THE SIMULATION OF THIRD ORDER SYSTEMS WITH A LEADING TIME-CONSTANT BY A SINGLE OPERATIONAL AMPLIFIER—II

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In the previous paper (Wadhwa 1964) on the same topic three basic circuits each employing three capacitors and six resistors and capable of simulating, under certain conditions, third order systems with a leading time-constant were presented. In this paper three more basic circuits for simulating the same type of third order systems are presented. The circuits are analysed, the design formulae obtained and the conditions of their physical realizability discussed.

Introduction

In Part I a network capable of simulating the general third order linear systems with only one operational amplifier was presented and the design of three circuits capable of simulating, under certain conditions, third order systems of the type

$$F(S) = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1} \quad \cdots \quad (1)$$

was discussed. The purpose of this paper is to present three more basic circuits for simulating the system of (1) with a certain arbitrary choice of resistor values and obtain the design formulae and conditions of their physical realizability.

Third Order System Simulation

Three basic circuits capable of simulating, under certain conditions, the system of (1) are shown in Fig. 1.

Transfer function for the circuit of Fig. 1(*a*) can be shown to be

$$\frac{E_0}{E_1} = -\frac{\frac{\alpha}{3}(RC_3S+1)}{\frac{\alpha\beta}{3}R^3C_3C_6C_8S^3+\left[\frac{(\alpha+2\alpha\beta+2\beta)}{3}R^2C_3C_6+\frac{\alpha}{3}R^2C_3C_8+\frac{(\alpha+\alpha\beta+2\beta)}{3}R^2C_6C_8\right]S^2+\left[\frac{1}{3}RC_3+\frac{(3\alpha+2\alpha\beta+6\beta)}{3}RC_6+\left(\frac{\alpha+1}{3}\right)RC_8\right]S+1} \quad \cdots \quad (2)$$

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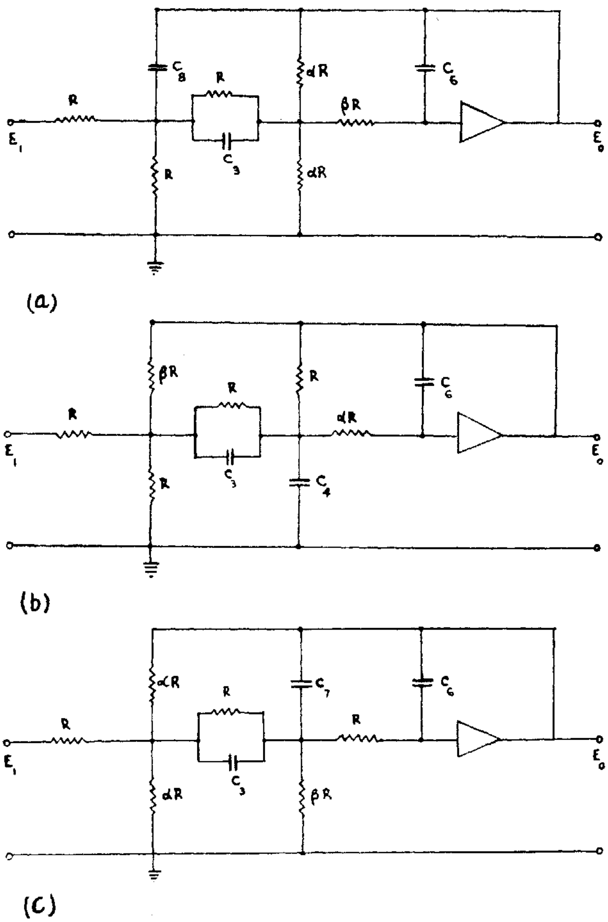

FIG. 1. Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}$.

(1) and (2) will be identical if

$$b_0 = \frac{\alpha}{3} \quad \ldots \quad (3)$$

$$b_1 = T_3 \quad \ldots \quad (4)$$

$$a_1 = \tfrac{1}{3}T_3 + \frac{(3\alpha+2\alpha\beta+6\beta)}{3}T_6 + \left(\frac{\alpha+1}{3}\right)T_8 \quad \ldots \quad (5)$$

$$a_2 = \frac{(\alpha+2\alpha\beta+2\beta)}{3}T_3T_6 + \frac{\alpha}{3}T_3T_8 + \frac{(\alpha+\alpha\beta+2\beta)}{3}T_6T_8 \quad \ldots \quad (6)$$

$$a_3 = \frac{\alpha\beta}{3}T_3T_6T_8 \quad \ldots \quad (7)$$

where

$$T_n = RC_n. \quad \ldots \quad (8)$$

Elimination of α, β, T_3 and T_6 from (3) to (7) gives a cubic

$$b_1(1+3b)T_8^3 - b_1(3a_1+6b_0b_1-2b_1)T_8^2 + (9a_2b_1+b_1^3-3a_1b_1^2-3a_3)T_8 - 12a_3b_1 = 0 \quad (9)$$

which will have at least one positive real root corresponding to which, as shown in the Appendix, a set of positive real α, β, T_3 and T_6 exists, provided that

$$\left.\begin{aligned} &(3a_1-b_1) > 0 \\ &b_0b_1(3a_1-b_1)^2 > 24a_3(1+3b_0)(1+b_0) \\ &\{3a_2b_0b_1-a_3(2+3b_0)\} > 0 \\ &\{3a_2b_0b_1-a_3(2+3b_0)\}^2 > 24a_3b_0^2b_1^3(1+3b_0) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (10)$$

and either

$$OB > OQ > OA > OP \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

or

$$OQ > OB > OP > OA \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

or

$$\left.\begin{aligned} &(3a_1+6b_0b_1-2b_1) > 0 \\ &(9a_2b_1+b_1^3-3a_1b_1^2-3a_3) > 0 \\ \text{and} \quad &\Delta = 4p^3+27q^2 < 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where

$$\left.\begin{aligned} OA &= \frac{b_0b_1(3a_1-b_1)-\sqrt{\Delta_1}}{2b_0b_1(1+3b_0)} \\ OB &= \frac{b_0b_1(3a_1-b_1)+\sqrt{\Delta_1}}{2b_0b_1(1+3b_0)} \\ OP &= \frac{3a_2b_0b_1-a_3(2+3b_0)-\sqrt{\Delta_2}}{6b_0^2b_1^2} \\ OQ &= \frac{3a_2b_0b_1-a_3(2+3b_0)+\sqrt{\Delta_2}}{6b_0^2b_1^2} \\ \Delta_1 &= b_0^2b_1^2(3a_1-b_1)^2-24a_3b_0b_1(1+b_0)(1+3b_0) \\ \Delta_2 &= \{3a_2b_0b_1-a_3(2+3b_0)\}^2-24a_3b_0^2b_1^3(1+3b_0) \\ p &= \frac{(9a_2b_1+b_1^3-3a_1b_1^2-3a_3)}{b_1(1+3b_0)}-\frac{1}{3}\left(\frac{3a_1+6b_0b_1-2b_1}{1+3b_0}\right)^2 \\ q &= -\frac{12a_3}{(1+3b_0)}+\frac{(3a_1+6b_0b_1-2b_1)(9a_2b_1+b_1^3-3a_1b_1^2-3a_3)}{3b_1(1+3b_0)^2} \\ &\qquad -\frac{2}{27}\left(\frac{3a_1+6b_0b_1-2b_1}{1+3b_0}\right)^3 \end{aligned}\right\} \quad \ldots \quad (14)$$

Hence, if the inequalities of (10) and either (11) or (12) or (13) are satisfied then at least one positive real set of α, β, T_3, T_6 and T_8 exists for which the circuit of Fig. 1(a) is physically realizable. The circuit component values may be obtained by first solving the cubic of (9), and then solving the simultaneous equations (5) and (7) to obtain β and T_6—as α and T_3 are known directly from (3) and (4).

Now, the transfer function for the circuit of Fig. 1(*b*) can be shown to be

$$\frac{E_0}{E_1} = -\frac{\frac{\beta}{(3\beta+2)}(RC_3S+1)}{\left(\frac{\alpha\beta}{3\beta+2}\right)R^3C_3C_4C_6S^3+\left[\left(\frac{\alpha+3\alpha\beta+\beta}{3\beta+2}\right)R^2C_3C_6+\frac{\alpha(3\beta+1)}{(3\beta+2)}R^2C_4C_6\right]S^2+\left[\left(\frac{\beta+1}{3\beta+2}\right)RC_3+\frac{(2\alpha+5\alpha\beta+3\beta+1)}{(3\beta+2)}RC_6\right]S+1} \quad \ldots \quad (15)$$

(1) and (15) will be identical if

$$b_0 = \frac{\beta}{(3\beta+2)} \quad \ldots \quad (16)$$

$$b_1 = T_3 \quad \ldots \quad (17)$$

$$a_1 = \left(\frac{\beta+1}{3\beta+2}\right)T_3+\frac{(2\alpha+5\alpha\beta+3\beta+1)}{(3\beta+2)}T_6 \quad \ldots \quad (18)$$

$$a_2 = \frac{(\alpha+3\alpha\beta+\beta)}{(3\beta+2)}T_3T_6+\frac{\alpha(3\beta+1)}{(3\beta+2)}T_4T_6 \quad \ldots \quad (19)$$

$$a_3 = \frac{\alpha\beta}{(3\beta+2)}T_3T_4T_6 \quad \ldots \quad (20)$$

where

$$T_n = RC_n. \quad \ldots \quad (21)$$

Elimination of β, T_3 and T_4 from (16), (17), (18), (20) and (16), (17), (19), (20) give the following two equations:

$$\alpha = \frac{1}{2(1+2b_0)}\left[\frac{(2a_1+b_0b_1-b_1)}{T_6}-(1+3b_0)\right] \quad \ldots \quad (22)$$

$$\alpha = \frac{2}{b_1(1+3b_0)}\left[\frac{2a_2b_0b_1-a_3(1+3b_0)}{2b_0b_1T_6}-b_0b_1\right] \quad \ldots \quad (23)$$

which on solution yield

$$\alpha = \frac{2b_0b_1\left\{a_2(1+3b_0)+b_0b_1^2\right\}-\left\{2b_1^2b_1^2(2a_1+b_0b_1)+a_3(1+3b_0)^2\right\}}{(k_1-k_2)} \quad \ldots \quad (24)$$

$$T_6 = \frac{(k_1-k_2)}{b_0b_1^2(1+b_0)^2}. \quad \ldots \quad (25)$$

Now, the solution of (16) gives

$$\beta = \frac{2b_0}{(1-3b_0)} \quad \ldots \quad (26)$$

and, since T_3 is known directly from (18), the solution of (20) gives

$$T_4 = \frac{a_3b_1(1+b_0)^2}{2b_0b_1\left\{a_2(1+3b_0)+b_0b_1^2\right\}-\left\{2b_0^2b_1^2(2a_1+b_0b_1)+a_3(1+3b_0)^2\right\}} \quad (27)$$

where

$$\left.\begin{aligned} k_1 &= (1+3b_0)\left[2a_3(1+2b_0)+b_0b_1^2(2a_1+b_0b_1)\right] \\ k_2 &= b_0b_1\left[4a_2(1+2b_0)+b_1^2(1+3b_0)\right] \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (28)$$

It is obvious from (24) to (27) that α, β, T_4 and T_6 are all real and these will be also positive, if

$$\left.\begin{aligned} 2b_0b_1\left[a_2(1+3b_0)+b_0b_1^2\right] &> \left[2b_0^2b_1^2(2a_1+b_0b_1)+a_3(1+3b_0)^2\right] \\ (1+3b_0)\left[2a_3(1+2b_0)+b_0b_1^2(2a_1+b_0b_1)\right] &> b_0b_1\left[4a_2(1+2b_0)+b_1^2(1+3b_0)\right] \\ b_0 &< \tfrac{1}{3} \end{aligned}\right\} \quad (29)$$

Hence, if the inequalities of expression (29) are satisfied then a set of positive real α, β, T_3, T_4 and T_6 exists for which the circuit of Fig. 2(*b*) is physically realizable.

Now, the transfer function for the circuit of Fig. 1(*c*) can be shown to be

$$\frac{E_0}{E_1} = -\frac{\alpha(RC_3S+1)}{\alpha R^3C_3C_6C_7S^3+\left[\frac{(\alpha+2\alpha\beta+2\beta)}{\beta}R^2C_3C_6+\alpha R^2C_3C_7+2(\alpha+1)R^2C_6C_7\right]S^2+\left[RC_3+\frac{(2\alpha+3\alpha\beta+4\beta+2)}{\beta}RC_6+2(\alpha+1)RC_7\right]S+1} \quad .. \quad (30)$$

(1) and (30) will be identical if

$$b_0 = \alpha \quad .. \quad (31)$$

$$b_1 = T_3 \quad .. \quad (32)$$

$$a_1 = T_3+\frac{(2\alpha+3\alpha\beta+4\beta+2)}{\beta}T_6+2(\alpha+1)T_7 \quad .. \quad (33)$$

$$a_2 = \frac{(\alpha+2\alpha\beta+2\beta)}{\beta}T_3T_6+\alpha T_3T_7+2(\alpha+1)T_6T_7 \quad .. \quad (34)$$

$$a_3 = \alpha T_3T_6T_7 \quad .. \quad (35)$$

where

$$T_n = RC_n. \quad .. \quad (36)$$

Elimination of α, T_3 and T_7 from (31), (32), (33), (35) and (31), (32), (34), (35) give the following two equations :

$$\frac{1}{\beta} = \frac{1}{2(1+b_0)}\left[\frac{(a_1-b_1)}{T_6}-\frac{2a_3(1+b_0)}{b_0b_1T_6^2}-(4+3b_0)\right] \quad .. \quad (37)$$

$$\frac{1}{\beta} = \frac{1}{b_0}\left[\frac{a_2b_0b_1-2a_3(1+b_0)}{b_0b_1^2T_6}-\frac{a_3}{b_1T_6^2}-2(1+b_0)\right] \quad .. \quad (38)$$

which on solution yield

$$\beta = \frac{2(1+b_0)(k_1'-k_2')^2}{a_1b_0b_1^2(2+b_0)^2(k_1'-k_2')-b_0b_1^3(2+b_0)^2\left\{(k_1'-k_2')+2a_3(1+b_0)(2+b_0)^2\right\}-(4+3b_0)(k_1'-k_2')^2} \quad \text{.. (39)}$$

$$T_6 = \frac{(k_1'-k_2')}{b_0b_1^2(2+b_0)^2}. \quad \text{.. .. .. .. (40)}$$

Since α and T_3 are known directly from (31) and (32), T_7 may be then obtained from (35)

$$T_7 = \frac{a_3b_1(2+b_0)^2}{(k_1'-k_2')} \quad \text{.. .. .. (41)}$$

where

$$\left.\begin{aligned} k_1' &= b_0b_1[2a_2(1+b_0)+b_0b_1^2] \\ k_2' &= a_1b_0^2b_1^2+4a_3(1+b_0)^2 \end{aligned}\right\}. \quad \text{.. .. .. (42)}$$

It is obvious from (39) to (41) that β, T_6 and T_7 are real and these will be also positive, if

$$\left.\begin{aligned} &(k_1'-k_2') > 0 \\ \text{and} \quad &[a_1b_0b_1^2(2+b_0)^2(k_1'-k_2')-b_0b_1^3(2+b_0)^2\left\{(k_1'-k_2')+2a_3(1+b_0)(2+b_0)^2\right\}- \\ &\qquad (4+3b_0)(k_1'-k_2')^2] > 0 \end{aligned}\right\} \quad (43)$$

Hence, if the inequalities of expression (43) are satisfied then a set of positive real α, β, T_3, T_6 and T_7 exists for which the circuit of Fig. 1(c) is physically realizable and the circuit component values may be determined with the aid of (31), (32), (39) to (41).

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Reference

Wadhwa, L. K. (1964). The simulation of third order systems with a leading time-constant by a single operational amplifier—I. *Proc. nat. Inst. Sci. India*, A **30**, 520–532.

APPENDIX

CONDITIONS UNDER WHICH THE CIRCUIT OF FIG. 1(*a*) IS PHYSICALLY REALIZABLE

Simulation of the system represented by (1) with the network of Fig. 1(*a*) is possible only if the values of α, β, T_3, T_6 and T_8 obtained as the solution of equations

$$b_0 = \frac{\alpha}{3} \quad \ldots \quad (A.1)$$

$$b_1 = T_3 \quad \ldots \quad (A.2)$$

$$a_1 = \tfrac{1}{3}T_3 + \frac{(3\alpha+2\alpha\beta+6\beta)}{3}T_6 + \frac{(\alpha+1)}{3}T_8 \quad \ldots \quad (A.3)$$

$$a_2 = \frac{(\alpha+2\alpha\beta+2\beta)}{3}T_3T_6 + \frac{\alpha}{3}T_3T_8 + \frac{(\alpha+\alpha\beta+2\beta)}{3}T_6T_8 \quad \ldots \quad (A.4)$$

$$a_3 = \frac{\alpha\beta}{3}T_3T_6T_8 \quad \ldots \quad (A.5)$$

are positive and real; where a's and b's are positive real constants.

Elimination of α, T_3 and T_6 from (A.1), (A.2), (A.3), (A.5) and (A.1), (A.2), (A.4), (A.5) give the following two equations:

$$\frac{1}{\beta} = \frac{b_1}{3a_3}\left[\left(\frac{3a_1-b_1}{3}\right)T_8 - \left(\frac{1+3b_0}{3}\right)T_8^2 - \frac{2a_3(1+b_0)}{b_0b_1}\right] \quad \ldots \quad (A.6)$$

$$\frac{1}{\beta} = \frac{b_1}{a_3(T_8+b_1)}\cdot\left[\frac{\{3a_2b_0b_1 - a_3(2+3b_0)\}}{3b_0b_1}T_8 - b_0b_1T_8^2 - \frac{2a_3(1+3b_0)}{3b_0}\right]. \quad (A.7)$$

The intersection of the curves of (A.6) and (A.7) in the first quadrant of the T_8—$\frac{1}{\beta}$ plane will give both T_8 and β as real and positive. It is obvious from (A.1), (A.2) and (A.5) that the corresponding α, T_3 and T_6 will be also real and positive.

Now, the curve of (A.6) will cut the T_8-axis $\left(\text{i.e. } \frac{1}{\beta} = 0\right)$ at two points A, B whose T_8-coordinates may be obtained by equating to zero the right-hand side of (A.6) and solving the resulting quadratic

$$b_0b_1(1+3b_0)T_8^2 - b_0b_1(3a_1-b_1)T_8 + 6a_3(1+b_0) = 0 \quad \ldots \quad (A.8)$$

the roots of which are

$$T_{8(A,\,B)} = \frac{b_0b_1(3a_1-b_1) \pm \sqrt{b_0^2b_1^2(3a_1-b_1)^2 - 24a_3b_0b_1(1+b_0)(1+3b_0)}}{2b_0b_1(1+3b_0)}. \quad (A.9)$$

These will be positive, if

$$(3a_1-b_1) > 0 \quad \ldots \quad (A.10)$$

and also real, if

$$b_0 b_1 (3a_1 - b_1)^2 > 24a_3(1+b_0)(1+3b_0). \qquad \ldots \quad (A.11)$$

Similarly, the curve of (A.7) will cut the T_8-axis at two points P, Q whose T_8-coordinates will be given by

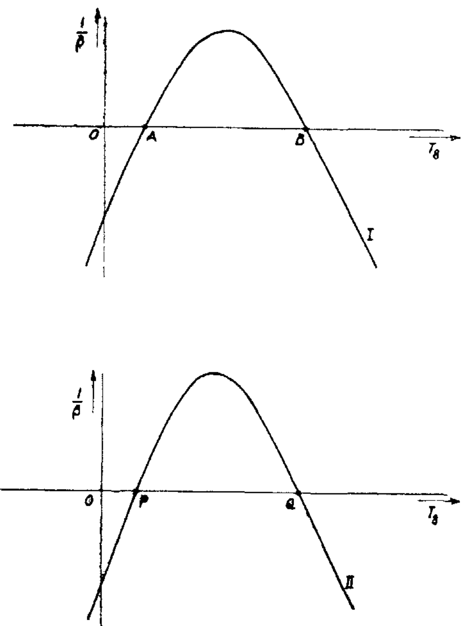
$$T_{8(P,Q)} = \frac{\{3a_2 b_0 b_1 - a_3(2+3b_0)\} \pm \sqrt{\{3a_2 b_0 b_1 - a_3(2+3b_0)\}^2 - 24a_3 b_0^2 b_1^3 (1+3b_0)}}{6b_0^2 b_1^2} \qquad \ldots \quad (A.12)$$

and these will be positive and real, if

$$\left.\begin{aligned} \{3a_2 b_0 b_1 - a_3(2+3b_0)\} &> 0 \\ \{3a_2 b_0 b_1 - a_3(2+3b_0)\}^2 &> 24a_3 b_0^2 b_1^3 (1+3b_0) \end{aligned}\right\} \qquad \ldots \quad (A.13)$$

Therefore, if the conditions of expressions (A.10), (A.11) and (A.13) are satisfied then a portion of the curves of (A.6) and (A.7) can exist in the first quadrant and the sketches of these curves for positive T_8 are shown in Fig. 2.

FIG. 2. Sketches of the curves for positive T_8.



$$\text{I} \quad \frac{1}{\beta} = \frac{b_1}{3a_3}\left[\left(\frac{3a_1 - b_1}{3}\right)T_8 - \left(\frac{1+3b_0}{3}\right)T_8^2 - \frac{2a_3(1+b_0)}{b_0 b_1}\right]$$

$$\text{II} \quad \frac{1}{\beta} = \frac{b_1}{a_3(T_8+b_1)}\left[\frac{\{3a_2 b_0 b_1 - a_3(2+3b_0)\}}{3b_0 b_1} T_8 - \frac{2a_3(1+3b_0)}{3b_0} - b_0 b_1 T_8^2\right]$$

Elimination of β from (A.6) and (A.7) gives a cubic

$$b_1(1+3b_0)T_8^3 - b_1(3a_1 + 6b_0 b_1 - 2b_1)T_8^2 + (9a_2 b_1 + b_1^3 - 3a_1 b_1^2 - 3a_3)T_8 - 12a_3 b_1 = 0 \qquad \ldots \quad (A.14)$$

which will have at least one positive and real root; it can have three real roots if its discriminant Δ is negative—the sign of the other two roots depending upon the sign of the coefficients of T_8 and T_8^2.

The real roots of (A.14) give the real points of intersection of the curves of (A.6) and (A.7). Now, the curves will intersect each other in the first quadrant provided that the conditions of (A.10), (A.11) and (A.13) are satisfied, and either

$$OB > OQ > OA > OP \quad \ldots \quad \ldots \quad \ldots (A.15)$$

or

$$OQ > OB > OP > OA \quad \ldots \quad \ldots \quad \ldots (A.16)$$

where

$$OA = \frac{b_0b_1(3a_1-b_1)-\sqrt{b_0^2b_1^2(3a_1-b_1)^2-24a_3b_0b_1(1+b_0)(1+3b_0)}}{2b_0b_1(1+3b_0)}$$

$$OB = \frac{b_0b_1(3a_1-b_1)+\sqrt{b_0^2b_1^2(3a_1-b_1)^2-24a_3b_0b_1(1+b_0)(1+3b_0)}}{2b_0b_1(1+3b_0)}$$

$$OP = \frac{\{3a_2b_0b_1-a_3(2+3b_0)\}-\sqrt{\{3a_2b_0b_1-a_3(2+3b_0)\}^2-24a_3b_0^2b_1^3(1+3b_0)}}{6b_0^2b_1^2}$$

$$OQ = \frac{\{3a_2b_0b_1-a_3(2+3b_0)\}+\sqrt{\{3a_2b_0b_1-a_3(2+3b_0)\}^2-24a_3b_0^2b_1^3(1+3b_0)}}{6b_0^2b_1^2}.$$

But, if

$$\left.\begin{array}{l} \Delta = 4p^3+27q^2 < 0 \\ (3a_1+6b_0b_1-2b_1) > 0 \\ (9a_2b_1+b_1^3-3a_1b_1^2-3a_3) > 0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots (A.17)$$

where

$$\left.\begin{array}{l} p = \dfrac{(9a_2b_1+b_1^3-3a_1b_1^2-3a_3)}{b_1(1+3b_0)}-\dfrac{1}{3}\left(\dfrac{3a_1+6b_0b_1-2b_1}{1+3b_0}\right)^2 \\ q = -\dfrac{12a_3}{(1+3b_0)}+\dfrac{(3a_1+6b_0b_1-2b_1)(9a_2b_1+b_1^3-3a_1b_1^2-3a_3)}{3b_1(1+3b_0)^2}-\dfrac{2}{27}\left(\dfrac{3a_1+6b_0b_1-2b_1}{1+3b_0}\right)^3 \end{array}\right\} \quad \ldots (A.18)$$

then (A.14) will have three real and positive roots corresponding to which at least one positive real set of α, β, T_3, T_6 and T_8 exists, provided that (A.10), (A.11) and (A.13) are also satisfied.

To summarize, if the inequalities of expressions (A.10), (A.11), (A.13) and either (A.15) or (A.16) or (A.17) are satisfied then at least one positive real set of α, β, T_3, T_6 and T_8 exists corresponding to which the circuit of Fig. 1(a) is physically realizable.