AN INTEGRAL INVOLVING G-FUNCTION

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In this note an integral

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G_{rs}^{\lambda\mu}\left[zx^{l-m}(x+y)^m \middle| \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\right]dx$$

has been evaluated for positive integral values of l and m, whether $l > m$ or $l < m$. Further, by considering its special cases, we have obtained the value of the integral

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}E\left[p ; \alpha_r : q ; \beta_s : Z\lambda^l(1-\lambda)^n\right] d\lambda$$

for all the cases whether l and n have opposite signs or same signs. The above integral had been evaluated by MacRobert (1958) when l and n are of the same sign.

1. MacRobert (1958) evaluated the integral

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}E[p;\ \alpha_r : q;\ \beta_s : Z\lambda^l(1-\lambda)^n]\, d\lambda, \quad \ldots \quad \ldots \quad (1)$$

where l and n are either both positive or both negative integers. In an attempt to give the values to the above integral, even when l and n are of opposite signs, we have evaluated an integral involving G-function, from which the value of (1) can be derived for every permutation of integral values of l and n, whether positive or negative. For the definition and properties of G-function one should see Meijer's papers (Meijer 1946).

The integral, evaluated in this note, is

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G_{rs}^{\lambda\mu}\left(z(x+y)^m x^{l-m} \middle| \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\right) dx, \quad \ldots \quad \ldots \quad (2)$$

where l and m are positive integers.

2. The following results will be required (Erdelyi *et al.* 1953, p. 207; Erdelyi *et al.* 1954, p. 233);

$$G_{pq}^{mn}\left(z \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\xi)\prod_{j=n+1}^{b}\Gamma(a_j-\xi)} z^\xi d\xi, \quad \ldots \quad \ldots \quad (3)$$

where L is a suitable contour,

$$\int_0^\infty x^{\nu-1}(x+y)^{-\rho}\,dx = \frac{\Gamma(\nu)\Gamma(\rho-\nu)}{\Gamma(\rho)}y^{\nu-\rho} \qquad \ldots \quad \ldots \quad (4)$$

where $\qquad R(\nu)>0,\ R(\rho-\nu)>0,$

and the well-known result

$$(2\pi)^{\frac{1}{2}-\frac{1}{2}n}\,n^{nz-\frac{1}{2}}\prod_{t=0}^{n-1}\Gamma\left(z+\frac{t}{n}\right)=\Gamma(nz),$$

from which it can be derived that, for positive integer n,

$$\Gamma(a+nz)=(2\pi)^{\frac{1}{2}-\frac{1}{2}n}\,n^{a+nz-\frac{1}{2}}\prod_{t=0}^{n-1}\Gamma\left(\frac{a+t}{n}+z\right), \qquad \ldots \quad \ldots \quad (5)$$

and

$$\Gamma(a-nz)=(2\pi)^{\frac{1}{2}-\frac{1}{2}n}\,n^{a-nz-\frac{1}{2}}\prod_{t=0}^{n-1}\Gamma\left(\frac{a+t}{n}-z\right). \qquad \ldots \quad \ldots \quad (6)$$

3. Proof: If we substitute for G-function in the integrand of (2) from the result (3), then change the order of integration, which is permissible under the conditions given with the results (7) and (8), and evaluate the inner integral with the help of (4), the value of (2) is found to be equal to

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{\lambda}\Gamma(b_j-\xi)\prod_{j=1}^{\mu}\Gamma(1-a_j+\xi)\Gamma\{\beta+(l-m)\xi\}\,\Gamma(\alpha-l\xi)}{\prod_{j=\lambda+1}^{s}\Gamma(1-b_j+\xi)\prod_{j=\mu+1}^{r}\Gamma(a_j-\xi)\Gamma(\alpha+\beta-m\xi)y^{\alpha-l\xi}}z^{\xi}d\xi,$$

the contour L is a suitable one, that of Barnes type. Due to the presence of the factor $\Gamma\{\beta+(l-m)\xi\}$ in the integrand we shall have to take slightly different contours in the two cases, one $l>m$ and the other $l<m$, and so would be the result.

Now using the results (5) and (6) and interpreting it with the help of (3), we get:

(i) if l, m are positive integers, $l<m$, and $2(\lambda+\mu)>r+s$, $|\arg zy^l|<(\lambda+\mu-\frac{1}{2}r-\frac{1}{2}s)\pi$, $R\left(1-a_j+\frac{\alpha}{l}\right)>0$, $R(1-a_j+\beta/(m-l))>0$, $(j=1,\ldots,\mu)$, then

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G_{rs}^{\lambda\mu}\left[zx^{l-m}(x+y)^m\left|\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right.\right]dx$$

$$=(2\pi)^{\frac{1}{2}}l^{\alpha-\frac{1}{2}}m^{\frac{1}{2}-\alpha-\beta}(m-l)^{\beta-\frac{1}{2}}y^{-\alpha}$$

$$\times\, G_{r+m\,s+m}^{\lambda+m\,\mu}\left(\frac{m^m(m-l)^{l-m}}{l^l}zy^l\left|\begin{matrix}a_1,\ldots,a_r,\frac{\alpha+\beta}{m},\ldots,\frac{\alpha+\beta+m-1}{m}\\ \frac{\beta}{m-l},\ldots,\frac{\beta+m-l-1}{m-l},\frac{\alpha}{l},\ldots,\frac{\alpha+l-1}{l},b_1,\ldots,b_s\end{matrix}\right.\right)$$

$\ldots \quad (7)$

and

(ii) if l, m are positive integers, but $l > m$, and $2(\lambda+\mu) > r+s$, $|\arg zy^l| < (\lambda+\mu-\frac{1}{2}r-\frac{1}{2}s)\pi$, $R\left(b_h+\frac{\beta}{l-m}\right) > 0$, $R\left(1-a_j+\frac{\alpha}{l}\right) > 0$, $(h = 1, \ldots, \lambda;$ $j = 1, \ldots, \mu)$, then

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G_{rs}^{\lambda\mu}\left(zx^{l-m}(x+y)^m \middle| \begin{matrix} a_1, \ldots, a_r \\ \beta_1 \ldots, \beta_s \end{matrix}\right) dx$$

$$= (2\pi)^{\frac{1}{2}+m-l}(l-m)^{\beta-\frac{1}{2}}l^{\alpha-\frac{1}{2}}m^{\frac{1}{2}-\alpha-\beta}y^{-\alpha}$$

$$\times G_{r+l\,s+l}^{\lambda+l\,\mu+l-m}\left[\frac{m^m(l-m)^{l-m}}{l^l}zy^l \middle| \begin{matrix} \frac{1-\beta}{l-m}, \ldots, \frac{l-m-\beta}{l-m}, a_1, \ldots, a_r, \frac{\alpha+\beta}{m}, \ldots, \frac{\alpha+\beta+m-1}{m} \\ \frac{\alpha}{l}, \ldots, \frac{\alpha+l-1}{l}, b_1, \ldots, b_s \end{matrix}\right]. \quad \ldots (8)$$

4. From the above two results the value of (1) can be obtained for every permutation of l and n, whether positive or negative, by substituting $x = y\left(\frac{1}{\lambda}-1\right)$, replacing z by zy^{-l} and taking suitable values of the parameters in the G-function as given below, since (Erdelyi *et al.* 1954, p. 444; Erdelyi *et al.* 1953, p. 115)

$$E(p\,;\ \alpha_r : q\,;\ \beta_s : z)$$

$$= G_{q+1\,p}^{p1}\left(z \middle| \begin{matrix} 1, \beta_1, \ldots, \beta_q \\ \alpha_1, \ldots, \alpha_p \end{matrix}\right)$$

$$= G_{pq+1}^{1p}\left(\frac{1}{z} \middle| \begin{matrix} 1-\alpha_1, \ldots, 1-\alpha_p \\ 0, 1-\beta_1, \ldots, 1-\beta_q \end{matrix}\right) \quad \ldots \quad \ldots (9)$$

(*a*) When we take $m = l+n$, $\mu = 1$, $\lambda = s = p$, $r = q+1$, $a_1 = 1$, $a_{j+1} = \beta_j$, $b_j = \alpha_j$ in (7), we get the value of (1) as given by MacRobert (1958), when l and n both are negative integers.

Similarly by giving other suitable values to the parameters we have—

(*b*) for $p+1 > q$, $|\arg z| < (p-q+1)\frac{\pi}{2}$, $R(\alpha+l\alpha_j) > 0$, and $R(\beta+n\alpha_j) > 0$, l and n being positive integers,

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}E[p\,;\ \alpha_r : q\,;\ \beta_s : z\lambda^l(1-\lambda)^n]\,d\lambda$$

$$= (2\pi)^{\frac{1}{2}}l^{\alpha-\frac{1}{2}}n^{\beta-\frac{1}{2}}(l+n)^{\frac{1}{2}-\alpha-\beta}$$

$$\times G_{q+l+n+1\,p+l+n}^{p\,l+n+1}\left(\frac{l^ln^n}{(l+n)^{l+n}}z \middle| \begin{matrix} 1, \frac{1-\beta}{n}, \ldots, \frac{n-\beta}{n}, \frac{1-\alpha}{l}, \ldots, \frac{l-\alpha}{l}, \beta_1, \ldots, \beta_q \\ \alpha_1, \ldots, \alpha_p, \frac{1-\alpha-\beta}{l+n}, \ldots, \frac{l+n-\alpha-\beta}{l+n} \end{matrix}\right) \quad (10)$$

(c) for $p+1 > q$, $|\arg z| < (p-q+1)\frac{\pi}{2}$, $R(\alpha) > 0$, $R(\beta+n\alpha_j) > 0$, $(j = 1, \ldots, p)$ and $l > n$, both being positive integers,

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}E[p\,;\alpha_r:q\,;\beta_s:z\lambda^{-l}(1-\lambda)^n]\,d\lambda$$

$$= (2\pi)^{\frac{1}{2}-n}l^{\alpha-\frac{1}{2}}n^{\beta-\frac{1}{2}}(l-n)^{\frac{1}{2}-\alpha-\beta}$$

$$\times G^{p+l\,n+1}_{q+l+1\,p+l}\left(\frac{(l-n)^{l-n}}{l^l n^{-n}}z\,\middle|\,\begin{matrix}\frac{1-\beta}{n},\ldots,\frac{n-\beta}{n},1,\beta_1,\ldots,\beta_q,\frac{\alpha+\beta}{l-n},\ldots,\frac{\alpha+\beta+l-n-1}{l-n}\\ \frac{\alpha}{l},\ldots,\frac{\alpha+l-1}{l},\alpha_1,\ldots,\alpha_p\end{matrix}\right) \quad \ldots (11)$$

(d) for $p+1 > q$, $|\arg z| < (p-q+1)\frac{\pi}{2}$, $R(\beta) > 0$, $R(\alpha+l\alpha_j) > 0$, $(j = 1, \ldots, p)$ and $l > n$, both being positive integers,

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}E[p\,;\alpha_r:q\,;\beta_s:z\lambda^{l}(1-\lambda)^{-n}]\,d\lambda$$

$$= (2\pi)^{\frac{1}{2}-n}l^{\alpha-\frac{1}{2}}n^{\beta-\frac{1}{2}}(l-n)^{\frac{1}{2}-\alpha-\beta}$$

$$\times G^{p+n\,l+1}_{q+l+1\,p+l}\left(\frac{l^l z}{n^n(l-n)^{l-n}}\,\middle|\,\begin{matrix}1,\frac{1-\alpha}{l},\ldots,\frac{l-\alpha}{l},\beta_1,\ldots,\beta_q\\ \frac{\beta}{n},\ldots,\frac{\beta+n-1}{n},\alpha_1,\ldots,\alpha_p,\frac{1-\alpha-\beta}{l-n},\ldots,\frac{l-n-\alpha-\beta}{l-n}\end{matrix}\right). \quad \ldots (12)$$

Values of the integrals in (11) and (12), when $l < n$ can be derived from them by substituting $\lambda = (1-t)$.

The right-hand side of the integrals (10), (11) and (12) can be expressed in terms of E-functions as well, since

$$G^{mn}_{pq}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$$

$$=\sum_{h=1}^{n}\frac{\pi^{m+n-q-1}\prod_{j=m+1}^{q}\sin(a_h-b_j)\pi}{\prod_{j=1,\,j\neq h}^{p}\sin(a_h-a_j)\pi}z^{a_h-1}$$

$$\times E\begin{bmatrix}1-a_h+b_1,\ldots,1-a_h+b_q\\ 1-a_h+a_1,\ldots,1-a_h+a_{h-1},1-a_h+a_{h+1},\ldots,1-a_h+a_p:(-1)^{q-m-n-1}z\end{bmatrix}, \quad (13)$$

and

$$G^{mn}_{pq}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$$

$$=\sum_{h=1}^{m}\frac{\pi^{m+n-p-1}\prod_{j=n+1}^{p}\sin(a_j-b_h)\pi}{\prod_{j=1,\,j\neq h}^{q}\sin(b_j-b_h)\pi}z^{b_h}$$

$$\times E\left[\begin{matrix}1+b_h-a_1, \ldots, 1+b_h-a_p \\ 1+b_h-b_1, \ldots, 1+b_h-b_{h-1}, 1+b_h-b_{h+1}, \ldots, 1+b_h-b_q\end{matrix} : (-1)^{m+n+p-1}\frac{1}{z}\right] \tag{14}$$

The value of the integral in (b) in terms of E-functions was given by MacRobert (1958).

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