SIMULATION OF THE GENERAL THIRD ORDER SYSTEMS BY A SINGLE OPERATIONAL AMPLIFIER—III

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In the previous papers (Wadhwa 1963*a*, *b*) on the same topic four basic circuits each employing three capacitors and seven resistors and capable of simulating, under certain conditions, the general third order linear systems were presented. In this paper two more basic circuits for simulating similar type of third order systems are presented. The circuits are analysed, design formulae obtained and the conditions of their physical realizability discussed.

INTRODUCTION

In previous communications (Wadhwa 1963*a*, *b*) on this subject four basic circuits each capable of simulating, under certain conditions, the general third order linear systems were presented.

The purpose of this paper is to present two more basic circuits, give the design formulae and discuss the conditions of their physical realizability.

THIRD ORDER SYSTEM SIMULATION

Two basic circuits, each employing three capacitors and seven resistors (with a certain arbitrary choice of resistor values) and capable of simulating, under certain conditions, the general third order linear systems, that is

$$F(s) = -\frac{b_0(b_2s^2+b_1s+1)}{a_3s^3+a_2s^2+a_1s+1}, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

are shown in Fig. 1.

The transfer function for the circuit of Fig. 1(*a*) can be shown to be

$$\frac{E_0}{E_1} = -\frac{\frac{\alpha\beta\gamma}{k}[R^2c_3c_5s^2+R(c_3+c_5)s+1]}{\frac{\alpha\beta\gamma}{k}R^3c_3c_5c_7s^3+\left[\frac{\beta(\alpha+\gamma)}{k}R^2c_3c_5+\frac{\alpha\beta(\gamma+1)}{k}R^2c_3c_7+\frac{2\beta\gamma(\alpha+1)}{k}R^2c_5c_7\right]s^2 + \left[\frac{(\alpha+2\alpha\beta+2\beta+\beta\gamma)}{k}Rc_3+\frac{\beta(2\alpha+\gamma+2)}{k}Rc_5+\frac{2\beta(\alpha+1)(\gamma+1)}{k}Rc_7\right]s+1} \quad \ldots \quad (2)$$

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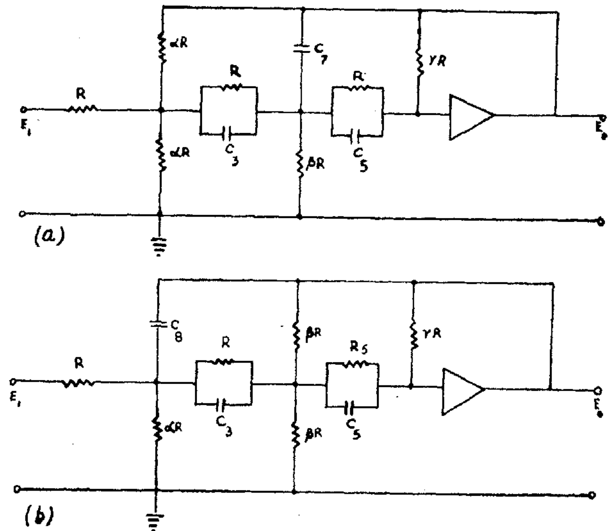
FIG. 1. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0(b_2s^2+b_1s+1)}{a_3s^3+a_2s^2+a_1s+1}$



Equations (1) and (2) will be identical if

$$b_0 = \frac{\alpha\beta\gamma}{k} \quad \ldots \quad (3)$$

$$b_1 = T_3+T_5 \quad \ldots \quad (4)$$

$$b_2 = T_3T_5 \quad \ldots \quad (5)$$

$$a_1 = \frac{(\alpha+2\alpha\beta+2\beta+\beta\gamma)}{k}T_3+\frac{\beta(2\alpha+\gamma+2)}{k}T_5+\frac{2\beta(\alpha+1)(\gamma+1)}{k}T_7 \quad \ldots \quad (6)$$

$$a_2 = \frac{\beta(\alpha+\gamma)}{k}T_3T_5+\frac{\alpha\beta(\gamma+1)}{k}T_3T_7+\frac{2\beta\gamma(\alpha+1)}{k}T_5T_7 \quad \ldots \quad (7)$$

$$a_3 = \frac{\alpha\beta\gamma}{k}T_3T_5T_7 \quad \ldots \quad (8)$$

where

$$k = (2\alpha+3\alpha\beta+4\beta+\beta\gamma+2) \quad \ldots \quad (9)$$

$$T_n = Rc_n \quad \ldots \quad (10)$$

Now, the solution of equations (4) and (5) gives

$$T_3 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2-4b_2}\right]. \quad \ldots \quad (11a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2-4b_2}\right]. \quad \ldots \quad (11b)$$

both of which will be positive real, provided that

$$b_1^2 \geqslant 4b_2. \quad \ldots \quad (12)$$

Elimination of β, γ, T_7 from equation (3) through (9) gives a cubic

$$\frac{P_1}{\alpha^3}+\frac{P_2}{\alpha^2}+\frac{P_3}{\alpha}-P_4=0 \quad \ldots \quad (13)$$

where

$$P_1=4lB_1;\; P_2=4lB_2-4nB_1-2mA_1;\; P_3=lB_3+mA_2-4nB_2;\; P_4=mA_3+nB_3;$$

$$A_1=\left(\frac{2a_3}{b_2}+b_0b_1\right);\; A_2=2\left(a_1-\frac{2a_3}{b_2}\right)+b_0T_3-2\left(\frac{2a_3}{b_2}+b_0b_1\right);$$

$$A_3=T_3-2\left(a_1-\frac{2a_3}{b_2}\right);\; B_1=\left(\frac{a_3}{b_2}+b_0b_1\right):\; B_2=2\left(\frac{a_3}{b_2}+b_0T_5\right)+b_0T_3;$$

$$B_3=4\left(\frac{a_3}{b_2}+b_0T_5\right)+b_0T_3;\; l=\left(\frac{2a_3T_5}{b_2}+b_0b_2\right);\; m=\left(\frac{a_3T_3}{b_2}+b_0b_2\right);$$

$$n=\left(a_2-\frac{a_3b_1}{b_2}-\frac{a_3T_5}{b_2}\right) \quad \ldots \quad (14)$$

As seen from equation (14), A_1, B_1, B_2, B_3, l, m and P_1 are all positive real and, if

$$n=\left(a_2-\frac{a_3b_1}{b_2}-\frac{a_3T_5}{b_2}\right)>0 \quad \ldots \quad (15)$$

then, as shown in Appendix I, a set of positive real α, γ exists; provided that

$$\text{either}\quad \left.\begin{array}{l} A_2>0,\; A_3>0,\; \Delta_1>0, \\ \text{and, either } \lambda_1>\frac{n}{l}>\lambda_2 \\ \text{or } P_2<0,\; P_3<0,\; \Delta<0,\; \frac{n}{l}>\lambda_1 \end{array}\right\} \quad \ldots \quad (16)$$

$$\text{or}\quad \left.\begin{array}{l} A_3<0 \\ \text{and, either } \left.\begin{array}{l}\frac{n}{l}>\lambda_2 \\ -\frac{A_3}{B_3}>\frac{n}{m}\end{array}\right\} \\ \text{or } \left.\begin{array}{l}\lambda_2>\frac{n}{l} \\ \frac{n}{m}>-\frac{A_3}{B_3}\end{array}\right\} \end{array}\right] \quad \ldots \quad (17)$$

Now, corresponding to positive real α and γ a positive real β and also T_7 exist, provided that

$$\text{either}\quad \left.\begin{array}{l} Q_2>0 \\ Q_3>0 \\ \Delta_2>0 \end{array}\right\}$$

$$\left.\begin{array}{r} \text{and, either } \mu_1 > p_1 > \mu_2 > \dfrac{1}{\alpha_0} \\ \text{or } p_1 > \dfrac{1}{\alpha_0} > \mu_1 > \mu_2 \\ \text{or } p_1 > \mu_1 > \mu_2 > \dfrac{1}{\alpha_0} \\ \text{or } \mu_1 > \mu_2 > p_1 > \dfrac{1}{\alpha_0} \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

or

$$\left.\begin{array}{r} Q_2 > 0 \\ Q_3 > 0 \\ \Delta_2 < 0 \\ p_1 > \dfrac{1}{\alpha_0} \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

or

$$\left.\begin{array}{r} Q_3 < 0 \\ p_1 > \dfrac{1}{\alpha_0} > \mu_1 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where

$$\lambda_1 = OM = \frac{A_2+\sqrt{\Delta_1}}{4A_1}; \; \lambda_2 = OL = \frac{A_2-\sqrt{\Delta_1}}{4A_1}; \; \mu_1 = \frac{Q_2+\sqrt{\Delta_2}}{2Q_1};$$

$$\mu_2 = \frac{Q_2-\sqrt{\Delta_2}}{2Q_1}; \; p_1 = OA = \frac{n}{l}; \; Q_1 = 4b_0 l; \; Q_2 = b_0(m+4n-3l);$$

$$Q_3 = m-3b_0 n; \; \Delta_1 = A_2^2-8A_1A_3; \; \Delta_2 = Q_2^2-4Q_1Q_3;$$

$$\frac{1}{\alpha_0} = \text{positive real root of equation (13).} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

For the design of the network component values are required to be determined. The proper procedure for design would be first to check and see if the inequalities of expressions (12) and (15) are satisfied; and then compute the values of A's and B's, etc., with the aid of equations (14) and (21) and see if a set of conditions in either expressions (16) or (17) is satisfied. Thereafter, solve the cubic of equation (13) and see if any one of its positive real roots $\frac{1}{\alpha_0}$, is such that a set of conditions in any one of the expressions (18) through (20) is satisfied. The satisfaction of these conditions signifies that the circuit of Fig. 1(a) for the simulation of the system of equation (1) is physically realizable. The circuit component values may then be obtained by computing T_3 and T_5 from equations (11a) and (11b) and substituting $\frac{1}{\alpha_0}$ into equations (1.11), (1.18) gives γ and β respectively; and then T_7 may be obtained from equation (8). Choosing arbitrarily a convenient value for any one of the capacitors the values for the resistors and the remaining capacitors can then be obtained with the aid of equation (10).

Referring to the circuit of Fig. 1(b) its transfer function can be shown to be

$$\frac{E_0}{E_1} = -\frac{\frac{\alpha\beta\gamma}{k}[R^2c_3c_5s^2+R(c_3+c_5)s+1]}{\frac{\alpha\beta\gamma}{k}R^3c_3c_5c_8s^3+\left[\frac{\alpha(\beta+\gamma)}{k}R^2c_3c_5+\frac{\alpha\beta(\gamma+1)}{k}R^2c_3c_8+\frac{\alpha(\beta+\beta\gamma+\gamma)}{k}R^2c_5c_8\right]s^2 +\left[\frac{(2\alpha+2\alpha\beta+\alpha\gamma+\beta)}{k}Rc_3+\frac{(2\alpha\beta+2\alpha\gamma+\beta+\gamma)}{k}Rc_5+\frac{\alpha(2\beta+\beta\gamma+\gamma+2)}{k}Rc_8\right]s+1} \quad (22)$$

Equations (1) and (22) will be identical, if

$$b_0 = \frac{\alpha\beta\gamma}{k} \quad (23)$$

$$b_1 = T_3+T_5 \quad (24)$$

$$b_2 = T_3T_5 \quad (25)$$

$$a_1 = \frac{(2\alpha+2\alpha\beta+\alpha\gamma+\beta)}{k}T_3+\frac{(2\alpha\beta+2\alpha\gamma+\beta+\gamma)}{k}T_5+\frac{\alpha(2\beta+\beta\gamma+\gamma+2)}{k}T_8 \quad (26)$$

$$a_2 = \frac{\alpha(\beta+\gamma)}{k}T_3T_5+\frac{\alpha\beta(\gamma+1)}{k}T_3T_8+\frac{\alpha(\beta+\beta\gamma+\gamma)}{k}T_5T_8 \quad (27)$$

$$a_3 = \frac{\alpha\beta\gamma}{k}T_3T_5T_8 \quad (28)$$

where

$$k = (4\alpha+3\alpha\beta+2\alpha\gamma+2\beta+\gamma+2) \quad (29)$$

$$T_n = Rc_n \quad (30)$$

The solution of equations (24) and (25) gives

$$T_3 = \tfrac{1}{2}\left[b_1\pm\sqrt{b_1^2-4b_2}\right] \quad (31a)$$

$$T_5 = \tfrac{1}{2}\left[b_1\mp\sqrt{b_1^2-4b_2}\right] \quad (31b)$$

from which it is evident that T_3 and T_5 will be positive real, if

$$b_1^2 \geqslant 4b_2. \quad (32)$$

Now, elimination of α, γ and T_8 from equations (23) through (29) gives

$$\frac{lB_4}{\beta^4}-(lB_3+mA_4+nB_4)\frac{1}{\beta^3}+(nB_3-lB_2-mA_3)\frac{1}{\beta^2}+(nB_2-lB_1-mA_2)\frac{1}{\beta} +(nB_1-mA_1) = 0 \quad (33)$$

where

$$A_1 = 2n\left(q_1-\frac{a_3}{b_2}\right);\ A_2 = \frac{A_1}{2n}Q_1-\frac{2nl}{T_5}-T_5(m+b_0n);\ A_3 = b_0lT_5-\frac{A_1l}{n}-\frac{Q_1l}{T_5};$$

$$A_4 = \frac{2l^2}{T_5}\,;\; B_1 = b_1 m + n\left(\frac{4a_3}{b_2} + b_0 b_1\right);\; B_2 = \frac{2a_3}{b_2}(2n + Q_1) + 4b_0 n T_3 - b_0 b_1 l\,;$$

$$B_3 = 2l\left(\frac{Q_1}{T_5} - \frac{2a_3}{b_2}\right);\; B_4 = \frac{4l^2}{T_5}\,;\; l = \left(\frac{a_3}{T_3} + b_0 b_2\right);\; m = \left(\frac{a_3 b_1}{b_2} + b_0 b_2\right);$$

$$n = \left(a_2 - \frac{a_3 b_1}{b_2}\right) \quad \ldots \quad (34)$$

As seen from equation (34), A_4, B_1, B_4, l, m are all positive real and, if

$$n = \left(a_2 - \frac{a_3 b_1}{b_2}\right) > 0 \quad \ldots \quad (35)$$

then, as is shown in Appendix II, a set of positive real β, γ exists provided that any one set of conditions listed under either one of the two cases mentioned below is satisfied.

I. Either $B_2 < 0$, $B_3 > 0$, $\Delta_2 > 0$, or $B_3 < 0$, or $B_2 > 0$, $B_3 > 0$.

Either $A_1 > 0$, $\dfrac{n}{m} > \dfrac{A_1}{B_1}$

- and, either $A_2 > 0$, $A_3 > 0$
- or $A_3 < 0$, $\Delta_1 > 0$
- or $A_2 < 0$, $A_3 > 0$, $\Delta_1 > 0$ .. .. (36a)

or $A_1 < 0$

- and, either $A_2 > 0$, $A_3 < 0$, $\Delta_1 > 0$
- or $A_3 > 0$
- or $A_2 < 0$, $A_3 < 0$ .. .. (36b)
- and, either $\dfrac{n}{l} > \lambda_0$, $\Delta_1 > 0$
- or $\dfrac{n}{l} > \lambda_3$, $\Delta_1 < 0$

or $A_1 > 0$, $\Delta_1 < 0$

- and, either $A_3 < 0$
- or $A_2 < 0$, $A_3 > 0$

and, either $\nu > \lambda_3 > \lambda_2$, $\frac{n}{l} > \lambda_2$

or $\lambda_3 > \nu > \lambda_2$

- and, either $\frac{n}{l} > \lambda_3$
- or $\frac{n}{l} > \lambda_2$, $\frac{A_1}{B_1} > \frac{n}{m}$
- or $\lambda_2 > \frac{n}{l}$, $\frac{n}{m} > \frac{A_1}{B_1}$

or $\lambda_3 > \lambda_2 > \nu$

- and, either $\frac{n}{m} > \frac{A_1}{B_1}$
- or $\lambda_3 > \frac{n}{l} > \lambda_2$

.. .. (36c)

or $A_1 < 0$, $A_2 > 0$, $A_3 < 0$, $\Delta_1 < 0$

and, either $\lambda_3 > \lambda_2 > \nu > \lambda_1$, $\frac{n}{l} > \lambda_1$

or $\lambda_3 > \nu > \lambda_2 > \lambda_1$

- and, either $\lambda_2 > \frac{n}{l} > \lambda_1$
- or $\frac{n}{l} > \lambda_3$

.. .. (36d)

$$\left.\begin{array}{ll} \text{or} & \left.\begin{array}{l} \nu > \lambda_3 > \lambda_2 > \lambda_1 \\ \text{and, either } \lambda_2 > \dfrac{n}{l} > \lambda_1 \\ \text{or } \dfrac{n}{l} > \lambda_3 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} \lambda_3 > \lambda_2 > \lambda_1 > \nu \\ \dfrac{n}{l} > \lambda_1 \end{array}\right\} \end{array}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36d)$$

II. $B_2 < 0,\ B_3 > 0,\ \Delta_2 < 0.$

$$\left.\begin{array}{ll} \text{Either} & \left.\begin{array}{l} A_1 > 0 \\ \dfrac{n}{m} > \dfrac{A_1}{B_1} \end{array}\right\} \\ \text{and, either} & \left.\begin{array}{l} A_2 > 0 \\ A_3 > 0 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} A_3 < 0 \\ \Delta_1 > 0 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} A_2 < 0 \\ A_3 > 0 \\ \Delta_1 > 0 \end{array}\right\} \end{array}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37a)$$

$$\left.\begin{array}{ll} \text{or} & A_1 < 0 \\ \text{and, either} & \left.\begin{array}{l} A_2 > 0 \\ A_3 < 0 \\ \Delta_1 > 0 \end{array}\right\} \\ \text{or} & A_3 > 0 \\ \text{or} & \left.\begin{array}{l} A_2 < 0 \\ A_3 < 0 \end{array}\right\} \\ \text{and, either} & \left.\begin{array}{l} \dfrac{n}{l} > \lambda_0 \\ \Delta_1 > 0 \end{array}\right\} \\ \text{or} & \left.\begin{array}{l} \dfrac{n}{l} > \lambda_3 \\ \Delta_1 < 0 \end{array}\right\} \end{array}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37b)$$

$$\left.\begin{array}{ll} \text{or} & \left.\begin{array}{l} A_1 > 0 \\ \Delta_1 < 0 \end{array}\right\} \\ \text{and, either} & A_3 < 0 \end{array}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37c)$$

or $\left.\begin{array}{l} A_2 < 0 \\ A_3 > 0 \end{array}\right\}$

and, either $\nu_1 > \lambda_3 > \lambda_2$

and, either $\left.\begin{array}{l} \frac{n}{l} > \lambda_2 \\ \frac{A_1}{B_1} > \frac{n}{m} \end{array}\right\}$

or $\left.\begin{array}{l} \lambda_2 > \frac{n}{l} \\ \frac{n}{m} > \frac{A_1}{B_1} \end{array}\right\}$

or $\frac{n}{l} > \lambda_3$

or $\nu_2 > \lambda_3 > \lambda_2 > \nu_1$

and, either $\frac{n}{m} > \frac{A_1}{B_1}$

or $\lambda_3 > \frac{n}{l} > \lambda_2$

or $\nu_3 > \lambda_3 > \lambda_2 > \nu_2$

and, either $\frac{n}{m} > \frac{A_1}{B_1}$

or $\frac{n}{l} > \lambda_2$

or $\lambda_3 > \lambda_2 > \nu_3$

and, either $\frac{n}{m} > \frac{A_1}{B_1}$

or $\frac{n}{l} > \lambda_2$

or $\nu_3 > \nu_2 > \lambda_3 > \nu_1 > \lambda_2$

and, either $\left.\begin{array}{l} \frac{A_1}{B_1} > \frac{n}{m} \\ \frac{n}{l} > \lambda_2 \end{array}\right\}$

or $\left.\begin{array}{l} \frac{n}{m} > \frac{A_1}{B_1} \\ \lambda_2 > \frac{n}{l} \end{array}\right\}$

or $\frac{n}{l} > \lambda_3$

.. .. (37c)

$$
\left.\begin{array}{l}
\text{or}\quad \nu_3 > \lambda_3 > \nu_2 > \nu_1 > \lambda_2 \\
\quad\text{and, either}\quad \left.\begin{array}{l} \frac{n}{l} > \lambda_2 \\ \frac{A_1}{B_1} > \frac{n}{m} \end{array}\right\} \\
\qquad\text{or}\quad \left.\begin{array}{l} \lambda_2 > \frac{n}{l} \\ \frac{n}{m} > \frac{A_1}{B_1} \end{array}\right\} \\
\qquad\text{or}\quad \frac{n}{l} > \lambda_3 \\
\text{or}\quad \lambda_3 > \nu_3 > \nu_2 > \nu_1 > \lambda_2 \\
\quad\text{and, either}\quad \left.\begin{array}{l} \frac{n}{l} > \lambda_2 \\ \frac{A_1}{B_1} > \frac{n}{m} \end{array}\right\} \\
\qquad\text{or}\quad \left.\begin{array}{l} \lambda_2 > \frac{n}{l} \\ \frac{n}{m} > \frac{A_1}{B_1} \end{array}\right\} \\
\qquad\text{or}\quad \frac{n}{l} > \lambda_3 \\
\text{or}\quad \nu_3 > \lambda_3 > \nu_2 > \lambda_2 > \nu_1 \\
\quad\text{and, either}\quad \frac{n}{l} > \lambda_2 \\
\qquad\text{or}\quad \frac{n}{m} > \frac{A_1}{B_1} \\
\text{or}\quad \lambda_3 > \nu_3 > \nu_2 > \lambda_2 > \nu_1 \\
\quad\text{and, either}\quad \frac{n}{l} > \lambda_2 \\
\qquad\text{or}\quad \frac{n}{m} > \frac{A_1}{B_1} \\
\text{or}\quad \lambda_3 > \nu_3 > \lambda_2 > \nu_2 > \nu_1 \\
\quad\text{and, either}\quad \frac{n}{l} > \lambda_2 \\
\qquad\text{or}\quad \frac{n}{m} > \frac{A_1}{B_1}
\end{array}\right] \quad \ldots \quad \ldots \quad \ldots \quad (37c)
$$

or

$$
\left.\begin{array}{l}
A_1 < 0 \\
A_2 > 0 \\
A_3 < 0 \\
\Delta_1 < 0
\end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (37d)
$$

and, either $\nu_1 > \lambda_3$

and, either $\lambda_2 > \frac{n}{l} > \lambda_1$

or $\frac{n}{l} > \lambda_3$

or $\frac{n}{l} > \lambda_1$

and, either

$\nu_2 > \lambda_3 > \lambda_2 > \lambda_1 > \nu_1$

or $\nu_2 > \lambda_3 > \lambda_2 > \nu_1 > \lambda_1$

or $\nu_3 > \lambda_3 > \nu_2 > \lambda_2 > \lambda_1 > \nu_1$

or $\lambda_3 > \nu_3 > \nu_2 > \lambda_2 > \lambda_1 > \nu_1$

or $\nu_3 > \lambda_3 > \lambda_2 > \nu_2 > \nu_1 > \lambda_1$

or $\nu_3 > \lambda_3 > \lambda_2 > \nu_2 > \lambda_1 > \nu_1$

or $\lambda_3 > \lambda_2 > \nu_3 > \nu_2 > \nu_1 > \lambda_1$

or $\lambda_3 > \lambda_2 > \nu_3 > \nu_2 > \lambda_1 > \nu_1$

or $\lambda_3 > \lambda_2 > \nu_3 > \lambda_1 > \nu_2 > \nu_1$

or $\nu_3 > \lambda_3 > \nu_2 > \lambda_2 > \nu_1 > \lambda_1$

or $\lambda_3 > \nu_3 > \nu_2 > \lambda_2 > \nu_1 > \lambda_1$

or $\lambda_3 > \nu_3 > \lambda_2 > \nu_2 > \nu_1 > \lambda_1$ .. .. (37*d*)

or $\lambda_3 > \nu_3 > \lambda_2 > \nu_2 > \lambda_1 > \nu_1$

or $\nu_3 > \lambda_3 > \lambda_2 > \lambda_1 > \nu_2$

and, either $\lambda_2 > \frac{n}{l} > \lambda_1$

or $\frac{n}{l} > \lambda_3$

or $\frac{n}{l} > \lambda_1 > \nu_3$

or $\nu_2 > \lambda_3 > \nu_1 > \lambda_2 > \lambda_1$

and, either $\lambda_2 > \frac{n}{l} > \lambda_1$

or $\frac{n}{l} > \lambda_3$

or $\nu_3 > \lambda_3 > \nu_2 > \nu_1 > \lambda_2 > \lambda_1$

and, either $\lambda_2 > \frac{n}{l} > \lambda_1$

or $\frac{n}{l} > \lambda_3$

$$\left.\begin{array}{l}
\text{or} \quad \lambda_3 > \nu_3 > \nu_2 > \nu_1 > \lambda_2 > \lambda_1 \\
\quad \text{and, either} \quad \lambda_2 > \frac{n}{l} > \lambda_1 \\
\quad\quad \text{or} \quad \frac{n}{l} > \lambda_3 \\
\text{or} \quad \lambda_3 > \nu_3 > \lambda_2 > \lambda_1 > \nu_2 > \nu_1 \\
\quad \text{and, either} \quad \lambda_2 > \frac{n}{l} > \lambda_1 \\
\quad\quad \text{or} \quad \frac{n}{l} > \lambda_3
\end{array}\right] \quad \ldots \quad \ldots (37d)$$

where

$$\left.\begin{array}{l}
\Delta_1 = 4p_1^3+27q_1^2;\ \Delta_2 = 4p_2^3+27q_2^2;\ p_1 = \frac{A_2}{A_4} - \frac{1}{3}\left(\frac{A_3}{A_4}\right)^2; \\
q_1 = \frac{A_1}{A_4} - \frac{A_2A_3}{3A_4^2} + \frac{2}{27}\left(\frac{A_3}{A_4}\right)^3;\ p_2 = -\frac{B_2}{B_4} - \frac{1}{3}\left(\frac{B_3}{B_4}\right)^2; \\
q_2 = -\frac{B_1}{B_4} - \frac{1}{3}\frac{B_2B_3}{B_4^2} - \frac{2}{27}\left(\frac{B_3}{B_4}\right)^3;\ \lambda_0 = \sqrt[3]{k_1} + \sqrt[3]{k_2} - \frac{1}{3}\frac{A_3}{A_4}. \\
\begin{array}{l} (\lambda_3, \lambda_2, \lambda_1) = \\ \lambda_3 > \lambda_2 > \lambda_1 \end{array} \left\{\begin{array}{l}
2\sqrt{\frac{-p_1}{3}}\cos\frac{\phi_1}{3} - \frac{1}{3}\frac{A_3}{A_4}. \\
-2\sqrt{\frac{-p_1}{3}}\cos\left(\frac{\pi}{3} - \frac{\phi_1}{3}\right) - \frac{1}{3}\cdot\frac{A_3}{A_4}. \\
-2\sqrt{\frac{-p_1}{3}}\cos\left(\frac{\pi}{3} + \frac{\phi_1}{3}\right) - \frac{1}{3}\frac{A_3}{A_4}.
\end{array}\right. \\
\nu_0 = \sqrt[3]{k_3} + \sqrt[3]{k_4} + \frac{1}{3}\cdot\frac{B_3}{B_4} \\
\begin{array}{l} (\nu_3, \nu_2, \nu_1) = \\ \nu_3 > \nu_2 > \nu_1 \end{array} \left\{\begin{array}{l}
2\sqrt{\frac{-p_2}{3}}\cos\frac{\phi_2}{3} + \frac{1}{3}\frac{B_3}{B_4} \\
-2\sqrt{\frac{-p_2}{3}}\cos\left(\frac{\pi}{3} - \frac{\phi_2}{3}\right) + \frac{1}{3}\cdot\frac{B_3}{B_4}. \\
-2\sqrt{\frac{-p_2}{3}}\cos\left(\frac{\pi}{3} + \frac{\phi_2}{3}\right) + \frac{1}{3}\cdot\frac{B_3}{B_4}.
\end{array}\right. \\
\phi_1 = \tan^{-1}\left[-\frac{\sqrt{-\Delta_1}}{q_1\sqrt{27}}\right]. \\
\phi_2 = \tan^{-1}\left[-\frac{\sqrt{-\Delta_2}}{q_2\sqrt{27}}\right]. \\
k_1 = -\frac{q_1}{2} + \sqrt{\frac{\Delta_1}{108}};\quad k_2 = -\frac{q_1}{2} - \sqrt{\frac{\Delta_1}{108}};\quad k_3 = -\frac{q_2}{2} + \sqrt{\frac{\Delta_2}{108}}; \\
k_4 = -\frac{q_2}{2} - \sqrt{\frac{\Delta_2}{108}};\quad \nu = \nu_0, \text{ when } \Delta_2 > 0;\quad \nu = \nu_3, \text{ when } \Delta_2 < 0
\end{array}\right] \quad \ldots (38)$$

Now, corresponding to positive real β, γ a positive real α and also T_8 exist, provided that

either

$$\left.\begin{array}{l} P_1 > 0 \\ P_2 > 0 \\ \Delta_3 > 0 \end{array}\right\} \text{ and, either } \left.\begin{array}{l} q > \frac{1}{\beta_0} > \mu_1 > \mu_2 \\ \text{or } q > \mu_1 > \mu_2 > \frac{1}{\beta_0} \\ \text{or } \mu_1 > \frac{1}{\beta_0} > q > \mu_2 \\ \text{or } \mu_1 > q > \mu_2 > \frac{1}{\beta_0} \\ \text{or } \mu_1 > \frac{1}{\beta_0} > \mu_2 > q \\ \text{or } \mu_1 > \mu_2 > q > \frac{1}{\beta_0} \end{array}\right\} \quad \ldots \quad (39)$$

or

$$\left.\begin{array}{l} P_1 > 0 \\ q > \frac{1}{\beta_0} \end{array}\right\} \text{ and, either } \left.\begin{array}{l} P_2 > 0 \\ \Delta_3 < 0 \end{array}\right\} \text{ or } P_2 < 0 \quad \ldots \quad (40)$$

or

$$P_1 < 0 \text{ and, either } q > \frac{1}{\beta_0} > \mu_1 \text{ or } \mu_1 > \frac{1}{\beta_0} > q \quad \ldots \quad (41)$$

where

$$\left.\begin{array}{l} P_1 = m-3b_0n\,;\ P_2 = b_0(2m+4n-3l)\,;\ Q_1 = m+2n-2l\,; \\ \Delta_3 = P_2^2-16P_1b_0l\,;\ q = \dfrac{Q_1+\sqrt{Q_1^2+16ln}}{4l}\,;\ \mu_1 = \dfrac{P_2+\sqrt{\Delta_3}}{8b_0l}\,; \\ \mu_2 = \dfrac{P_2-\sqrt{\Delta_3}}{8b_0l}\,;\ \dfrac{1}{\beta_0} = \text{positive real root of equation (33)} \end{array}\right\} \quad \ldots \quad (42)$$

The proper procedure for design would be to first check and see if the inequalities of expressions (32) and (35) are satisfied. Thereafter, compute the values of A's, B's, etc., with the aid of equations (34) and (38) and see if either one of the two cases mentioned above is applicable and that any one set of conditions listed therein is satisfied. Then solve the quartic of equation (33) and see if any one of its positive real root(s), $\frac{1}{\beta_0}$, satisfies any one set of conditions listed in any one of the equations (39) through (41). The satisfaction of these conditions signifies that the circuit of Fig. 1(*b*) for the simulation of the system of equations (1) is physically realizable. The circuit component values may be then obtained with the aid of equations (31*a*), (32*b*), (33), (2.11), (2.12) and (28). Choosing arbitrarily a convenient value for any one of the capacitors the values of the resistors and remaining capacitors may be obtained with the aid of equation (30).

References

Uspensky, J. V. (1948). Theory of Equations. McGraw-Hill, New York, 82–96.

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Appendix I

Conditions for Positive Real Roots

Simulation of the system represented by equation (1) with the network of Fig. 1(*a*) is possible only if the values of α, β, γ, T_3, T_5, T_7 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta\gamma}{k} \quad \ldots \quad (1.1)$$

$$b_1 = T_3 + T_5 \quad \ldots \quad (1.2)$$

$$b_2 = T_3 T_5 \quad \ldots \quad (1.3)$$

$$a_1 = \frac{(\alpha+2\alpha\beta+2\beta+\beta\gamma)}{k} T_3 + \frac{\beta(2\alpha+\gamma+2)}{k} T_5 + \frac{2\beta(\alpha+1)(\gamma+1)}{k} T_7 \quad \ldots \quad (1.4)$$

$$a_2 = \frac{\beta(\alpha+\gamma)}{k} T_3 T_5 + \frac{\alpha\beta(\gamma+1)}{k} T_3 T_7 + \frac{2\beta\gamma(\alpha+1)}{k} T_5 T_7 \quad \ldots \quad (1.5)$$

$$a_3 = \frac{\alpha\beta\gamma}{k} T_3 T_5 T_7 \quad \ldots \quad (1.6)$$

where

$$k = (2\alpha+3\alpha\beta+4\beta+\beta\gamma+2) \quad \ldots \quad (1.7)$$

are positive and real; where a's and b's are non-zero positive real constants.

The solution of equations (1.2) and (1.3) gives

$$T_3 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (1.8a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (1.8b)$$

from which it is evident that T_3 and T_5 will be positive real, provided that

$$b_1^2 > 4b_2 \quad \ldots \quad (1.9)$$

Elimination of β, T_7 from equations (1.1), (1.2), (1.3), (1.4), (1.6), (1.7) and (1.1), (1.2), (1.3), (1.5), (1.6), and (1.7) gives the following two equations:

$$\frac{1}{\gamma}=\frac{\frac{-2A_1}{\alpha^2}+\frac{A_2}{\alpha}-A_3}{\frac{4B_1}{\alpha^2}+\frac{4B_2}{\alpha}+B_3} \quad \cdots \quad (1.10)$$

$$\frac{l}{\alpha}+\frac{m}{\gamma}=n. \quad \cdots \quad (1.11)$$

where A's, B's, l, m, n are all real constants and their values in terms of the known a's and b's are given in equation (14).

Now, the function of equation (1.10) has a pair of zeros (λ_1, λ_2) given by the roots of

$$\frac{2A_1}{\alpha^2}-\frac{A_2}{\alpha}+A_3=0 \quad \cdots \quad (1.12)$$

both of which will be positive real, if

$$\left.\begin{aligned} A_2 &> 0 \\ A_3 &> 0 \\ \Delta_1=(A_2^2-8A_1A_3) &> 0 \end{aligned}\right\} \quad \cdots \quad (1.13)$$

and one of which will be positive and the other negative real, if

$$\left.\begin{aligned} A_2 &\gtrless 0 \\ A_3 &< 0 \end{aligned}\right\} \quad \cdots \quad (1.14)$$

as A_1 is already positive real according to equation (14). Since B_1, B_2 and B_3 are all positive real, therefore the poles of the function of equation (1.10) will be both either negative real or complex conjugate depending on whether the discriminant of the equation

$$\frac{4B_1}{\alpha^2}+\frac{4B_2}{\alpha}+B_3=0 \quad \cdots \quad (1.15)$$

is positive or negative.

The straight line of equation (1.11) will cut the $\frac{1}{\alpha}$ and $\frac{1}{\gamma}$ axes at the points A and B respectively, such that

$$\left.\begin{aligned} OA &= p_1=\frac{n}{l} \\ OB &= \frac{n}{m} \end{aligned}\right\} \quad \cdots \quad (1.16)$$

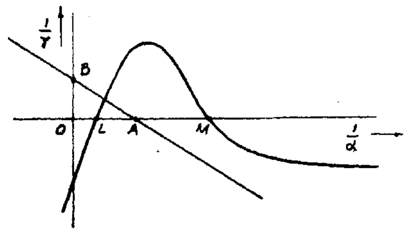
and these intercepts will be positive, provided that

$$n=\left(a_2-\frac{a_3b_1}{b_2}-\frac{a_3T_5}{b_2}\right)>0 \quad \cdots \quad (1.17)$$

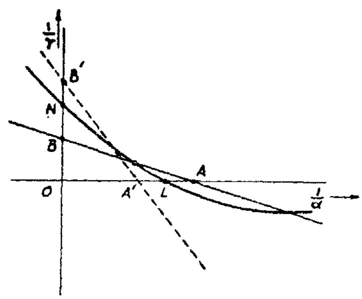
as l, m are both positive.

Therefore, if the conditions of expressions (1.9), (1.17) and either (1.13) or (1.14) are satisfied then it is possible for a portion of the curve of equation (1.10) and the straight line of equation (1.11) to exist in the first quadrant of the $\frac{1}{\alpha} - \frac{1}{\gamma}$ plane and the sketches of these functions for positive α are shown in Fig. 1.1.

FIG. 1.1. Sketches of the curves $\frac{1}{\gamma} = \dfrac{\frac{-2A_1}{\alpha^2} + \frac{A^2}{\alpha} - A_3}{\frac{4B_1}{\alpha^2} + \frac{4B_2}{\alpha} + B_3}$ for positive α.



(a) $A_2 > 0$, $A_3 > 0$, $\Delta_1 > 0$
$OM > OA > OL$



(b) $A_2 \gtrless 0$, $A_3 < 0$
either $OA > OL$, $ON > OB$
or $OL > OA'$, $OB' > ON$.

It is possible for the curve and the straight line to intersect each other at a point in the first-quadrant yielding a set of positive real α, γ provided that the conditions of expressions (1.9), (1.17), and either (16) or (17) are satisfied.

Elimination of γ from equations (1.1) and (1.11) gives

$$\frac{1}{\beta} = \frac{\frac{Q_1}{\alpha^2} - \frac{Q_2}{\alpha} + Q_3}{2b_0\left(\frac{1}{\alpha}+1\right)\left(n - \frac{l}{\alpha}\right)} \qquad \ldots \quad \ldots \quad \ldots (1.18)$$

which has a pair of positive real zeros, if

$$\left.\begin{aligned} Q_2 &> 0 \\ Q_3 &> 0 \\ \Delta_2 = (Q_2^2 - 4Q_1Q_3) &> 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots (1.19)$$

or one positive and one negative real zeros, if

$$\left.\begin{aligned} Q_2 &\gtrless 0 \\ Q_3 &< 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.20)$$

or a pair of complex conjugate zeros, if

$$\left.\begin{aligned} Q_2 &> 0 \\ Q_3 &> 0 \\ \Delta_2 &< 0. \end{aligned}\right\}$$

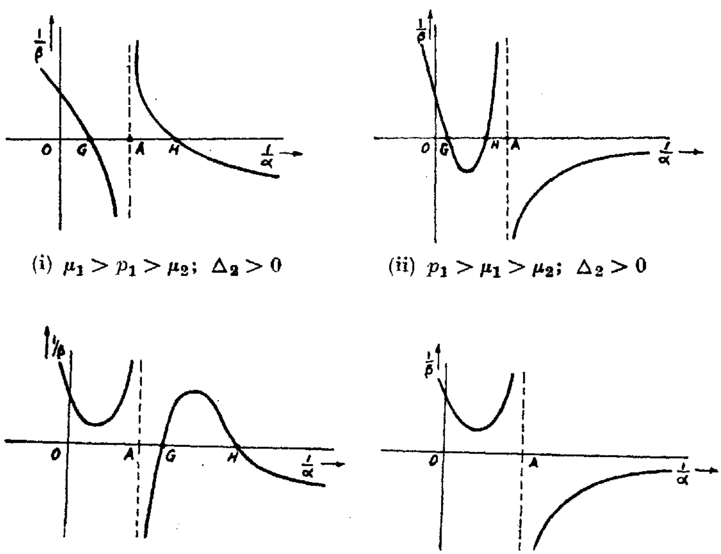
The function of equation (1.18) has a pair of real poles at

$$\frac{1}{\alpha'} = -1$$

$$\frac{1}{\alpha''} = OA = p_1 = \frac{n}{l}$$

and the function for positive α is sketched in Fig. 1.2. It will be obvious from the sketches of Fig. 1.2 that if any one set of conditions in any one of the expressions (18) through (20) is satisfied then corresponding to positive real α, γ a positive real β and also T_7 exist.

FIG. 1.2. Sketches of the curve $\dfrac{1}{\beta} = \dfrac{\dfrac{Q_1}{\alpha^2} - \dfrac{Q_2}{\alpha} + Q_3}{2b_0\left(\dfrac{1}{\alpha}+1\right)\left(n - \dfrac{l}{\alpha}\right)}$ for positive α.



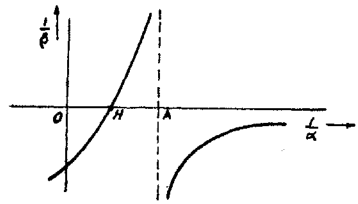
(i) $\mu_1 > p_1 > \mu_2$; $\Delta_2 > 0$

(ii) $p_1 > \mu_1 > \mu_2$; $\Delta_2 > 0$

(iii) $\mu_1 > \mu_2 > p_1$; $\Delta_2 > 0$

(iv) $\Delta_2 < 0$

(a) $Q_2 > 0$; $Q_3 < 0$



(b) $Q_2 \gtrless 0,\ Q_3 < 0;\ p_1 > \mu_1$

To summarize, if the inequalities of expressions (1.9), (1.17) together with a set of conditions from either expression (16) or (17) and also any one set of conditions from any one of the expressions (18) through (20) are satisfied then a set of positive real α, β, γ, T_3, T_5, T_7 exists and the circuit of Fig. 1(a) for simulating the system of equation (1) is physically realizable.

APPENDIX II

CONDITIONS FOR POSITIVE REAL ROOTS

It is possible to simulate the system represented by equation (1) with the network of Fig. 1(b) only if the values of α, β, γ, T_3, T_5 and T_8 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta\gamma}{k} \quad \ldots \quad (2.1)$$

$$b_1 = T_3 + T_5 \quad \ldots \quad (2.2)$$

$$b_2 = T_3 T_5 \quad \ldots \quad (2.3)$$

$$a_1 = \frac{(2\alpha+2\alpha\beta+\alpha\gamma+\beta)}{k} T_3 + \frac{(2\alpha\beta+2\alpha\gamma+\beta+\gamma)}{k} T_5 + \frac{\alpha(2\beta+\beta\gamma+\gamma+2)}{k} T_8 \quad (2.4)$$

$$a_2 = \frac{\alpha(\beta+\gamma)}{k} T_3 T_5 + \frac{\alpha\beta(\gamma+1)}{k} T_3 T_8 + \frac{\alpha(\beta+\beta\gamma+\gamma)}{k} T_5 T_8 \quad \ldots \quad (2.5)$$

$$a_3 = \frac{\alpha\beta\gamma}{k} T_3 T_5 T_8 \quad \ldots \quad (2.6)$$

where

$$k = (4\alpha+3\alpha\beta+2\alpha\gamma+2\beta+\gamma+2) \quad \ldots \quad (2.7)$$

are positive real.

The solution of equations (2.2) and (2.3) gives

$$T_3 = \tfrac{1}{2}\left[b_1 \pm \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (2.8a)$$

$$T_5 = \tfrac{1}{2}\left[b_1 \mp \sqrt{b_1^2 - 4b_2}\right] \quad \ldots \quad (2.8b)$$

from which it is evident that T_3 and T_5 will be positive real, if

$$b_1^2 > 4b_2 \quad \ldots \quad (2.9)$$

Elimination of α, T_8 from equations (2.1), (2.2), (2.3), (2.4), (2.6), (2.7) and (2.1), (2.2), (2.3), (2.5), (2.6), (2.7) give the following two equations

$$\frac{1}{\gamma}=\frac{\frac{A_4}{\beta^3}+\frac{A_3}{\beta^2}+\frac{A_2}{\beta}+A_1}{\frac{-B_4}{\beta^3}+\frac{B_3}{\beta^2}+\frac{B_2}{\beta}+B_1} \qquad \ldots \quad (2.10)$$

$$\frac{l}{\beta}+\frac{m}{\gamma}=n \qquad \ldots \quad (2.11)$$

where A's, B's, l, m, n are all real constants and their values in terms of the known a's and b's are given in equation (34).

Now, the function of equation (2.10) has either no positive real zeros, if

either $\left.\begin{array}{l} A_1>0 \\ A_2>0 \\ A_3>0 \end{array}\right\}$

or $\left.\begin{array}{ll} & \left.\begin{array}{l} A_1>0 \\ \Delta_1>0 \end{array}\right\} \\ \text{and, either} & A_3<0 \\ \text{or} & \left.\begin{array}{l} A_2<0 \\ A_3>0 \end{array}\right\} \end{array}\right\}$

or has one positive real zero, if

$$\left.\begin{array}{ll} & A_1<0 \\ \text{and, either} & \left.\begin{array}{l} A_2>0 \\ A_3<0 \\ \Delta_1>0 \end{array}\right\} \\ \text{or} & A_3>0 \\ \text{or} & \left.\begin{array}{l} A_2<0 \\ A_3<0 \end{array}\right\} \end{array}\right\}$$

or has two positive real zeros, if

$$\left.\begin{array}{ll} & \left.\begin{array}{l} A_1>0 \\ \Delta_1<0 \end{array}\right\} \\ \text{and, either} & A_3<0 \\ \text{or} & \left.\begin{array}{l} A_2<0 \\ A_3>0 \end{array}\right\} \end{array}\right\}$$

or has three positive real zeros, if

$$\left.\begin{array}{l} A_1<0 \\ A_2>0 \\ A_3<0 \\ \Delta_1<0 \end{array}\right\}$$

and has either one positive real pole, if

either $$\left.\begin{array}{l} B_2 < 0 \\ B_3 > 0 \\ \Delta_2 > 0 \end{array}\right\}$$

or $$B_3 < 0$$

or $$\left.\begin{array}{l} B_2 > 0 \\ B_3 > 0 \end{array}\right\}$$

or has three positive real poles, if

$$\left.\begin{array}{l} B_2 < 0 \\ B_3 > 0 \\ \Delta_2 < 0 \end{array}\right\}$$

since A_4, B_1 and B_4, as seen from equation (34), are positive real constants.

In view of what has been stated above it should be simple enough to sketch the nature of the function of equation (2.10) and therefore these will not be shown here. With the aid of these sketches it can be seen that the curve of equation (2.10) and the straight line of equation (2.11) will intersect each other at a point in the first quadrant of the $\frac{1}{\beta} - \frac{1}{\gamma}$ plane if any one set of conditions listed in either expressions (36) or (37) together with the inequalities of expressions (32) and (35) is satisfied.

Elimination of γ from equations (2.1) and (2.11) gives

$$\frac{1}{\alpha} = \frac{\frac{4b_0 l}{\beta^2} - \frac{P_2}{\beta} + P_1}{b_0\left(\frac{-2l}{\beta^2} + \frac{Q_1}{\beta} + 2n\right)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

which has one positive real pole and either two positive real zeros, if

$$\left.\begin{array}{l} P_1 > 0 \\ P_2 > 0 \\ \Delta_3 \geqslant 0 \end{array}\right\}$$

or no positive real zeros, if

either $$\left.\begin{array}{l} P_1 > 0 \\ P_2 > 0 \\ \Delta_3 < 0 \end{array}\right\}$$

or $$\left.\begin{array}{l} P_1 > 0 \\ P_2 < 0 \end{array}\right\}$$

or one positive and one negative real zeros, if

$$\left.\begin{array}{l} P_1 < 0 \\ P_2 \gtrless 0 \end{array}\right\}$$

It should be easy enough, in view of the above statements, to sketch the function of equation (2.12) and it can be seen that if any one set of conditions listed in any one of the expressions (39) through (42) is satisfied then corresponding to positive real T_3, T_5, β, γ a positive real α and also T_8 exist.

To summarize, if the inequalities of expressions (32) and (35) together with a set of conditions from either expressions (36) or (37) and a set of conditions from any one of the three expressions (39) through (41) are satisfied then a set of positive real α, β, γ, T_3, T_5, T_8 exists for which the circuit of Fig. 1(b) is physically realizable.