5+\left\{2+\frac{\beta\gamma(3\alpha+2)}{K}\right\}RC_6\right]S+1} \quad (24)$$

Eqns. (1) and (24) will be identical if

$$b_0 = \frac{\alpha\beta\gamma}{K} \quad \ldots (25)$$

$$b_1 = T_3+T_5 \quad \ldots (26)$$

$$b_2 = T_3 T_5 \quad \ldots \quad (27)$$

$$a_1 = \frac{\alpha(\beta+\gamma)}{K} T_3 + T_5 + \left\{2 + \frac{\beta\gamma(3\alpha+2)}{K}\right\} T_6 \quad \ldots \quad (28)$$

$$a_2 = \frac{\alpha(\beta+\gamma)}{K} T_3 T_5 + \frac{(\alpha\beta+2\alpha\beta\gamma+2\alpha\gamma+\beta\gamma)}{K} T_3 T_6 + \frac{\beta(\alpha+2\alpha\gamma+\gamma)}{K} T_5 T_6 \quad (29)$$

$$a_3 = \frac{\alpha\beta\gamma}{K} T_3 T_5 T_6 \quad \ldots \quad (30)$$

where

$$K = (\alpha+\alpha\beta+2\alpha\gamma+\gamma) \quad \ldots \quad (31)$$

$$T_n = RC_n. \quad \ldots \quad (32)$$

The solution of eqns. (26) and (27) gives

$$T_3 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (33a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (33b)$$

from which it is obvious that T_3 and T_5 will be positive real, provided that

$$b_1^2 \geqslant 4b_2. \quad \ldots \quad (34)$$

Elimination of α, γ and T_6 from eqns. (25) through (30) gives a cubic

$$P_1\beta^3 + P_2\beta^2 + P_3\beta + P_4 = 0, \quad \ldots \quad (35)$$

where

$$\left.\begin{aligned} P_1 &= a_3 T_5\{a_3 b_1 + b_1 b_2(a_1-b_1) + 2b_2^2 - 2a_2 b_2\} \\ P_2 &= a_3 b_1 b_2 T_5(a_1-b_1) + a_3 T_5(a_3 b_1 + b_0 b_2^2 + 4b_2^2) + b_2(a_2 b_0 b_2^2 + 4a_3^2 + a_3 b_1 b_2) \\ &\quad - \{b_2^2(a_3 b_0 b_1 + a_3 b_1 + b_0 b_2^2) + b_2 T_5[4a_2 a_3 + b_0 b_2^2(a_1-b_1)]\} \\ P_3 &= b_2[a_2 b_0 b_2^2 + 8a_3^2 + 2a_3 b_2(b_0 b_1 + T_5) - \{a_3(2a_2 T_5 + 5b_0 b_2 T_3) \\ &\quad + b_0 b_2^2 T_5(a_1-b_1) + b_0 b_2^3\}] \end{aligned}\right\} \quad (36)$$

$$P_4 = 2a_3 b_2\{2a_3 + b_0 b_2(T_6 - T_3)\}.$$

As shown in Appendix II, positive real α, β exist provided that any one of the twelve cases mentioned below is applicable and any one set of conditions listed therein is satisfied.

(i) $A_0 > 0,\ A_1 > 0;\ B_1 > 0.$

Either $\quad q_1 > \mu_2 > \lambda_2 \quad \ldots \quad (37a)$

or $\quad \lambda_2 > \mu_2,\ \dfrac{A_1}{2} > \dfrac{B_1}{b_1} \quad \ldots \quad (37b)$

or $\quad \mu_2 > q_1,\ \mu_2 > \lambda_2,\ \dfrac{B_1}{b_1} > \dfrac{A_1}{2} \quad \ldots \quad (37c)$

(ii) $A_0 > 0.\ A_1 > 0;\ B_1 < 0,\ B_2 > 0.$

Either $\mu_1 > \lambda_2 > \mu_2$ .. .. (38a)

or $q_1 > \mu_1 > \mu_2 > \lambda_2$ .. .. (38b)

or $\mu_1 > q_1 > \mu_2 > \lambda_2$ .. .. (38c)

(iii) $A_0 > 0.\ A_1 < 0,\ A_2 > 0,\ \Delta_1 > 0;\ B_1 > 0.$

$\lambda_1 > \mu_2 > \lambda_2$ .. .. (39)

(iv) $A_0 > 0.\ A_1 < 0,\ A_2 > 0.\ \Delta_1 > 0;\ B_1 < 0,\ B_2 > 0.$

Either $\lambda_1 > \mu_1 > \lambda_2 > \mu_2$ .. .. (40a)

or $\mu_1 > \lambda_1 > \mu_2 > \lambda_2$ .. .. (40b)

or $\left.\begin{array}{ll} & \lambda_1 > \mu_1 > \mu_2 > \lambda_2 \\ \text{and either} & q_1 > \mu_1 > \mu_2 \\ \text{or} & \mu_1 > q_1 > \mu_2 \end{array}\right\}$ .. .. (40c)

(v) $A_0 < 0.\ A_1 > 0;\ B_1 > 0.$

Either $\left.\begin{array}{ll} & \frac{A_1}{2} > \frac{B_1}{b_1} \\ \text{and either} & q_1 > \mu_2 > p_1 > \lambda_2, \text{ or } q_1 > p_1 > \mu_2 > \lambda_2, \\ \text{or} & q_1 > p_1 > \lambda_2 > \mu_2, \text{ or } q_1 > \lambda_2 > \mu_2 > p_1, \\ \text{or} & q_1 > \lambda_2 > p_1 > \mu_2, \text{ or } \lambda_2 > q_1 > p_1 > \mu_2, \\ \text{or} & \lambda_2 > p_1 > \mu_2 > q_1, \text{ or } \lambda_2 > \mu_2 > p_1 > q_1, \\ \text{or} & \lambda_2 > \mu_2 > q_1 > p_1 \end{array}\right\}$ .. (41a)

or $\left.\begin{array}{ll} & \frac{B_1}{b_1} > \frac{A_1}{2} \\ \text{and either} & p_1 > \lambda_2 > q_1 > \mu_2, \text{ or } p_1 > q_1 > \lambda_2 > \mu_2, \\ \text{or} & p_1 > q_1 > \mu_2 > \lambda_2, \text{ or } p_1 > \mu_2 > \lambda_2 > q_1, \\ \text{or} & p_1 > \mu_2 > q_1 > \lambda_2, \text{ or } \mu_2 > p_1 > \lambda_2 > q_1, \\ \text{or} & \mu_2 > p_1 > q_1 > \lambda_2, \text{ or } \mu_2 > \lambda_2 > p_1 > q_1, \\ \text{or} & \mu_2 > q_1 > \lambda_2 > p_1, \text{ or } \mu_2 > \lambda_2 > q_1 > p_1 \end{array}\right\}$ .. (41b)

or $p_1 > \lambda_2 > \mu_2 > q_1$ .. .. (41c)

or $\mu_2 > q_1 > p_1 > \lambda_2$ .. .. (41d)

or $q_1 > \mu_2 > \lambda_2 > p_1$ .. .. (41e)

or $\left.\begin{array}{l} \lambda_2 > p_1 \\ \mu_2 > q_1 > p_1 \end{array}\right\}$ .. .. (41f)

(vi) $A_0 < 0,\ A_1 > 0;\ B_1 < 0,\ B_2 < 0.$

Either $q_1 > p_1 > \lambda_2$ .. .. (42a)

or $\lambda_2 > q_1 > p_1$ .. .. (42b)

or $q_1 > \lambda_2 > p_1$ .. .. (42c)

(vii) $A_0 < 0,\ A_1 > 0;\ B_1 < 0,\ B_2 > 0,\ \Delta_2 > 0.$

$$\left.\begin{array}{l}
\text{Either} \quad p_1 > \lambda_2 > q_1 > \mu_1 > \mu_2 \\
\text{or any one of the following:} \\
p_1 > q_1 > \lambda_2 > \mu_1 > \mu_2,\ q_1 > p_1 > \lambda_2 > \mu_1 > \mu_2,\ q_1 > \mu_1 > p_1 > \lambda_2 > \mu_2, \\
q_1 > \mu_1 > p_1 > \mu_2 > \lambda_2,\ q_1 > \mu_1 > \mu_2, > p_1 > \lambda_2,\ \lambda_2 > p_1 > q_1 > \mu_1 > \mu_2, \\
q_1 > \mu_1 > \lambda_2 > p_1 > \mu_2,\ \lambda_2 > p_1 \text{ and } q_1 > \mu_1 > \mu_2 > p_1,\ p_1 > \mu_1 > \lambda_2 > \mu_2 > q_1, \\
\mu_1 > p_1 > \lambda_2 > \mu_2 > q_1,\ \mu_1 > \mu_2 > q_1 > p_1 > \lambda_2,\ \mu_1 > \lambda_2 > p_1 > \mu_2 > q_1, \\
\mu_1 > \lambda_2 > \mu_2 > p_1 > q_1,\ \mu_1 > \lambda_2 > \mu_2 > q_1 > p_1,\ p_1 > \lambda_2 > \mu_1 > q_1 > \mu_2, \\
\mu_1 > q_1 > p_1 > \lambda_2 > \mu_2,\ \mu_1 > q_1 > p_1 > \mu_2 > \lambda_2,\ \mu_1 > q_1 > \mu_2 > p_1 > \lambda_2, \\
\lambda_2 > p_1 > \mu_1 > q_1 > \mu_2,\ \mu_1 > \lambda_2 > p_1 > q_1 > \mu_2,\ \mu_1 > q_1 > \lambda_2 > p_1 > \mu_2, \\
\mu_1 > \lambda_2 > q_1 > p_1 > \mu_2,\ \mu_1 > q_1 > \mu_2 > \lambda_2 > p_1,\ \mu_1 > q_1 > \lambda_2 > \mu_2 > p_1, \\
\mu_1 > \lambda_2 > q_1 > \mu_2 > p_1
\end{array}\right\} \quad \ldots (43)$$

(viii) $A_0 < 0,\ A_1 > 0;\ B_1 < 0,\ B_2 > 0,\ \Delta_2 < 0.$

$$q_1 > p_1 > \lambda_2 \quad \ldots \quad (44)$$

(ix) $A_0 < 0,\ A_1 < 0,\ A_2 > 0,\ \Delta_1 > 0;\ B_1 > 0.$

Either $\left.\begin{array}{l} p_1 > \lambda_1 > \lambda_2 \\ p_1 > q_1 > \mu_2 \end{array}\right\}$ .. (45a)

or $q_1 > p_1 > \mu_2 > \lambda_1 > \lambda_2$ .. (45b)

or $q_1 > \mu_2 > p_1 > \lambda_1 > \lambda_2$ .. (45c)

or $p_1 > \lambda_1 > \mu_2 > q_1$ .. (45d)

or $\mu_2 > q_1 > p_1 > \lambda_1 > \lambda_2$ .. (45e)

or $\left.\begin{array}{ll} & \lambda_1 > p_1 > \lambda_2 \\ \text{and either} & p_1 > q_1 > \mu_2 \\ \text{or} & p_1 > \mu_2 > q_1 \\ \text{or} & \lambda_1 > \mu_2 > p_1 > q_1 \\ \text{or} & \mu_2 > q_1 > p_1 \end{array}\right\}$ .. (45f)

or $q_1 > \mu_2 > \lambda_1 > p_1 > \lambda_2$ .. (45g)

or $\lambda_1 > \lambda_2 > p_1 > q_1 > \mu_2$ .. (45h)

or $q_1 > \lambda_1 > \mu_2 > \lambda_2 > p_1$ .. (45i)

or $\lambda_1 > \mu_2 > \lambda_2 > p_1 > q_1$ .. (45j)

or $\lambda_1 > \mu_2 > \lambda_2 > q_1 > p_1$ .. (45k)

or $\lambda_1 > \mu_2 > q_1 > \lambda_2 > p_1$ .. (45l)

(x) $A_0 < 0, A_1 < 0, A_2 > 0, \Delta_1 > 0$; $B_1 < 0$ and either $B_2 < 0$ or $B_2 > 0, \Delta_2 < 0$.

Either $q_1 > p_1 > \lambda_1 > \lambda_2$ .. .. (46a)

or $\lambda_1 > q_1 > p_1 > \lambda_2$ .. .. (46b)

or $q_1 > \lambda_1 > p_1 > \lambda_2$ .. .. (46c)

(xi) $A_0 < 0, A_1 < 0, A_2 > 0, \Delta_1 > 0$; $B_1 < 0, B_2 > 0, \Delta_2 > 0$.

Either $p_1 > \lambda_1 > \lambda_2 > q_1 > \mu_1 > \mu_2$, or $p_1 > q_1 > \lambda_1 > \lambda_2 > \mu_1 > \mu_2$,

or $q_1 > p_1 > \lambda_1 > \mu_1 > \mu_2 > \lambda_2$, or $q_1 > p_1 > \lambda_1 > \mu_1 > \lambda_2 > \mu_2$,

or $q_1 > p_1 > \lambda_1 > \lambda_2 > \mu_1 > \mu_2$, or $q_1 > \mu_1 > p_1 > \mu_2 > \lambda_1 > \lambda_2$,

or $q_1 > \mu_1 > \mu_2 > p_1 > \lambda_1 > \lambda_2$, or $q_1 > p_1 > \mu_1 > \mu_2 > \lambda_1 > \lambda_2$,

or $p_1 > \mu_1 > \lambda_1 > \mu_2 > q_1 > \lambda_2$, or $p_1 > \mu_1 > \lambda_1 > \lambda_2 > \mu_2 > q_1$,

or $p_1 > \lambda_1 > \mu_1 > \lambda_2 > \mu_2 > q_1$, or $\mu_1 > p_1 > \lambda_1 > \mu_2 > q_1 > \lambda_2$,

or $\mu_1 > p_1 > \lambda_1 > \mu_2 > \lambda_2 > q_1$, or $\mu_1 > p_1 > \lambda_1 > \lambda_2 > \mu_2 > q_1$,

or $\mu_1 > \mu_2 > q_1 > p_1 > \lambda_1 > \lambda_2$,

or $\left.\begin{array}{l} p_1 > \lambda_1 > \lambda_2 \\ p_1 > \mu_1 > q_1 > \mu_2 \\ \mu_1 > \lambda_1 \end{array}\right\}$,

or $\left.\begin{array}{l} p_1 > \lambda_1 > \lambda_2 \\ \mu_1 > p_1 > q_1 > \mu_2 \end{array}\right\}$,

or $\mu_1 > q_1 > p_1 > \mu_2 > \lambda_1 > \lambda_2$, or $\mu_1 > q_1 > \mu_2 > p_1 > \lambda_1 > \lambda_2$.

or $\lambda_1 > p_1 > \lambda_2 > q_1 > \mu_1 > \mu_2$, or $\lambda_1 > p_1 > q_1 > \lambda_2 > \mu_1 > \mu_2$

or $\lambda_1 > p_1 > \lambda_2$

and either $q_1 > p_1 > \mu_1 > \mu_2$

or $\left.\begin{array}{l} q_1 > \mu_1 > p_1 > \mu_2 \\ \lambda_1 > \mu_1 \end{array}\right\}$,

or $\left.\begin{array}{l} q_1 > \mu_1 > \mu_2 > p_1 \\ \lambda_1 > \mu_1 \end{array}\right\}$,

or $\mu_1 > \mu_2 > q_1 > p_1$, or $\mu_1 > \lambda_1 > p_1 > q_1 > \mu_2$,

or $\lambda_1 > \mu_1 > q_1 > p_1 > \mu_2$

or $\lambda_1 > p_1 > \mu_1 > \lambda_2 > \mu_2 > q_1$, or $\lambda_1 > \mu_1 > p_1 > \lambda_2 > \mu_2 > q_1$,

or $\mu_1 > \lambda_1 > p_1 > \lambda_2 > \mu_2 > q_1$, or $\mu_1 > \lambda_1 > p_1 > \mu_2 > \lambda_2 > q_1$.

or $\mu_1 > \lambda_1 > p_1 > \mu_2 > q_1 > \lambda_2$, or $\mu_1 > \lambda_1 > \mu_2 > p_1 > \lambda_2 > q_1$.

or $\mu_1 > \lambda_1 > \mu_2 > p_1 > q_1 > \lambda_2$, or $\lambda_1 > p_1 > \lambda_2 > \mu_1 > q_1 > \mu_2$,

or $\lambda_1 > \mu_1 > q_1 > \mu_2 > p_1 > \lambda_2$, or $\mu_1 > q_1 > \mu_2 > \lambda_1 > p_1 > \lambda_2$,

$$
\left.
\begin{array}{ll}
\text{or} & \left.\begin{array}{l}
\lambda_1 > \lambda_2 > p_1 \\
\text{and either } p_1 > \mu_1 > q_1 > \mu_2 \\
\text{or } \mu_1 > p_1 > q_1 > \mu_2 \\
\text{or } \mu_1 > q_1 > \mu_2 > p_1 \\
\quad \text{and either } \lambda_1 > \mu_2 > \lambda_2 \\
\quad \text{or } \lambda_1 > \mu_1 > \lambda_2
\end{array}\right\} \\
\text{or} & \lambda_1 > \mu_1 > \lambda_2 > q_1 > p_1 > \mu_2. \text{ or } \lambda_1 > \mu_1 > q_1 > \lambda_2 > p_1 > \mu_2
\end{array}
\right\} \quad (47)
$$

(xii) $A_0 < 0$, $A_1 < 0$; and either $A_2 > 0$, $\Delta_1 < 0$ or $A_2 < 0$; $B_1 < 0$, $B_2 > 0$, $\Delta_2 > 0$.

Either $$p_1 > q_1 > \mu_1 > \mu_2 \quad \ldots \quad (48a)$$

or $$p_1 > \mu_1 > q_1 > \mu_2 \quad \ldots \quad (48b)$$

or $$\mu_1 > p_1 > q_1 > \mu_2 \quad \ldots \quad (48c)$$

The corresponding γ and also T_6 will be positive real if any one set of inequalities mentioned in the six expressions given below is satisfied.

Either

$$
\left.
\begin{array}{l}
\left.\begin{array}{l} A_0 > 0 \\ \Delta_3 > 0 \end{array}\right\} \\
\text{and either } R_2 > 0 \\
\text{or } \left.\begin{array}{l} R_1 < 0 \\ R_2 < 0 \end{array}\right\}
\end{array}
\right\} \quad \ldots \quad (49)
$$

or

$$
\left.
\begin{array}{l}
A_0 > 0 \\
\Delta_3 < 0 \\
\text{and either } \nu_2 > \beta_0 \\
\text{or } \beta_0 > \nu_3 \\
\text{and either } R_2 > 0 \\
\text{or } \left.\begin{array}{l} R_1 < 0 \\ R_2 < 0 \end{array}\right\}
\end{array}
\right\} \quad \ldots \quad (50)
$$

or $$A_0 > 0,\ R_1 > 0,\ R_2 < 0,\ \Delta_3 \gtrless 0 \quad \ldots \quad (51)$$

or $$A_0 < 0,\ \Delta_3 > 0,\ \beta_0 > \frac{A_0}{2a_3} > p_1 \quad \ldots \quad (52)$$

or

$$
\begin{array}{l}
A_0 < 0 \\
\Delta_3 < 0 \\
\text{and either } \left.\begin{array}{l}
p_1 > \nu_3 > \beta_0 > \nu_2, \text{ or } \beta_0 > p_1 > \nu_3 > \nu_2, \\
\text{or } \nu_3 > p_1 > \beta_0 > \nu_2, \text{ or } \beta_0 > \nu_3 > p_1 > \nu_2, \\
\text{or } \nu_3 > \nu_2 > \beta_0 > p_1, \text{ or } \beta_0 > \nu_3 > \nu_2 > p_1
\end{array}\right\} \quad \ldots \quad (53)
\end{array}
$$

or $$A_0 < 0,\ R_1 > 0,\ R_2 < 0,\ \Delta_3 \gtrless 0,\ \beta_0 > p_1 \quad \ldots \quad (54)$$

where

$$
\left.\begin{aligned}
&A_0 = (2a_3 - b_0b_2T_3);\ A_1 = b_0b_2(a_1-b_1) - a_3(2+3b_0);\ A_2 = A_1 + b_0b_2T_3(1+b_0);\\
&B_1 = T_5\left[a_2b_0b_2 - a_3b_1 - b_0(2a_3b_1 + b_2^2)\right];\ B_2 = b_0b_2[T_5(a_2+b_0b_2) - 2a_3];\\
&R_1 = 4a_3 - b_0(a_1b_2 + a_3 - b_2T_5);\ R_2 = b_0(a_1b_2 + a_3 - b_0b_2T_3 - b_2T_5) - 2a_3;\\
&\Delta_1 = A_2^2 + 4A_1b_0^2b_2T_3;\ \ \Delta_2 = B_2^2 + 4B_1b_0b_2(2a_3 + b_0b_2T_5);\ \Delta_3 = 4p^3 + 27q^2;\\
&\lambda_1 = \frac{-A_2 - \sqrt{\Delta_1}}{2A_1};\ \lambda_2 = \frac{-A_2 + \sqrt{\Delta_1}}{2A_1};\ \mu_1 = \frac{-B_2 - \sqrt{\Delta_2}}{2B_1};\\
&\mu_2 = \frac{-B_2 + \sqrt{\Delta_2}}{2B_1};\ \begin{matrix}(\nu_3, \nu_2, \nu_1) = \\ \nu_3 > \nu_2 > \nu_1\end{matrix} \begin{Bmatrix} 2\sqrt{\dfrac{-p}{3}}\cos\dfrac{\phi}{3} - \dfrac{1}{3}\cdot\dfrac{R_1}{2a_3} \\ -2\sqrt{\dfrac{-p}{3}}\cos\left(\dfrac{\pi}{3} - \dfrac{\phi}{3}\right) - \dfrac{1}{3}\cdot\dfrac{R_1}{2a_3} \\ -2\sqrt{\dfrac{-p}{3}}\cos\left(\dfrac{\pi}{3} + \dfrac{\phi}{3}\right) - \dfrac{1}{3}\cdot\dfrac{R_1}{2a_3}\end{Bmatrix};\\
&\phi = \tan^{-1}\left[\frac{-\sqrt{-\Delta_3}}{q\sqrt{27}}\right];\ p = -\frac{R_2}{2a_3} - \frac{1}{3}\left(\frac{R_1}{2a_3}\right)^2;\ q = \frac{b_0^2b_2T_3}{2a_3} + \frac{R_1R_2}{12a_3^2} + \frac{2}{27}\left(\frac{R_1}{2a_3}\right)^3;\\
&p_1 = -\frac{A_0}{2a_3};\ q_1 = \frac{b_0b_2^2}{a_3b_1}
\end{aligned}\right\} \quad (55)
$$

Now, the proper design procedure would be to first check and see if the inequality of expression (34) is satisfied; and then compute the values of A's, B's, etc., with the aid of eqn. (55) and see if any one of the twelve cases listed above is applicable and a set of conditions mentioned therein is satisfied. Thereafter, solve (Uspensky 1948) the cubic of eqn. (35) and see if any one of its positive real roots, β_0, satisfies any one set of conditions listed in any one of the expressions (49) through (54). The satisfaction of these conditions signifies that the circuit of Fig. 1(*b*) for the simulation of the system of eqn. (1) is physically realizable. The circuit component values may then be obtained by computing T_3 and T_5 with the aid of eqns. (33*a*) and (33*b*) and substituting β_0 into eqns. (2.10) and (2.19) to obtain α and γ respectively; T_6 may then be obtained with the aid of eqn. (31). Choosing arbitrarily a convenient value for any one of the capacitors, the values of the resistors and the remaining capacitors can then be obtained with the aid of eqn. (32).

References

Uspensky, J. V. (1948). Theory of Equations, pp. 82–96. McGraw-Hill, N.Y.

Wadhwa, L. K. (1963). *Proc. nat. Inst. Sci. India*, **A 29**, 597–618.

APPENDIX I

CONDITIONS FOR POSITIVE REAL ROOTS

Simulation of the system represented by eqn. (1) with the network of Fig. 2(*a*) is possible only if the values of α, β, γ, T_1, T_5 and T_7 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta\gamma}{K} \quad \ldots \quad (1.1)$$

$$b_1 = T_1 + T_5 \quad \ldots \quad (1.2)$$

$$b_2 = T_1 T_5 \quad \ldots \quad (1.3)$$

$$a_1 = \frac{\alpha(2\beta+1)}{K} T_1 + \frac{\beta(2\alpha+\gamma+2)}{K} T_5 + \frac{2\beta(\alpha+1)(\gamma+1)}{K} T_7 \quad \ldots \quad (1.4)$$

$$a_2 = \frac{\alpha\beta}{K} T_1 T_5 + \frac{\alpha\beta(\gamma+1)}{K} T_1 T_7 + \frac{2\beta\gamma(\alpha+1)}{K} T_5 T_7 \quad \ldots \quad (1.5)$$

$$a_3 = \frac{\alpha\beta\gamma}{K} T_1 T_5 T_7 \quad \ldots \quad (1.6)$$

where

$$K = (2\alpha + 3\alpha\beta + 4\beta + \beta\gamma + 2) \quad \ldots \quad (1.7)$$

are positive and real, where a's and b's are non-zero positive real constants.

The solution of eqns. (1.2) and (1.3) gives

$$T_1 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (1.8a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (1.8b)$$

from which it is evident that T_1 and T_5 will be positive real, provided that

$$b_1^2 \geqslant 4b_2 \quad \ldots \quad (1.9)$$

Elimination of β and T_7 from eqns. (1.1), (1.2), (1.3), (1.4), (1.6), (1.7) and (1.1), (1.2), (1.3), (1.5), (1.6), (1.7) give the following two equations

$$\frac{1}{\gamma} = -\frac{m\left(\frac{1}{\alpha} - z_1\right)}{\left(\frac{A_1}{\alpha^2} + \frac{A_2}{\alpha} + A_3\right)} \quad \ldots \quad (1.10)$$

$$\frac{p}{\alpha} + \frac{q}{\gamma} = k \quad \ldots \quad (1.11)$$

where A_1, A_2, A_3, k, m, p, q are real constants and their values in terms of the known a's and b's are given in eqn. (14).

Now, the function of eqn. (1.10) has a zero at

$$\frac{1}{\alpha'} = OP = z_1 = \left(\frac{l-m}{m}\right) \quad \ldots \quad (1.12)$$

and two poles (λ_1, λ_2) which are the roots of

$$\frac{A_1}{\alpha^2}+\frac{A_2}{\alpha}+A_3=0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.13)$$

and are given by

$$(\lambda_1, \lambda_2)=\frac{1}{\alpha_{(L,\, M)}}=\frac{-A_2\pm\sqrt{A_2^2-4A_1A_3}}{2A_1}. \qquad \ldots \quad \ldots \quad (1.14)$$

As seen from eqn. (14), A_1 is positive, therefore the roots of eqn. (1.13) will be negative, if

$$\left.\begin{array}{l} A_2>0 \\ A_3>0 \end{array}\right\}$$

and both these roots will be real, if

$$\Delta_1=(A_2^2-4A_1A_3)>0$$

or complex, if

$$\Delta_1<0.$$

But, if

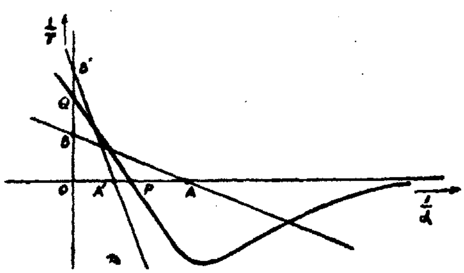
$$\left.\begin{array}{l} A_2 \gtrless 0 \\ A_3<0 \end{array}\right\}$$

then eqn. (1.13) will have one negative and one positive real root.
And, if

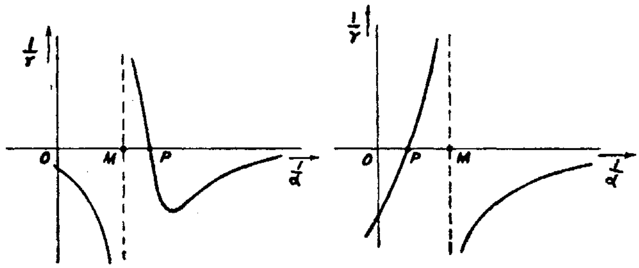
$$A_2<0$$
$$A_3>0$$
$$\Delta_1>0$$

then both the roots of eqn. (1.13) will be positive real. The function of eqn. (1.10) for positive α and for various possible cases under which a portion of the curves can lie in the first quadrant of the $\frac{1}{\alpha}-\frac{1}{\gamma}$ plane is sketched in Fig. 1.1.

Fig. 1.1. The sketches of $\frac{1}{r}=\dfrac{-m\left(\frac{1}{\alpha}-z_1\right)}{\left(\frac{A_1}{\alpha^2}+\frac{A_2}{\alpha}+A_3\right)}$ for positive α.

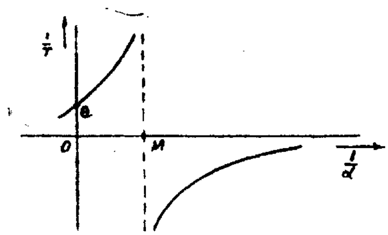


(a) $A_3>0$, $(l-m)>0$ and either $A_2>0$ or $A_2<0$, $\Delta_1<0$

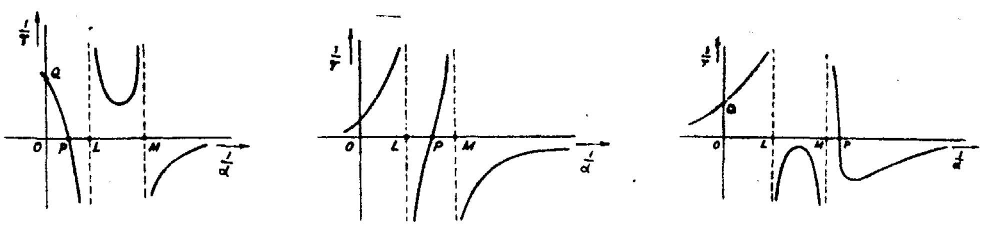


(i) $z_1 > \lambda_2$ (ii) $z_1 < \lambda_2$

(*b*) $A_3 < 0$, $(l-m) > 0$

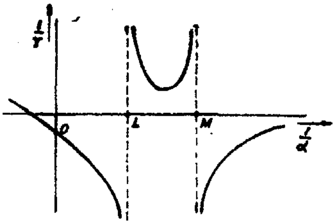


(c) $A_3 < 0$, $(l-m) < 0$



(i) $z_1 < \lambda_1 < \lambda_2$ (ii) $\lambda_1 < z_1 < \lambda_2$ (iii) $\lambda_1 < \lambda_2 < z_1$

(*d*) $A_2 < 0$, $A_3 > 0$, $\Delta_1 > 0$, $(l-m) < 0$



(*e*) $A_2 < 0$, $A_3 > 0$, $\Delta_1 > 0$, $(l-m) < 0$

Straight line of eqn. (1.11) will cut the $\frac{1}{\alpha}$ and $\frac{1}{\gamma}$ axes at the points A and B respectively, such that

$$OA = z_2 = \frac{k}{p}$$

$$OB = \frac{k}{q}$$

and these intercepts will be positive, that is a portion of the straight line can exist in the first quadrant, provided that

$$k = \left(a_2 - \frac{a_3 b_1}{b_2} - \frac{a_3 T_5}{b_2}\right) > 0 \quad \ldots \quad \ldots \quad \ldots (1.15)$$

as p and q are both positive.

Elimination of γ from eqns. (1.1) and (1.11) gives

$$\beta = -\frac{\left(\frac{1}{\alpha}+1\right)\left(\frac{1}{\alpha}-z_2\right)}{2\left[\frac{1}{\alpha^2}-\frac{B_1}{\alpha}+B_2\right]} \quad \ldots \quad \ldots \quad \ldots (1.16)$$

which has zeros at

$$\frac{1}{\alpha'} = -1$$

$$\frac{1}{\alpha''} = OA = z_2 = \frac{k}{p}$$

and two poles which are the roots of

$$\frac{1}{\alpha^2} - \frac{B_1}{\alpha} + B_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (1.17)$$

and are given by

$$(\mu_1, \mu_2) = \frac{1}{\alpha_{(G,\,H)}} = \frac{B_1 \pm \sqrt{B_1^2 - 4B_2}}{2}. \quad \ldots \quad \ldots (1.18)$$

Now, μ_1 and μ_2 will be both positive real, if

$$\left.\begin{array}{r} B_1 > 0 \\ B_2 > 0 \\ \Delta_2 = (B_1^2 - 4B_2) > 0 \end{array}\right\}$$

and either negative real or complex, if

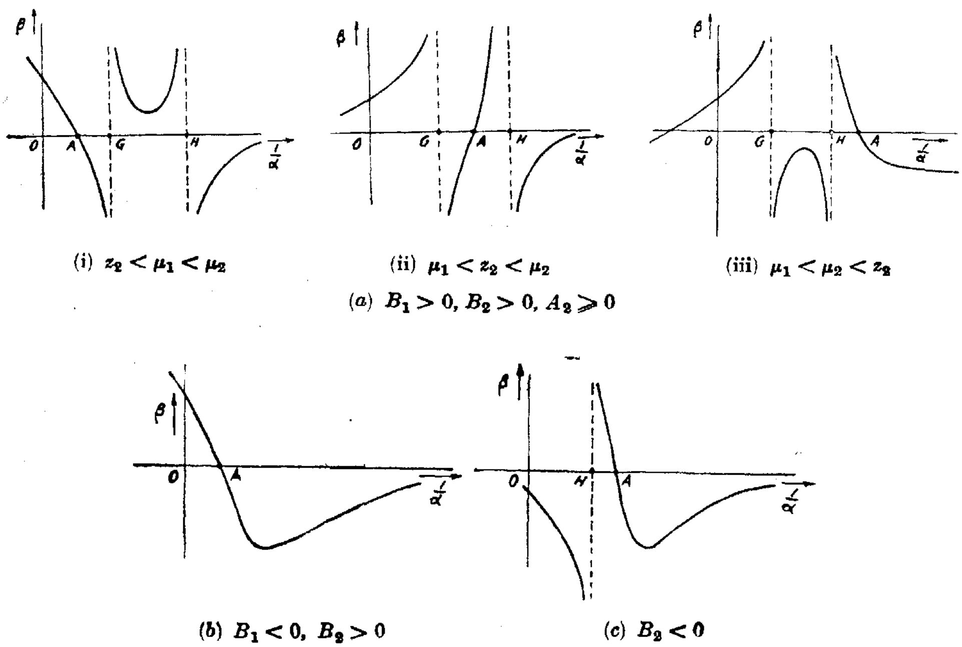
$$\left.\begin{array}{r} B_1 < 0 \\ B_2 > 0 \end{array}\right\}$$

and one will be positive real and the other negative real, if

$$\left.\begin{array}{r} B_1 \gtrless 0 \\ B_2 < 0 \end{array}\right\}.$$

The function of eqn. (1.16) for positive α and for various possible cases under which a portion of the curve can exist in the first quadrant of the $\frac{1}{\alpha} - \beta$ plane is sketched in Fig. 1.2.

FIG. 1.2. The sketches of $\beta = \dfrac{-\left(\frac{1}{\alpha}+1\right)\left(\frac{1}{\alpha}-z_2\right)}{2\left[\frac{1}{\alpha^2}-\frac{B_1}{\alpha}+B_2\right]}$ for positive α.



Now, α and γ will be positive real if the straight line of eqn. (1.11) and the curve of eqn. (1.10) intersect each other in the first quadrant of the $\frac{1}{\alpha} - \frac{1}{\gamma}$ plane; and the conditions under which that can happen are given in expressions (16) through (19) listed under four cases mentioned in the text. The corresponding β will be positive real only for a certain discrete value of α. Therefore, if the real root, $\frac{1}{\alpha_0}$, of eqn. (13) is positive and satisfies any one set of conditions listed in expressions (20) through (22) then β will be positive real and hence the corresponding T_7, as will be evident from eqn. (1.6), will be also positive real.

To summarize, if the conditions of expressions (1.9) and (1.15) together with any one set of conditions from expressions (16) through (19) and also any one set of conditions from (20) through (22) are satisfied then the network of Fig. 1(a) for the simulating of the system of eqn. (1) is physically realizable.

APPENDIX II

CONDITIONS FOR POSITIVE REAL ROOTS

It is possible to simulate the system represented by eqn. (1) with the network of Fig. 1(*b*) only if the values of α, β, γ, T_3, T_5, T_6 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta\gamma}{K} \quad (2.1)$$

$$b_1 = T_3 + T_5 \quad (2.2)$$

$$b_2 = T_3 T_5 \quad (2.3)$$

$$a_1 = \frac{\alpha(\beta+\gamma)}{K} T_3 + T_5 + \left\{2 + \frac{\beta\gamma(3\alpha+2)}{K}\right\} T_6 \quad (2.4)$$

$$a_2 = \frac{\alpha(\beta+\gamma)}{K} T_3 T_5 + \frac{(\alpha\beta+2\alpha\beta\gamma+2\alpha\gamma+\beta\gamma)}{K} T_3 T_6 + \frac{\beta(\alpha+2\alpha\gamma+\gamma)}{K} T_5 T_6 \quad (2.5)$$

$$a_3 = \frac{\alpha\beta\gamma}{K} T_3 T_5 T_6 \quad (2.6)$$

where

$$K = (\alpha+\alpha\beta+2\alpha\gamma+\gamma) \quad (2.7)$$

are positive and real.

The solution of eqns. (2.2) and (2.3) gives

$$T_3 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \quad (2.8a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \quad (2.8b)$$

from which it is evident that T_3 and T_5 will be positive real, provided that

$$b_1^2 > 4b_2. \quad (2.9)$$

Elimination of γ and T_6 from eqns. (2.1), (2.2), (2.3), (2.4), (2.6), (2.7) and (2.1), (2.2), (2.3), (2.5), (2.6), (2.7) give the following two equations

$$\frac{1}{\alpha} = \frac{A_1\beta^2 + A_2\beta - b_0^2 b_2 T_3}{b_0\beta(2a_3\beta + A_0)} \quad (2.10)$$

$$\frac{1}{\alpha} = \frac{B_1\beta^2 + B_2\beta - b_0 b_2(2a_3 + b_0 b_2 T_5)}{b_0 T_5 \beta(a_3 b_1 \beta - b_0 b_2^2)} \quad (2.11)$$

where A's and b's are real constants and their values in terms of the known a's and b's are given in eqn. (55).

Now, the function of eqn. (2.10) has a pair of zeros (λ_1, λ_2) which are the roots of

$$A_1\beta^2 + A_2\beta - b_0^2 b_2 T_3 = 0 \quad (2.12)$$

and are given by

$$(\lambda_1, \lambda_2) = \beta_{(A,\,B)} = \frac{-A_2 \pm \sqrt{A_2^2 + 4A_1 b_0^2 b_2 T_3}}{2A_1} \qquad \ldots \quad \ldots \quad (2.13)$$

and has a pair of poles, one at $\beta = 0$ and the other at

$$OL = p_1 = -\frac{A_0}{2a_3} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.14)$$

which is real and will be either negative or positive depending respectively on whether A_0 is positive or negative.

Now, if

$$A_1 > 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.15)$$

then A_2 is also positive and the function of eqn. (2.10) has two real zeros one of which is positive and the other negative; but if

$$\left.\begin{aligned} A_1 &< 0 \\ A_2 &> 0 \\ \Delta_1 = (A_2^2 + 4A_1 b_0^2 b_2 T_3) &\geqslant 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (2.16)$$

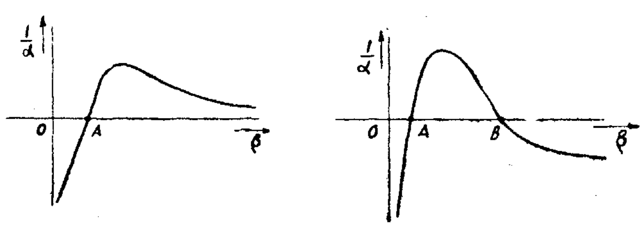
then both the zeros are positive real, and if

$$\left.\begin{aligned} & A_1 < 0 \\ \text{and either} \quad & A_2 > 0 \\ & \Delta_1 < 0 \\ \text{or} \quad & A_2 < 0 \\ & \Delta_1 \gtrless 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.17)$$

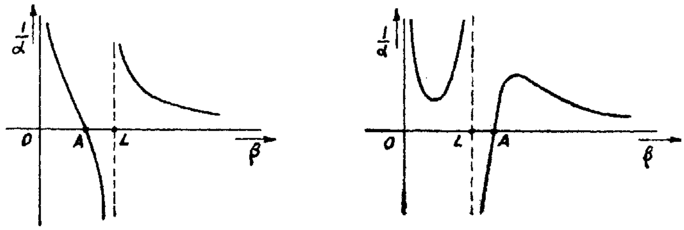
then both the roots are either negative real or complex conjugate.

The function of eqn. (2.10) for positive β and for various possible cases under which a portion of the curves can lie in the first quadrant of the $\beta - \frac{1}{\alpha}$ plane is sketched in Fig. 2.1.

FIG. 2.1. Sketches of $\frac{1}{\alpha} = \frac{A_1\beta^2 + A_2\beta - b_0^2 b_2 T_3}{b_0\beta(2a_3\beta + A_0)}$ for positive β.

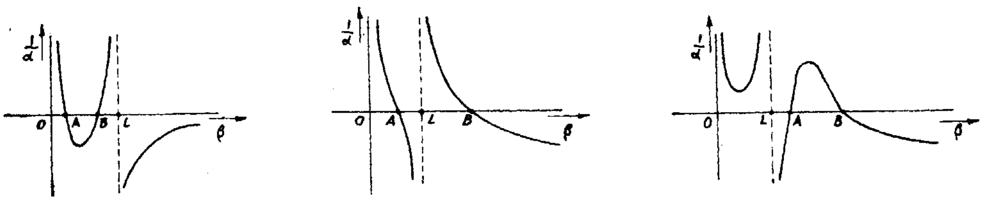


(a) $A_0 > 0,\ A_1 > 0$ (b) $A_0 > 0,\ A_1 < 0,\ A_2 > 0,\ \Delta > 0$



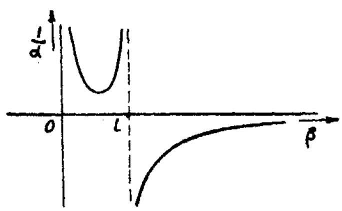
(i) $OL > OA$ (ii) $OA > OL$

(c) $A_0 < 0,\ A_1 > 0$



(i) $OL > OB > OA$ (ii) $OB > OL > OA$ (iii) $OB > OA > OL$

(d) $A_0 < 0,\ A_1 < 0,\ A_2 > 0,\ \Delta_1 > 0$



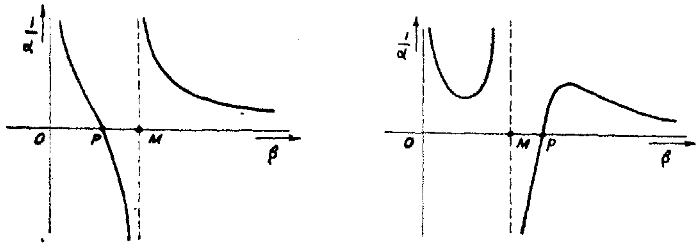
(e) $A_0 < 0,\ A_1 < 0$
and either $A_2 > 0,\ \Delta_1 < 0$
or $A_2 < 0,\ \Delta_1 \gtrless 0$

Similarly, the function of eqn. (2.11) has a pair of zeros (μ_1, μ_2) which may be positive or negative, real or complex, depending on the sign of B_1 and B_2; and has a pair of real poles one at $\beta = 0$ and the other at

$$OM = q_1 = \frac{b_0 b_2^2}{a_3 b_1}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (2.18)$$

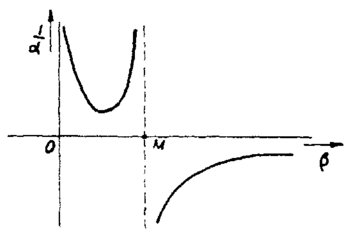
The function of eqn. (2.11) for positive β and for various possible cases under which a portion of the curves can exist in the first quadrant of the $\beta - \frac{1}{\alpha}$ plane is sketched in Fig. 2.2.

FIG. 2.2. Sketches of $\frac{1}{\alpha} = \frac{B_1\beta^2 + B_2\beta - b_0b_2(2a_3 + b_0b_2T_5}{b_0T_5\beta\left(a_3b_1\beta - b_0b_2^2\right)}$ for positive β.

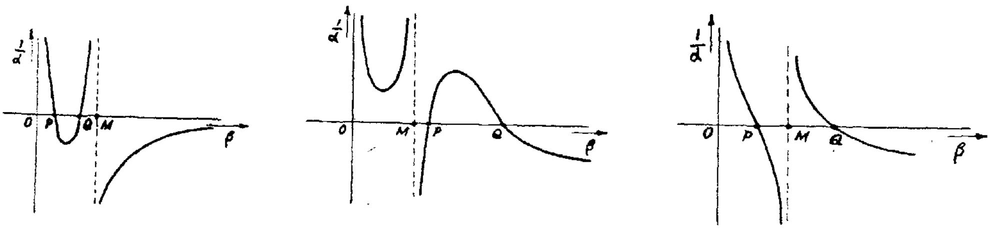


(i) $OM > OP$ (ii) $OP > OM$

(a) $B_1 > 0,\ B_2 \gtrless 0$

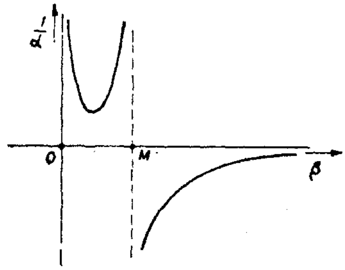


(b) $B_1 < 0,\ B_2 < 0,\ \Delta_2 \gtrless 0$



(i) $OM > OQ > OP$ (ii) $OQ > OP > OM$ (iii) $OQ > OM > OP$

(c) $B_1 < 0,\ B_2 > 0,\ \Delta_2 > 0$



(d) $B_1 < 0,\ B_2 > 0,\ \Delta_2 < 0$

Now, the conditions under which the curves of eqns. (2.10) and (2.11) can exist in the first quadrant of the $\beta-\frac{1}{\alpha}$ plane are listed as the twelve cases mentioned in the text, and these curves can intersect each other in that region provided that any one set of conditions listed under the relevant case is satisfied.

Elimination of $\frac{1}{\alpha}$ from eqns. (2.1) and (2.10) gives

$$\frac{1}{\gamma}=\frac{2a_3\beta^3+R_1\beta^2-R_2\beta+b_0^2b_2T_3}{b_0\beta(\beta+1)(2a_3\beta+A_0)} \quad \ldots \quad \ldots \quad \ldots \quad (2.19)$$

which has three real poles the locations of which are given by

$$\left.\begin{aligned} \beta' &= 0 \\ \beta'' &= -1 \\ \beta''' &= -\frac{A_0}{2a_3} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.20)$$

and either one or three real zeros depending on whether discriminant Δ_3 of the equation

$$2a_3\beta^3+R_1\beta^2-R_2\beta+b_0^2b_2T_3=0 \quad \ldots \quad \ldots \quad \ldots \quad (2.21)$$

is positive or negative.

Therefore, if

either $$\left.\begin{aligned} R_1 &\gtrless 0 \\ R_2 &> 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.22a)$$

or $$\left.\begin{aligned} R_1 &< 0 \\ R_2 &< 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.22b)$$

then eqn. (2.21) will have one negative real and either none or two positive real roots depending on whether

$$\Delta_3=4p^3+27q^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.23)$$

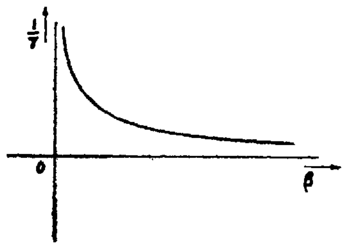
is either positive or negative. But if

$$\left.\begin{aligned} R_1 &> 0 \\ R_2 &< 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.24)$$

then eqn. (2.21) will have either one or three negative real roots.

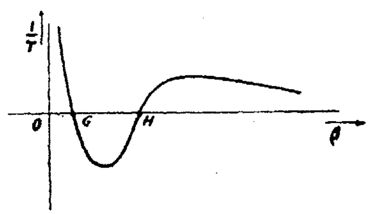
The function of eqn. (2.19) for positive β and for various possible cases under which a portion of the curves can exist in the first quadrant of the $\beta-\frac{1}{\gamma}$ plane is sketched in Fig. 2.3; and the conditions under which corresponding to a set of positive real α, β a positive real γ can exist are listed in eqns. (49) through (54).

FIG. 2.3. Sketches of $\dfrac{1}{\alpha} = \dfrac{2a_3\beta^3 + R_1\beta^2 - R_2\beta + b_0^2 b_2 T_3}{b_0\beta(1+\beta)(2a_3\beta + A_0)}$ for positive β.



(*a*) Either $\left.\begin{array}{r} \Delta_3 > 0 \\ A_0 > 0 \end{array}\right\}$ and either $\left.\begin{array}{r} R_1 \gtrless 0 \\ R_2 > 0 \end{array}\right\}$ or $\left.\begin{array}{r} R_1 < 0 \\ R_2 < 0 \end{array}\right\}$

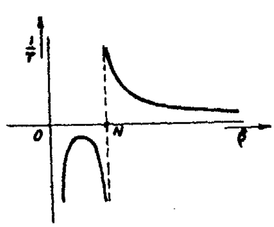
or $\left.\begin{array}{r} \Delta_3 \gtrless 0 \\ A_0 > 0 \\ R_1 > 0 \\ R_2 < 0 \end{array}\right\}$



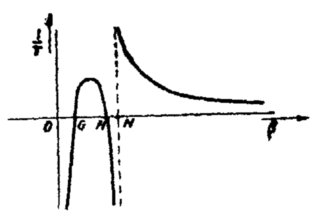
(*b*) $\Delta_3 > 0$

$A_0 > 0$

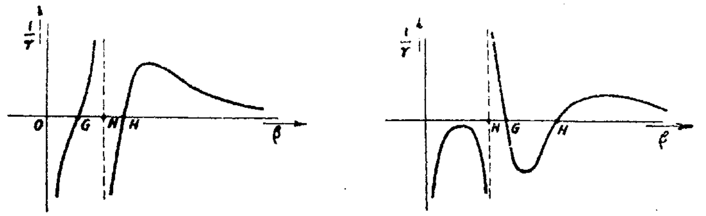
and either $\left.\begin{array}{r} R_1 \gtrless 0 \\ R_2 < 0 \end{array}\right\}$, or $\left.\begin{array}{r} R_1 < 0 \\ R_2 < 0 \end{array}\right\}$



(i) $\Delta_3 > 0$

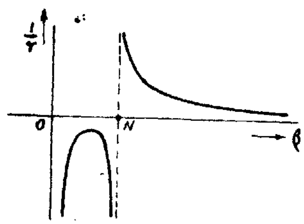


(ii) $\Delta_3 < 0$, $ON > OH > OG$



(iii) $\Delta_3 < 0$, $OH > ON > OG$ (iv) $\Delta_3 < 0$, $OH > OG > ON$

(c) $A_0 < 0$ and either $R_1 \gtrless 0$, $R_2 > 0$ or $R_1 < 0$, $R_2 < 0$



(d) $\Delta_3 \gtrless 0$, $A_0 < 0$, $R_1 > 0$, $R_2 < 0$

To summarize, if any one of the twelve cases mentioned in the text is applicable, and any one set of conditions listed under the relevant case is satisfied together with the inequality of expression (34) and any one set of conditions listed in any one of the inequalities of expressions (49) through (54), then the circuit of Fig. 1(b) for the simulation of the system of eqn. (1) is physically realizable.