THE SIMULATION OF THIRD ORDER SYSTEMS WITH A LEADING TIME-CONSTANT BY A SINGLE OPERATIONAL AMPLIFIER—I

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A method for the simulation of third order linear systems with a single operational amplifier is outlined. A particular case of the general third order systems, that is systems with a leading time-constant, is considered. Ten basic circuits are possible—three of which, each consisting of one operational amplifier, three capacitors and six resistors, are presented here and the remaining seven will be presented in Parts II, III and IV of this paper. In these parts, circuits have been analysed and the conditions of physical realizability discussed and obtained. The design formulae and procedure have also been given.

INTRODUCTION

In previous communications (Wadhwa 1962*a*, *b*, *c*, 1963) on this general topic three particular classes of the general third order linear systems were considered for simulation by a single operational amplifier. The purpose of this paper is to consider yet another particular class of the general third order systems, that is systems with a leading time-constant which are characterized by a transfer function of the form

$$F(S) = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1} \quad \ldots \quad (1)$$

A block diagram of a network capable of simulating third order systems is shown in Fig. 1; and it should be obvious, by inspection, that ten basic circuits, each employing three capacitors and six resistors, are possible. For convenience and other practical considerations the presentation and discussion of these basic circuits will be divided into four parts. In this paper, which is Part I of the series, only three basic circuits will be presented and discussed.

THIRD ORDER SYSTEM SIMULATION

A network for the simulation of third order systems is shown in Fig. 1 and its transfer function has been shown (Wadhwa 1963) to be

$$\frac{E_0}{E_1} = -\frac{Y_1Y_3Y_5}{Y_6(Y_1+Y_2+Y_8)(Y_3+Y_4+Y_5+Y_7)+Y_3Y_6(Y_4+Y_5+Y_7)+ +Y_5Y_7(Y_1+Y_2+Y_3+Y_8)+Y_3Y_5Y_8} \quad \ldots \quad (2)$$

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Simulation of the system of (1) with the network of Fig. 1 is possible if the admittances (Y's) are properly chosen. Furthermore, it should be obvious from (2) that at least three of appropriate admittances (Y's) will be required to be capacitative. An inspection of Fig. 1 will indicate that ten basic circuits each employing three capacitors and six resistors are possible.

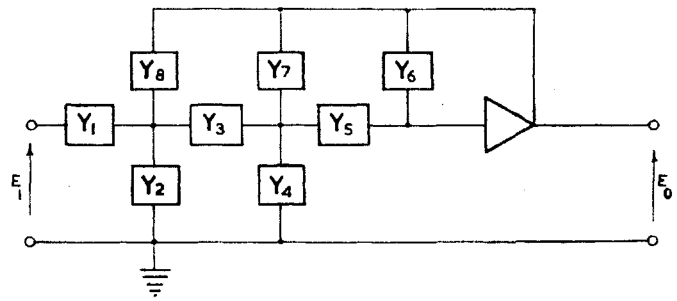

FIG. 1. Network for the simulation of third order systems.

Each basic circuit has nine parameters and as the number of constants in the system represented by (1) is five, the choice of resistor values is required to be made so as to reduce the number of circuit parameters to the number of constants in (1). This requirement offers considerable latitude in the choice of the intended design values of the resistors as a result of which a number of circuits are possible which are essentially a variation of the basic circuit employing three capacitors and six resistors. Each circuit will require, for its physical realizability, satisfaction of a set of conditions which will be different for every circuit.

Three basic circuits, each with certain arbitrary choice of resistor values, are shown in Fig. 2.

(i) *Y_1, Y_4 and Y_6 capacitative*

A possible circuit for simulating the system of (1) is shown in Fig. 2(a), in which

$$\left.\begin{aligned} Y_1 &= \left(SC_1+\frac{1}{R}\right) \\ Y_4 &= SC_4 \\ Y_6 &= SC_6 \\ Y_3 = Y_5 &= \frac{1}{R} \\ Y_2 = Y_8 &= \frac{1}{\alpha R} \\ Y_7 &= \frac{1}{\beta R} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Substituting (3) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\left(\frac{\alpha\beta}{2\alpha+\beta+2}\right)(RC_1S+1)}{\left(\frac{\alpha\beta}{2\alpha+\beta+2}\right)R^3C_1C_4C_6S^3+\left[\frac{\alpha(2\beta+1)}{(2\alpha+\beta+2)}R^2C_1C_6+\frac{2\beta(\alpha+1)}{(2\alpha+\beta+2)}R^2C_4C_6\right]S^2+ +\left[\frac{\alpha}{(2\alpha+\beta+2)}RC_1+\frac{(2\alpha+3\alpha\beta+4\beta+2)}{(2\alpha+\beta+2)}RC_6\right]S+1} \quad .. \quad (4)$$

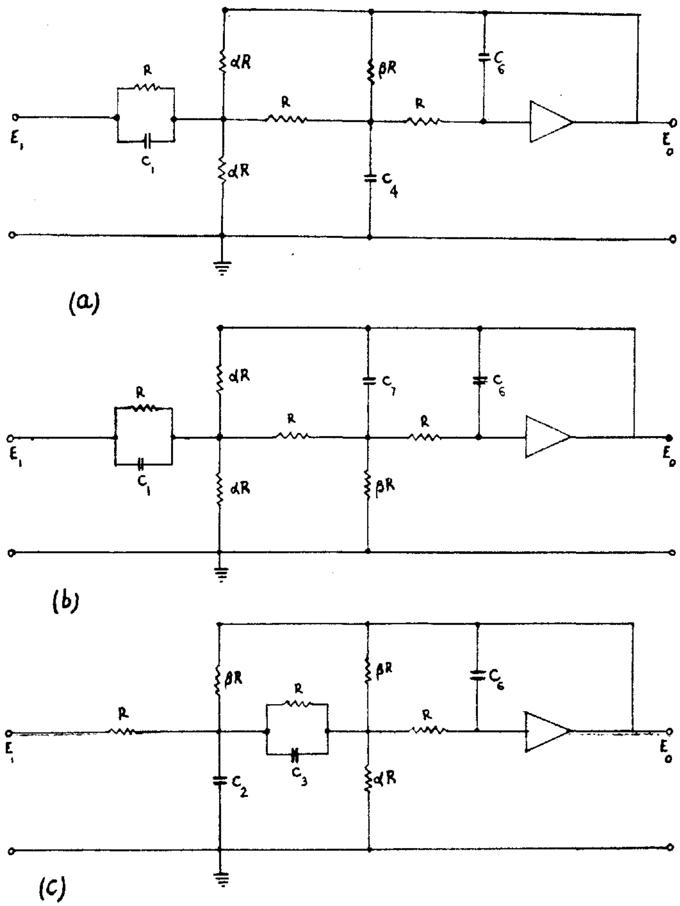

FIG. 2. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}$.

(1) and (4) will be identical, if

$$b_0 = \frac{\alpha\beta}{(2\alpha+\beta+2)} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

$$b_1 = T_1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (6)$$

$$a_1 = \frac{\alpha}{(2\alpha+\beta+2)}T_1+\frac{(2\alpha+3\alpha\beta+4\beta+2)}{(2\alpha+\beta+2)}T_6 \quad .. \quad .. \quad (7)$$

$$a_2 = \frac{\alpha(2\beta+1)}{(2\alpha+\beta+2)} T_1 T_6 + \frac{2\beta(\alpha+1)}{(2\alpha+\beta+2)} T_4 T_6 \quad \quad (8)$$

$$a_3 = \frac{\alpha\beta}{(2\alpha+\beta+2)} T_1 T_4 T_6 \quad \quad (9)$$

where

$$T_n = RC_n \quad \quad (10)$$

Now, simulation of the system of (1) with the network of Fig. 2(a) is possible only if the values of α, β, T_1, T_4 and T_6 obtained as the solution of (5) to (9) are positive and real. It is required to determine, therefore, in terms of the given positive and real a's and b's, the values of α, β, T_1, T_4 and T_6 and find the conditions, if any, under which these can be positive and real.

Elimination of β, T_1, T_4 and T_6 from (5) to (9) gives a quartic

$$A_1\alpha^4 + A_2\alpha^3 + A_3\alpha^2 - A_4\alpha + 24a_3b_0 = 0 \quad \quad (11)$$

where

$$\left.\begin{aligned}
A_1 &= b_1^2(1+4b_0)(2a_1-b_1) - 4(1+3b_0)(a_2b_1-2a_3) \\
A_2 &= 2b_1^2(a_1+7a_1b_0+2b_0^2b_1-b_0b_1) - 4\{2(a_2b_1-3a_3)+3b_0(3a_2b_1-8a_3)\} \\
A_3 &= 3b_0b_1^2(2a_1+b_0b_1) - 4\{(a_2b_1-2a_3)(1+9b_0)-2a_3(2+9b_0)\} \\
A_4 &= 4\{3b_0(a_2b_1-2a_3)-2a_3(1+9b_0)\}
\end{aligned}\right\} \quad (12)$$

which will have at least one positive and real root corresponding to which, as shown in Appendix I, a set of positive real β, T_1, T_4 and T_6 exists, provided that

$$\left.\begin{aligned} \alpha &> b_0 \\ (a_2b_1-2a_3) &> 0 \end{aligned}\right\} \quad \quad (13)$$

and, either

$$\left.\begin{aligned} (2a_1-b_1) &> 0 \\ \frac{2(a_2b_1-2a_3)}{b_1^2(1+4b_0)} &> \frac{(2a_1-b_1)}{2(1+3b_0)} \end{aligned}\right\} \quad \quad (14)$$

or

$$\left.\begin{aligned} (2a_1-b_1) &< 0 \\ \left(\frac{2a_1+b_0b_1}{b_1-2a_1}\right) &> \left(\frac{2a_3}{a_2b_1-2a_3}\right) \end{aligned}\right\} \quad \quad (14a)$$

For the design of the network, circuit component values are required to be determined. The proper procedure for design would be first to solve the quartic of (11) by methods that are well known and discussed at length in textbooks (Uspensky 1948) on higher algebra and obtain α. The next step would be to check and see if the inequalities of (13) and either (14) or (14a) are satisfied. Satisfaction of these conditions signifies the existence of at least one positive real set of α, β, T_1, T_4 and T_6 for which the circuit of Fig. 2(a) is physically realizable. The solution of (11) will give α, substitution of the positive real value(s) of α into (5) will give β; T_1, T_6 and T_4 may then be conveniently obtained from (6), (7) and (8). Having thus determined α, β, T_1, T_4 and T_6 and choosing arbitrarily a convenient value for any one of the capacitors the values of the remaining capacitors and resistors may then be obtained with the aid of (10) and (3).

(ii) Y_1, Y_6 *and* Y_7 *capacitative*

Another basic circuit in which

$$\left.\begin{aligned} Y_1 &= \left(SC_1+\frac{1}{R}\right) \\ Y_6 &= SC_6 \\ Y_7 &= SC_7 \\ Y_3 = Y_5 &= \frac{1}{R} \\ Y_2 = Y_8 &= \frac{1}{\alpha R} \\ Y_4 &= \frac{1}{\beta R} \end{aligned}\right\} \qquad (15)$$

is shown in Fig. 2(*b*).

Substituting (15) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\alpha(RC_1S+1)}{\alpha R^3C_1C_6C_7S^3+\left[\frac{\alpha(2\beta+1)}{\beta}R^2C_1C_6+\alpha R^2C_1C_7+2(\alpha+1)R^2C_6C_7\right]S^2+ +\left[\frac{(2\alpha+3\alpha\beta+4\beta+2)}{\beta}RC_6+2(\alpha+1)RC_7\right]S+1} \qquad (16)$$

(1) and (16) will be identical if

$$b_0 = \alpha \qquad (17)$$

$$b_1 = T_1 \qquad (18)$$

$$a_1 = \frac{(2\alpha+3\alpha\beta+4\beta+2)}{\beta}T_6+2(\alpha+1)T_7 \qquad (19)$$

$$a_2 = \frac{\alpha(2\beta+1)}{\beta}T_1T_6+\alpha T_1T_7+2(\alpha+1)T_6T_7 \qquad (20)$$

$$a_3 = \alpha T_1T_6T_7 \qquad (21)$$

where

$$T_n = RC_n \qquad (22)$$

Now, α and T_1 are known directly from (17) and (18) to be positive and real and the solution of (17) to (21) gives

$$T_6 = \frac{K}{b_0^3b_1^2} \qquad (23)$$

$$T_7 = \frac{a_3 b_0^2 b_1}{K} \quad \cdots \quad (24)$$

$$\beta = \frac{2(1+b_0)K^2}{a_1 b_0^3 b_1^2 K - (4+3b_0)K^2 - 2a_3 b_0^5 b_1^3 (1+b_0)} \quad \cdots \quad (25)$$

where

$$K = 2a_2 b_0 b_1 (1+b_0) - \left\{ a_1 b_0^2 b_1^2 + 4a_3 (1+b_0)^2 \right\}$$

from which it is evident that T_6, T_7 and β are also real, and will be positive, if

$$\left.\begin{aligned} &K > 0 \\ \text{and} \quad &\left[a_1 b_0^3 b_1^2 K - (4+3b_0)K^2 - 2a_3 b_0^5 b_1^3 (1+b_0)\right] > 0 \end{aligned}\right\} \quad \cdots \quad (26)$$

Therefore, if the inequalities of (26) are satisfied then a set of positive real α, β, T_1, T_6 and T_7 exists for which the circuit of Fig. 2(*b*) is physically realizable and the circuit component values may be determined with the aid of (17), (18), (23), (24), (25), (22) and (15).

(iii) *Y_2, Y_3 and Y_6 capacitative*

In Fig. 2(*c*) is shown a third basic circuit with

$$\left.\begin{aligned} Y_2 &= SC_2 \\ Y_3 &= \left(SC_3 + \frac{1}{R}\right) \\ Y_6 &= SC_6 \\ Y_1 = Y_5 &= \frac{1}{R} \\ Y_4 &= \frac{1}{\alpha R} \\ Y_7 = Y_8 &= \frac{1}{\beta R} \end{aligned}\right\} \quad \cdots \quad (27)$$

Substituting (27) into (2) and simplifying

$$\frac{E_0}{E_1} = - \frac{\frac{\beta^2}{(3\beta+1)}(RC_3 S+1)}{\frac{\beta^2}{(3\beta+1)} R^3 C_2 C_3 C_6 S^3 + \left[\frac{\beta(\alpha+2\alpha\beta+\beta)}{\alpha(3\beta+1)} R^2 C_2 C_6 + \frac{\beta(2\alpha+2\alpha\beta+\beta)}{\alpha(3\beta+1)} R^2 C_3 C_6\right] S^2 + \left[\frac{\beta}{(3\beta+1)} RC_2 + \frac{2\beta}{(3\beta+1)} RC_3 + \frac{(\beta+1)(\alpha+2\alpha\beta+\beta)+\beta(\alpha+\alpha\beta+\beta)}{\alpha(3\beta+1)} RC_6\right] S + 1} \quad \cdots \quad (28)$$

(1) and (28) will be identical if

$$b_0 = \frac{\beta^2}{(3\beta+1)} \quad \cdots \quad (29)$$

$$b_1 = T_3 \quad \cdots \quad (30)$$

$$a_1 = \frac{\beta}{(3\beta+1)}T_2 + \frac{2\beta}{(3\beta+1)}T_3 + \frac{(\beta+1)(\alpha+2\alpha\beta+\beta)+\beta(\alpha+\alpha\beta+\beta)}{\alpha(3\beta+1)}T_6 \quad .. \quad (31)$$

$$a_2 = \frac{\beta(\alpha+2\alpha\beta+\beta)}{\alpha(3\beta+1)}T_2T_6 + \frac{\beta(2\alpha+2\alpha\beta+\beta)}{\alpha(3\beta+1)}T_3T_6 \quad .. \quad .. \quad .. \quad (32)$$

$$a_3 = \frac{\beta^2}{(3\beta+1)}T_2T_3T_6 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (33)$$

where

$$T_n = RC_n \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (34)$$

The solution of (29) gives

$$\beta = \frac{3b_0 + \sqrt{9b_0^2+4b_0}}{2} \quad .. \quad .. \quad .. \quad .. \quad (35)$$

as the negative root is inadmissible.

Elimination of α, T_2 and T_3 from (30) to (33) gives a cubic

$$b_0^2b_1^3(\beta+1)^2T_6^3 - b_0b_1\beta\left\{a_2b_1(2\beta+1)-a_3\beta-b_1^2(a_1\beta-2b_0b_1)\right\}T_6^2 +$$
$$+a_3b_1\beta(a_1\beta-3b_0b_1)T_6 - a_3^2\beta = 0 \quad .. \quad .. \quad .. \quad .. \quad (36)$$

which has at least one positive real root corresponding to which, as shown in Appendix II, a set of positive real α, β, T_2 and T_3 exists, provided that

$$\left.\begin{aligned} (a_1\beta-2b_0b_1) &> 0 \\ [b_1\beta(a_1\beta-2b_0b_1)^2-4a_3b_0(3\beta+1)(\beta+1)] &> 0 \\ [a_2b_1\beta-a_3(2\beta+1)] &> 0 \\ OB > OP &> OA \end{aligned}\right\} \quad .. \quad .. \quad (37)$$

and

where

$$\left.\begin{aligned} OA &= \frac{b_1\beta(a_1\beta-2b_0b_1)-\sqrt{b_1^2\beta^2(a_1\beta-2b_0b_1)^2-4a_3b_0b_1\beta(3\beta+1)(\beta+1)}}{2b_0b_1(3\beta+1)(\beta+1)} \\ OB &= \frac{b_1\beta(a_1\beta-2b_0b_1)+\sqrt{b_1^2\beta^2(a_1\beta-2b_0b_1)^2-4a_3b_0b_1\beta(3\beta+1)(\beta+1)}}{2b_0b_1(3\beta+1)(\beta+1)} \end{aligned}\right\} \quad (38)$$

$$OP = \frac{a_2b_1\beta-a_3(2\beta+1)}{2b_0b_1^2(\beta+1)} \quad .. \quad .. \quad .. \quad .. \quad (39)$$

Hence, if the inequalities of (37) are satisfied then a positive set of α, β, T_2, T_3 and T_6 exists for which the circuit of Fig. 2(*c*) is physically realizable. The circuit component values may then be obtained by solution of (35), (36), (30), (31), (33), (34) and (27).

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Appendix I

Conditions under which the Circuit of Fig. 2(*a*) is Physically Realizable

Simulation of the system represented by (1) with the network of Fig. 2(*a*) is possible only if the values of α, β, T_1, T_4 and T_6 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta}{(2\alpha+\beta+2)} \qquad \text{(1.1)}$$

$$b_1 = T_1 \qquad \text{(1.2)}$$

$$a_1 = \frac{\alpha}{(2\alpha+\beta+2)} T_1 + \frac{(2\alpha+3\alpha\beta+4\beta+2)}{(2\alpha+\beta+2)} T_6 \qquad \text{(1.3)}$$

$$a_2 = \frac{\alpha(2\beta+1)}{(2\alpha+\beta+2)} T_1 T_6 + \frac{2\beta(\alpha+1)}{(2\alpha+\beta+2)} T_4 T_6 \qquad \text{(1.4)}$$

$$a_3 = \frac{\alpha\beta}{(2\alpha+\beta+2)} T_1 T_4 T_6 \qquad \text{(1.5)}$$

are positive and real; where a's and b's are positive and real constants.

It is, therefore, required to determine the conditions under which α, β, T_1, T_4 and T_6 can be positive and real; and graphical methods may perhaps be a convenient means of obtaining these.

Elimination of β, T_1 and T_4 from (1.1), (1.2), (1.3), (1.5) and (1.1), (1.2), (1.4), (1.5) gives the following two equations

$$T_6 = \frac{\alpha}{2} \cdot \left[\frac{(2a_1-b_1)\alpha+(2a_1+b_0b_1)}{(1+3b_0)\alpha^2+(1+6b_0)\alpha+3b_0}\right] \qquad \text{(1.6)}$$

$$T_6 = \frac{2}{b_1^2}\left[\frac{(a_2b_1-2a_3)\alpha^2+(a_2b_1-4a_3)\alpha-2a_3}{\alpha\{(1+4b_0)\alpha+3b_0\}}\right] \qquad \text{(1.7)}$$

The intersection of the curves of (1.6) and (1.7) in the first quadrant of the α—T_6 plane will give both α and T_6 as positive and real. From (1.1)

$$\beta = \frac{2b_0(\alpha+1)}{(\alpha-b_0)} \qquad \text{(1.8)}$$

and it is obvious that β will be positive, if

$$\alpha > b_0 \quad \ldots \quad (1.9)$$

It is evident from (1.2) and (1.5) that the corresponding T_1 and T_4 are also positive real.

It should be clear, therefore, that only the portions of the curve lying on the right of the T_6-axis are of interest.

Now, the curve of (1.6) will intersect the α-axis (i.e. $T_6 = 0$) at a point A whose α-coordinate is given by

$$OA = -\frac{(2a_1+b_0b_1)}{(2a_1-b_1)} \quad \ldots \quad (1.10)$$

and the value of T_6 as $\alpha \to \infty$ is

$$T'_{6\infty} = \frac{(2a_1-b_1)}{2(1+3b_0)} \quad \ldots \quad (1.11)$$

Therefore, if

$$(2a_1-b_1) > 0 \quad \ldots \quad (1.12)$$

then one branch of the curve of (1.6) cuts the α-axis on the left of the T_6-axis and tends to a positive value as α approaches infinity. But, if

$$(2a_1-b_1) < 0 \quad \ldots \quad (1.13)$$

then the curve cuts the α-axis on the right of the T_6-axis and finally tends to a negative value. The branch of the curve lying on the right of the T_6-axis for either of the two possibilities is sketched in Figs. 1.1 (i) and (ii).

Similarly, the curve of (1.7) will cut the α-axis at two points, one of which (i.e. P) will be on the right of the T_6-axis if

$$(a_2b_1-2a_3) > 0 \quad \ldots \quad (1.14)$$

and its α-coordinate will be given by

$$OP = \frac{2a_3}{(a_2b_1-2a_3)} \quad \ldots \quad (1.15)$$

The value of T_6 as α approaches infinity is

$$T''_{6\infty} = \frac{2(a_2b_1-2a_3)}{b_1^2(1+4b_0)} \quad \ldots \quad (1.16)$$

A branch of the curve lying on the right of the T_6-axis is sketched in Fig. 1.1 (iii). Therefore, if the conditions of either expression (1.12) or (1.13) and (1.14) are satisfied then it is possible for the curves of (1.6) and (1.7) to exist in the first quadrant of the α-T_6 plane and it may be possible, under certain conditions, for these to intersect each other at one or more points in that region.

Now, if either the inequalities of (1.12) and (1.14) are satisfied and

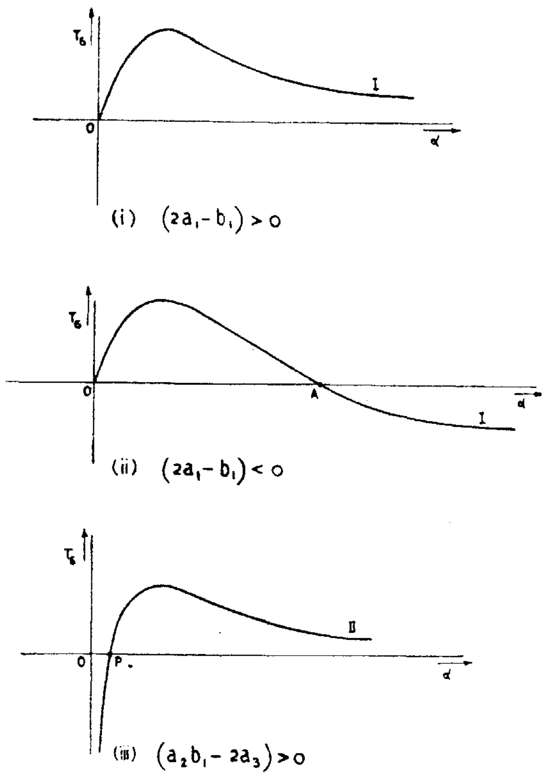
$$T''_{6\infty} > T'_{6\infty}$$

$$\text{i.e.} \quad \frac{2(a_2b_1-2a_3)}{b_1^2(1+4b_0)} > \frac{(2a_1-b_1)}{2(1+3b_0)} \quad \ldots \quad \ldots \quad \ldots (1.17)$$

or that of (1.13) and (1.14) are satisfied, and

$$\left(\frac{2a_1+b_0b_1}{b_1-2a_1}\right) > \left(\frac{2a_3}{a_2b_1-2a_3}\right) \quad \ldots \quad \ldots \quad \ldots (1.18)$$

then a portion of the curves exists in the first quadrant and these can intersect each other in that region.



$$\text{I} \quad T_6 = \frac{\alpha}{2}\left[\frac{(2a_1-b_1)\alpha+(2a_1+b_0b_1)}{(1+3b_0)\alpha^2+(1+6b_0)\alpha+3b_0}\right]$$

$$\text{II} \quad T_6 = \frac{2}{b_1^2}\left[\frac{(a_2b_1-2a_3)\alpha^2+(a_2b_1-4a_3)\alpha-2a_3}{\alpha\{(1+4b_0)\alpha+3b_0\}}\right]$$

FIG. 1.1. Sketches of the curves for positive α.

To summarize, if the inequalities (1.9) and (1.14) and either (1.12) and (1.17) or (1.13) and (1.18) are satisfied then a positive real set of α, β, T_1, T_4 and T_6 exists for which the circuit of Fig. 2(a) is physically realizable.

Appendix II

Conditions under which the Circuit of Fig. 2(c) is Physically Realizable

It is possible to simulate the system of (1) with the network of Fig. 2(c) only if the values of α, β, T_2, T_3 and T_6 obtained as the solution of equations

$$b_0 = \frac{\beta^2}{(3\beta+1)} \qquad (2.1)$$

$$b_1 = T_3 \qquad (2.2)$$

$$a_1 = \frac{\beta}{(3\beta+1)} T_2 + \frac{2\beta}{(3\beta+1)} T_3 + \frac{(\beta+1)(\alpha+2\alpha\beta+\beta)+\beta(\alpha+\alpha\beta+\beta)}{\alpha(3\beta+1)} T_6 \qquad (2.3)$$

$$a_2 = \frac{\beta(\alpha+2\alpha\beta+\beta)}{\alpha(3\beta+1)} T_2 T_6 + \frac{\beta(2\alpha+2\alpha\beta+\beta)}{\alpha(3\beta+1)} T_3 T_6 \qquad (2.4)$$

$$a_3 = \frac{\beta^2}{(3\beta+1)} T_2 T_3 T_6 \qquad (2.5)$$

are positive and real.

Solving (2.1) for β gives

$$\beta = \frac{3b_0 + \sqrt{9b_0^2+4b_0}}{2} \qquad (2.6)$$

as the negative root is inadmissible.

Elimination of T_2 and T_3 from (2.2), (2.3), (2.5) and (2.2), (2.4), (2.5) gives

$$\frac{1}{\alpha} = \frac{1}{b_0(2\beta+1)} \cdot \left[\frac{(a_1\beta - 2b_0b_1)}{T_6} - \frac{a_3}{b_1T_6^2} - \frac{b_0}{\beta}(3\beta+1)(\beta+1)\right] \qquad (2.7)$$

$$\frac{1}{\alpha} = \frac{b_1}{(a_3+b_0b_1^2T_6)} \cdot \left[a_2 - \frac{a_3(2\beta+1)}{\beta b_1} - \frac{2b_0b_1(\beta+1)}{\beta} T_6\right] \qquad (2.8)$$

The intersection of the curves of (2.7) and (2.8) in the first quadrant of the $T_6 - \frac{1}{\alpha}$ plane will give both T_6 and α as positive real. It is obvious from (2.2), (2.5) and (2.6) that the corresponding T_2, T_3 and β will be also positive real.

Now, the curve of (2.7) will cut the T_6-axis at two points (A, B) whose coordinates may be obtained by equating to zero the right-hand side of (2.7) and solving the resultant quadratic

$$b_0b_1(3\beta+1)(\beta+1)T_6^2 - b_1\beta(a_1\beta - 2b_0b_1)T_6 + a_3\beta = 0. \qquad (2.9)$$

The roots of (2.9) are

$$T_{6(A,\,B)} = \frac{b_1\beta(a_1\beta-2b_0b_1)\pm\sqrt{b_1^2\beta^2(a_1\beta-2b_0b_1)^2-4a_3b_0b_1\beta(3\beta+1)(\beta+1)}}{2b_0b_1(3\beta+1)(\beta+1)} \qquad (2.10)$$

which will be positive, if

$$(a_1\beta-2b_0b_1) > 0 \quad .. \quad .. \quad .. \quad .. \quad (2.11)$$

and real, if

$$[b_1\beta(a_1\beta-2b_0b_1)^2-4a_3b_0(3\beta+1)(\beta+1)] > 0 \quad .. \quad .. \quad (2.12)$$

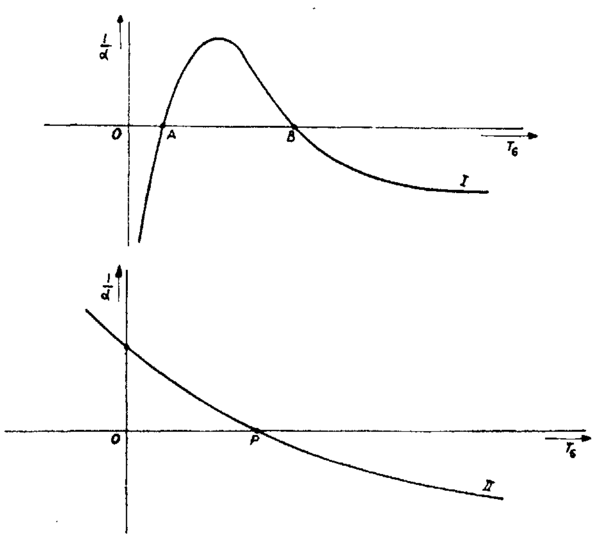
Similarly, the curve of (2.8) will cut the T_6-axis at a point P on the right of the $\frac{1}{\alpha}$-axis, if

$$[a_2b_1\beta-a_3(2\beta+1)] > 0 \quad .. \quad .. \quad .. \quad .. \quad (2.13)$$

and its T_6-coordinate is given by

$$OP = \frac{a_2b_1\beta-a_3(2\beta+1)}{2b_0b_1^2(\beta+1)} \quad .. \quad .. \quad .. \quad (2.14)$$

Therefore, if the inequalities (2.11) to (2.13) are satisfied then it is possible for a portion of the curves of (2.7) and (2.8) to exist in the first quadrant. The sketches of the curves for positive T_6 are shown in Fig. 2.1.



I $\quad \frac{1}{\alpha} = \frac{1}{b_0(2\beta+1)}\left[\frac{(a_1\beta-2b_0b_1)}{T_6} - \frac{a_3}{b_1T_6^2} - \frac{b_0}{\beta}(3\beta+1)(\beta+1)\right]$

II $\quad \frac{1}{\alpha} = \frac{b_1}{\left(a_3+b_0b_1^2T_6\right)}\left[a_2 - \frac{a_3(2\beta+1)}{\beta b_1} - \frac{2b_0b_1(\beta+1)}{\beta}T_6\right]$

FIG. 2.1. Sketches of the curves for positive T_6.

Now, it will be possible (Uspensky 1948) for the curves to intersect each other in the first quadrant if the point P lies in between A and B. That is, if

$$\left[\frac{b_1\beta(a_1\beta-2b_0b_1)+\sqrt{\Delta}}{(3\beta+1)}\right] > \left[\frac{a_2b_1\beta-a_3(2\beta+1)}{b_1}\right] > \left[\frac{b_1\beta(a_1\beta-2b_0b_1)-\sqrt{\Delta}}{(3\beta+1)}\right] \quad (2.15)$$

where

$$\Delta = b_1^2\beta^2(a_1\beta-2b_0b_1)^2-4a_3b_0b_1\beta(3\beta+1)(\beta+1).$$

Hence, if the inequalities of (2.11), (2.12), (2.13) and (2.15) are sat'sfied, then a positive real set of α, β, T_2, T_3 and T_6 exists for which the circuit of Fig. 2(c) is physically realizable.