TRANSFORMATION OF BIRKHOFF'S RINGS INTO THEMSELVES AND ROTATIONAL NUMBERS

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This paper aims to find out the transformation of Birkhoff's rings and the rotational numbers defined thereby. To achieve the goal a transformation corresponding to the ring for $\mu = 0$ has been obtained and then by analytical continuation this has been extended for $\mu \neq 0$. The properties, which are of fundamental importance, of the transformation of the ring into itself have also been studied in detail. Ultimately it is seen that the above transformation defines two rotational numbers, one corresponding to the inner and the other corresponding to the outer boundary of the ring. These rotational numbers play an important role in showing the existence of symmetrical periodic orbits and thereby in stability of the motion in space restricted problem of three bodies.

1. INTRODUCTION

This work is primarily based on Merman's long paper on the coplanar restricted problem of three bodies. Here I am presenting its three-dimensional generalization. I shall be exploiting Merman's works and my two papers published in this direction (Choudhry 1963, 1964: hereafter referred to as I and II respectively) at every step. Some of the main results obtained in the previous papers are given here in a nutshell.

In I the analytical continuation for $\mu \neq 0$ for the direct and retrograde circular orbits corresponding to $\mu = 0$ has been studied. The parametric representation of such a continuation is given as follows:

$$\left.\begin{aligned} \rho' &= \rho\left(\sqrt{1-\mu}+\mu\rho^2 f(\rho,\theta,\mu)\right) \\ \theta' &= \theta+\rho^2\mu g(\rho,\theta,\mu) \\ z &= \rho^2\mu h(\rho,\theta,\mu) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where ρ', θ', z coincide with ρ, θ, O for $\mu = 0$.

In II the solution has been regularized and the equations to the trajectory are found in the form

$$\left.\begin{aligned} p &= \rho' \cos\frac{\theta'}{2} \\ q &= \rho' \sin\frac{\theta'}{2} \\ z &= \mu\rho'^2 h_6(\rho', \theta', \mu) \\ p' &= -\sqrt{\frac{4(2\Omega-C)r_1^2}{p^2+q^2}-\frac{1}{4(p^2+q^2)}\left(\frac{dz}{d\tau}\right)^2}\sin\frac{\theta''}{2} \\ q' &= \sqrt{\frac{4(2\Omega-C)r_1^2}{p^2+q^2}-\frac{1}{4(p^2+q^2)}\left(\frac{dz}{d\tau}\right)^2}\cos\frac{\theta''}{2} \\ z' &= \mu\rho'\bar{h}(\rho', \theta', \mu). \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (2)$$

This paper concludes with one-to-one correspondence between the points on the trajectory (2) and the rings

$$\left.\begin{aligned} x &= (A+\rho)\cos\theta \\ y &= (A+\rho)\sin\theta \end{aligned}\right.$$

where

$$\left.\begin{aligned} -\sqrt{a_1(C)} &\leqslant \rho \leqslant \sqrt{a_2(C)} \\ 0 &\leqslant \theta < 2\pi. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

These rings are called Birkhoff's rings for $\mu \neq 0$.

In the present paper we shall be taking up the study of the rings (3) in detail. The knowledge of my previous papers will be assumed and sometime even the references will be made to these articles without much explanation. Our aim will be to show the existence of the symmetrical periodic orbits in the restricted problem of three bodies in three-dimensional coordinate system. It will be accomplished in some other paper.

2. Transformation of the ring into itself for $\mu = 0$

We shall consider here the trajectories of the motion inside the space of the zero velocity in the coordinate system of Levi-Civita (p, q, z). We know that

$$p+iq = \sqrt{x+iy} = \sqrt{r}e^{\frac{iw}{2}}, \; w = u-t, \; u = v+\omega$$

where w, u are the longitudes w.r.t. the rotating and the fixed system of axes respectively and v and w are the true anomaly and the argument of the perihelion. Thus

$$p\cos\frac{t-\omega}{2} - q\sin\frac{t-\omega}{2} = \sqrt{r}\cos\frac{v}{2} = \sqrt{a(1-e)}\cos\frac{E}{2}$$

$$p\sin\frac{t-\omega}{2} + q\cos\frac{t-\omega}{2} = \sqrt{r}\sin\frac{v}{2} = \sqrt{a(1+e)}\sin\frac{E}{2}$$

where E is the eccentric anomaly and $E = 4na(\tau-\tau_0)$, τ_0 being some preassigned value for τ and

$$\left.\begin{aligned} p &= \sqrt{a}\left(\sqrt{1-e}\cos 2na(\tau-\tau_0)\cos\frac{t-\omega}{2}+\sqrt{1+e}\sin 2na(\tau-\tau_0)\sin\frac{t-\omega}{2}\right) \\ q &= \sqrt{a}\left(-\sqrt{1-e}\cos 2na(\tau-\tau_0)\sin\frac{t-\omega}{2}+\sqrt{1+e}\sin 2na(\tau-\tau_0)\cos\frac{t-\omega}{2}\right) \end{aligned}\right\} \quad (2.1)$$

$$t = 4a(\tau-\tau_0)+\frac{e}{n}\sin 4na(\tau-\tau_0).$$

Using

$$\bar{p} = p\cos\frac{t-\omega}{2}-q\sin\frac{t-\omega}{2}$$

$$\bar{q} = p\sin\frac{t-\omega}{2}+q\cos\frac{t-\omega}{2}$$

one finds that in the case $o < e < 1$, the trajectory takes the form of the fixed ellipse

$$\frac{\bar{p}^2}{a(1-e)}+\frac{\bar{q}^2}{a(1+e)} = 1$$

with

$$\bar{p} = \sqrt{a(1-e)}\cos 2na(\tau-\tau_0)$$

$$\bar{q} = \sqrt{a(1+e)}\sin 2na(\tau-\tau_0).$$

In the case $e = 1$, the trajectories w.r.t. the axes reduce to the st. line

$$\bar{p} = 0\,;\ \bar{q} = \sqrt{2a}\sin 2na(\tau-\tau_0).$$

In the case $e = 0$ in both of the coordinates the trajectory will be a circle

$$p^2+q^2 = a = \bar{p}^2+\bar{q}^2$$

where

$$p = \sqrt{a}\cos\left[2(n-1)a(\tau-\tau_0)+\frac{\omega}{2}\right],\ \bar{p} = \sqrt{a}\cos 2na(\tau-\tau_0)$$

$$q = \sqrt{a}\sin\left[2(n-1)a(\tau-\tau_0)+\frac{\omega}{2}\right],\ \bar{q} = \sqrt{a}\sin 2na(\tau-\tau_0).$$

From § 4 of I we find that these orbits involve C through a.

Now we shall prove the following theorem:

Theorem 1.

Each trajectory inside the circles $p^2+q^2 = a_1$ and $p^2+q^2 = a_2$ (for the retrograde and the direct motions respectively) is touched by some circle which becomes a trajectory for some other value of C greater than the fixed one corresponding to the trajectory.

Taking $\sqrt{a(1-e)}$ for the independent parameter in place of a in determining the orbit it is clear that on pq-plane or $\bar{p}\bar{q}$-plane the motion is possible only inside the circles

$$p^2+q^2 = a_1 \quad \text{(for retrograde motion)}$$

and

$$p^2+q^2 = a_2 \quad \text{(for direct motion).}$$

From (2.1), we find that

$$\begin{aligned} p^2+q^2 &= a[(1-e)\cos^2 2na(\tau-\tau_0)+(1+e)\sin^2 2na(\tau-\tau_0)] \\ &= a[1-e\cos 4na(\tau-\tau_0)]. \end{aligned}$$

For the tangency (or positive tangency in the words of Birkhoff)

$$\frac{d}{d\tau}(p^2+q^2) = 0 = 4na^2e\sin 4na(\tau-\tau_0)$$

$$\frac{d^2}{d\tau^2}(p^2+q^2) > 0$$

will give

$$\sin 4na(\tau-\tau_0) = 0, \quad \cos 4na(\tau-\tau_0) = 1$$

and hence

$$\tau_k = \tau_0+\frac{k\pi}{2na} = \tau_0+k\frac{\pi\sqrt{a}}{2} \quad (k = 1, 2 \ldots).$$

In this way we find that a trajectory will be touching after each interval of $\frac{\pi\sqrt{a}}{2}$ for $o < e < 1$.

For $e = 1$, this property of min (p^2+q^2) is preserved and then min $(p^2+q^2) = 0$. Thus our theorem is proved for $o \leqslant e \leqslant 1$.

Theorem 2.

For any point K of a circle of the ring we have a positively tangential orbit which will be again positively tangential at a point K_1 for the first time. In this way a one-to-one transformation of the ring into itself, taking any point K into the corresponding point K_1 is obtained which leaves radial distance from J unchanged and regresses each point by a central angle $2\pi a^{3/2}$ about J, where a is the semi-major axis of the ellipse of motion of the negligible body.

We have seen in the last theorem that to any circle a positively tangential orbit is available at the moments

$$\tau = \tau_0+\frac{k\pi}{2na} \quad (k = 1, 2, 3, \ldots).$$

The corresponding p, q given by (2.1) are

$$p_k = (-1)^k\sqrt{a(1-e)}\cos\left(\frac{\omega}{2}-\frac{k\pi}{n}\right)$$

$$q_k = (-1)^k\sqrt{a(1-e)}\sin\left(\frac{\omega}{2}-\frac{k\pi}{n}\right).$$

We know by § 4 of II that

$$p = \rho \cos \frac{\theta}{2}, \; q = \rho \sin \frac{\theta}{2}$$

$$\rho = \sqrt{p^2+q^2}, \; \theta = 2 \text{ arc tg } \frac{q}{p}$$

whence

$$K(\rho, \theta): \; \rho = \sqrt{a(1-e)}, \; \theta = \omega - \frac{2k\pi}{n}$$

$$K_1(\rho_1, \theta_1) = TK(\rho, \theta): \; \rho_1 = \sqrt{a(1-e)}, \; \theta_1 = \omega - \frac{2(k+1)\pi}{n}.$$

Subtracting the two, we shall get the form of the transformation T in the form

$$\boxed{\rho_1 = \rho \; ; \; \theta_1 = \theta - 2\pi a^{\frac{3}{2}}} \qquad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

which shows that the radial distances from J remain unchanged and the central angle θ regresses by the amount $2\pi a^{3/2}$ which proves the theorem completely.

3. Positively tangential orbits for $\mu \neq 0$

Theorem 1.

For μ sufficiently small, every orbit becomes positively tangent to one of the auxiliary curves of the ring (the analytical continuation of a circle) within a suitable interval of time τ, where τ is independent of the orbit.

For $\mu \neq 0$, let us assume that the theorem does not hold good. It will mean that however large τ may be and however small μ may be there exists no auxiliary curve which can touch an orbit in the space, i.e. there exists a sequence $\{\mu_i\}$ ($\lim \mu_i \to 0$) and $\{\tau_i\}$ ($\lim \tau_i \to \infty$) that corresponding to these μ_i and the arcs of the orbit $A_i B_i$ of length τ_i, the orbit has got no point of contact with the auxiliary curves. Let us consider the set of points $\{A_i\}$ ($i = 1, 2, 3, \ldots$) satisfying the 1st axiom of countability. As this set is bounded by the surface of zero velocity and thus by Bolzano-Weierstrass's theorem, the set $\{A_i\}$ must have at least one limit point inside the limiting surface corresponding to $\mu = 0$ for all A_i lie inside the surface of zero velocity for μ_i and $\mu_i \to 0$ as $i \to \infty$. Let us select the subsequence from $\{A_i\}$ which tend to the limit L and let it be denoted again by $\{A_i\}$.

Let us first prove the theorem for $z = 0$.

We have three cases corresponding to the orbits for $\mu = 0$ passing through the limiting point L. The orbit may be a rotating ellipse, a rotating st. line or a circle. Since $\tau_i \to \infty$ as $i \to \infty$, let us choose $\tau_i - \tau_0 > \frac{\pi}{2}\sqrt{a_2(c)}$ for $i \geqslant i_0$ where i_0 is some preassigned number.

Case I.—Let the limiting trajectory passing through L for $\mu = 0$ be a rotating ellipse or a rotating st. line. Let LM be the limiting trajectory for $\mu_i \to 0$ and thus LM has got K, a point of tangency with some circle for $LM > \frac{\pi}{2}\sqrt{a_2(c)}$ and at any other in the neighbourhood of the point K, however

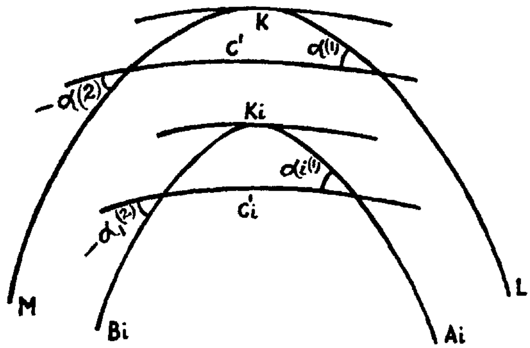

FIG. 1.

small may be, the trajectory LM will be intersected by circles and thus the angles of intersection $\alpha^{(1)}$ and $\alpha^{(2)}$ must be of opposite sign.

Since all the conditions of Lipschitz (Bellman 1953) are satisfied by the expressions on the right-hand side of (3.6) of II, this shows that the solutions will depend on the initial conditions and parameter μ continuously. Thus for the trajectories passing through A_i sufficiently near to L, the coordinates and the velocity component for any $\tau \leqslant \frac{\pi}{2}\sqrt{a_2(c)}$ will be very near to the coordinates and velocity components for the same τ for the limiting trajectory passing through L. The same relation will hold good between the coordinates and the velocity components for the circles and their analytical continuation for sufficiently small μ.

Taking into consideration the above statement, we find that the trajectories A_i B_i are being intersected by analytical continuation or (auxiliary curves) at angles which are opposite in sign. Regarding these angles as functions of the variable τ, we find that there must exist a point K_i at which these angles are zero. Thus each trajectory will be touched at some point K_i by an auxiliary curve. Taking the point of contact at the origin and remembering the st. line orbit, we find that the tangency is preserved for the case when the orbit is a rotating st. line.

Since for any point on the trajectory in the space

$$z = \mu(p^2+q^2)h_6'(p, q, \mu)$$

$$z' = \mu\sqrt{p^2+q^2}\bar{h}'(p, q, \mu)$$

where h_6' and $\bar{h}'$ are single-valued analytic functions in p, q and μ and thus if the trajectory is touched by an auxiliary curve on pq-plane, then the property will be preserved even in space.

Case II.—Here the limiting trajectory reduces to a circle LM (e.g. of radius $\sqrt{a_2}$). The differential equation in variation along the normal to the circular trajectory for $\mu = 0$ can be written in the form (Merman 1961)

$$\frac{d^2\delta p_n}{d\tau^2}+\frac{16}{a_2}\,\delta p_n = 0$$

which gives the expression for the normal variation for the circular orbit for $\mu = 0$.

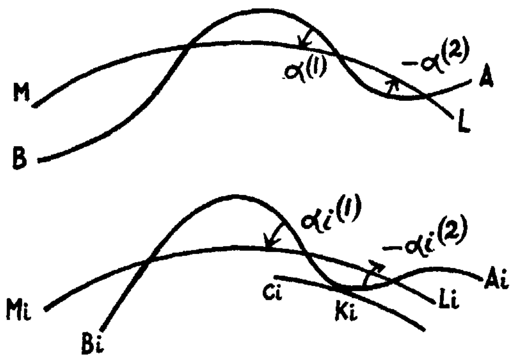

FIG. 2.

Thus

$$\delta p_n = A\,\sin\frac{4}{\sqrt{a_2}}(\tau-\tau_0) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

This function reduces to zero for $\tau = \tau_0+\frac{\pi\sqrt{a_2}}{4}k$ as after the interval $\frac{\pi\sqrt{a_2}}{4}$ the function δp_n changes its sign. It shows that after each interval of $\frac{\pi\sqrt{a_2}}{4}$ the trajectory will be intersected by the limiting circle LM. These points of intersection may be internal or external. Between each two internal type of points of intersection, there will be a moment corresponding to the minimum of the radius vector and between each two consecutive external type of points of intersection, there will be a moment for the maximum of the radius vector. The minimum radius vector gives us the positive tangency and in any neighbourhood of this minimum radius vector the orbit will be intersected by the circle at angles which are opposite in sign.

Just as in case I, here also we find that for a trajectory A_i B_i we must find at least a point K_i at which it touches the analytical continuation (or the auxiliary curve). Thus the tangential character is proved for pq-plane and then through the transformation for z and z' in terms of p, q, μ the property is generalized for space.

The existence of such a positively tangential orbit to an analytical continuation contradicts our hypothesis and this contradiction proves our theorem.

Theorem 2.

There is a one-to-one correspondence between the points of contact of the trajectory with the analytical continuation and the points of the ring (7.3) of II.

To prove the theorem let us show that the point of contact satisfies all the equations (5.8) of II. For any fixed value of C, the coordinates of the point of contact must satisfy $F = 0$ which is Jacobi's integral since the trajectory lies on the surface $F = 0$. This point of contact is of the trajectory with the analytical continuation and hence the point will satisfy the equations

$$\left.\begin{aligned} p^2+q^2+\mu(p^2+q^2)^2\Phi(p, q, \mu) &= (1-\mu)\rho^2 \\ z &= \mu(p^2+q^2)h_6'(p, q, \mu) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.2)$$

Again for the point of contact p, p', q, q', z, z' are the same for the trajectory and the analytical continuation and thus it shows that the coordinates of the point of contact will satisfy the eq.

$$G \equiv pp'+qq'+\mu(p^2+q^2)[A(p, q, \mu)p'+B(p, q, \mu)q'] = 0$$

also. Thus all the eqns. (5.8) of II are satisfied. The eqns. (3.2) determine the region of variation for ρ:

$$-\sqrt{a_1} < \rho < \sqrt{a_2}$$

and so the eqns. (5.4, II) give a definite (ρ, θ) for a point of contact and corresponding to this (ρ, θ), we shall have a corresponding point on the ring (7.3, II).

Thus we established a one-to-one correspondence between the points of contact of the trajectory with the analytic continuation and the points of the ring (7.3, II).

4. Some of the properties of the transformation of the ring into itself for $\mu \neq 0$

It can be easily seen for pq-plane as well as for space that for a given μ, taken sufficiently small, no points of positive tangency on an orbit can coincide or disappear as the orbit undergoes (Birkhoff 1915) continuous variation, whence it is clear that any orbit will touch an analytical continuation infinitely many times and all the points of contact will be distinct.

Let $K(\rho, \theta)$ be an arbitrary point inside the ring (7.3, II) with the restrictions (7.1, II). With this (ρ, θ) we can find a point $M(p, q, z)$ where the auxiliary curve has got a positively tangent orbit and let $M_1(p_1, q_1, z_1)$ be the next point of contact and the corresponding point of the ring be $K_1(\rho_1, \theta_1)$. Let us assume that there exists a transformation T which is such that

$$K_1(\rho_1, \theta_1) = TK(\rho, \theta).$$

Here let us note down some of its properties:

Property 1.

$T(K) \neq K$ or equivalently $T(K') \not\rightarrow K$ when $K' \rightarrow K$ or T is continuous throughout the ring.

Let us consider the trajectory in the phase space $p = p(\tau)$, $q = q(\tau)$, $z = z(\tau)$, $p' = p'(\tau)$, $q' = q'(\tau)$, $z' = z'(\tau)$ passing through $M_0(p_0, q_0, z_0, p_0', q_0', z_0')$ corresponding to some point $K_0(\rho_0, \theta_0)$ of the inside of the circular ring. Let

$$p_0 = p(\tau_0),\ q_0 = q(\tau_0),\ z_0 = z(\tau_0) \ldots$$

Take $K_1(\rho_1, \theta_1) = TK(\rho_0, \theta_0)$ and let to the point $K_1(\rho_1, \theta_1)$ correspond the point $M_1(p_1, q_1, z_1, p_1', q_1', z_1')$ of the phase space. We have already seen that both the points M_0 and M_1 lie on the surface $G = 0$. Let the point $K_1 \rightarrow K_0$, but $K_1 \neq K_0$. Then $M_1 \rightarrow M_0$, but $M_1 \neq M_0$ and $\tau_1 \rightarrow \tau_0$ but $\tau_1 \neq \tau_0$. This shows that

$$G(\tau_0) = G(\tau_1) = 0$$

$$O = G(\tau_1) - G(\tau_0) = \frac{dG(\tau)}{d\tau}\bigg|_{\tau_0+\epsilon(\tau_1-\tau_0)} (\tau_1 - \tau_0) \quad (|\epsilon| \leqslant 1).$$

This will mean that $\frac{dG}{d\tau} = 0$ if $\tau_1 \neq \tau_0$ at $\tau = \tau_0$.

For $\mu = 0$, the equations of motion are

$$\frac{d^2p}{d\tau^2} - 8(p^2+q^2)\frac{dq}{d\tau} = \bar{\Omega}_p(p, q, C)$$

$$\frac{d^2q}{d\tau^2} + 8(p^2+q^2)\frac{dp}{d\tau} = \bar{\Omega}_q(p, q, C)$$

$$\left(\frac{dp}{d\tau}\right)^2 + \left(\frac{dq}{d\tau}\right)^2 = 2\bar{\Omega}(p, q, C)$$

$$\bar{\Omega}(p, q, C) = 2(p^2+q^2)[(p^2+q^2)^2 - C] + 4$$

and
$$G \equiv p\frac{dp}{d\tau} + q\frac{dq}{d\tau}.$$

Here we find that

$$\frac{dG}{d\tau} = \left(\frac{dp}{d\tau}\right)^2 + \left(\frac{dq}{d\tau}\right)^2 + p\frac{d^2p}{d\tau^2} + q\frac{d^2q}{d\tau^2}$$

$$= 2\bar{\Omega}(p, q, C) + p\bar{\Omega}_p(p, q, C) + q\bar{\Omega}_q(p, q, C) + 8(p^2+q^2)\left(p\frac{dq}{d\tau} - q\frac{dp}{d\tau}\right).$$

Putting

$$p = \rho\cos\frac{\theta}{2},\ q = \rho\sin\frac{\theta}{2},\ \frac{dp}{d\tau} = -\sqrt{2\bar{\Omega}}\sin\frac{\theta}{2},\ \frac{dq}{d\tau} = \sqrt{2\bar{\Omega}}\cos\frac{\theta}{2}$$

we find that

$$\frac{dG}{d\tau} = -8\frac{(2\rho^3+C\rho^2+1)(2\rho^3-C\rho^2+1)}{2\rho^6-C\rho^2+2-2\rho^3\sqrt{\rho^6-C\rho^2+2}}$$

which shows that

$\frac{dG}{d\tau} = 0$ when $C = \pm 2\rho + \frac{1}{\rho^2}$, i.e. when $\rho^2 = a_1$ or a_2, i.e. on the boundaries of the ring.

Hence $\frac{dG}{d\tau} \neq 0$ inside the ring. This contradicts our assumption which is based on the statement that $\tau_1 \to \tau_0$ and thus $\tau_1 \not\to \tau_0$ inside the ring. Therefore, a number $\mu^{(0)}(\epsilon)$ can always be chosen, however small ϵ may be, such that in the region

$$-\sqrt{a_1}+\epsilon \leqslant \rho \leqslant \sqrt{a_2}+\epsilon, |\mu| \leqslant \mu^{(0)}(\epsilon),$$

we shall have

$$\frac{dG}{d\tau} \neq 0.$$

The above statement proves our theorem inside the ring

$$-a_1 < \rho < a_2.$$

Now it remains to prove the theorem in the neighbourhood of the boundaries of the rings, i.e. in the regions

$$-\sqrt{a_1} < \rho < -\sqrt{a_1}+\epsilon, \sqrt{a_2}-\epsilon < \rho_2 < \sqrt{a_2}.$$

For this let us find out an approximate expression of the transformation T for $\mu \neq 0$. From the above statement it is clear that the point of contact will satisfy the conditions

$$G(\tau) = 0 \text{ and } \frac{dG}{d\tau} \neq 0$$

for any τ.

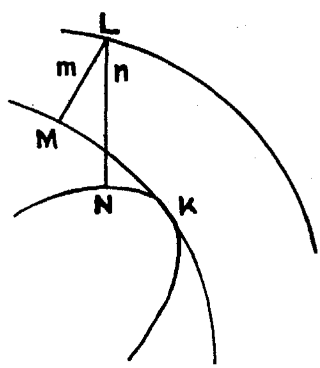

FIG. 3.

There is no loss of generality if we concentrate ourselves only on the projections of the trajectory and the analytical continuation on the pq-plane. Let $L(\tau)$, $M(\tau)$ and $N(\tau)$ be the points on the boundary of the ring, the analytical continuation and the trajectory respectively at the same moment τ. Let $m(\tau)$ and $n(\tau)$ be the normal distances from L to the points M and N. Thus for the point of contact K, we shall have the equations

$$n(\tau) = m(\tau)\,; \; \frac{dn(\tau)}{d\tau} = \frac{dm(\tau)}{d\tau} \qquad \text{.. .. ..} \qquad (4.1)$$

By (3.1), we can say that $n(\tau)$ and $m(\tau)$ are nearly given by

$$n(\tau) = \delta p_n = A \sin \frac{4}{\sqrt{a_2}}(\tau - \tau_0),\; m = \delta m_0 = \text{const}$$

during any interval of time if $|\mu|$ and ϵ are very small. Consequently, the solution τ for the eqns. (4.1) will be very near to

$$A \sin \frac{4}{\sqrt{a_2}}(\tau - \tau_0) = \delta m_0$$

$$\frac{4}{\sqrt{a_2}} A \cos \frac{4}{\sqrt{a_2}}(\tau - \tau_0) = 0.$$

Since A is of invariable sign, so

$$\tau = \tau_k = \tau_0 + \frac{(4k+1)\sqrt{a_2}}{8} = \text{const} + \frac{k\sqrt{a_2}}{2}\; (k = 0, 1, 2 \ldots) \qquad \text{..} \qquad (4.2)$$

and thus we found that a simple root for (4.1) exists.

It means that there exists such sufficiently small numbers $\mu_0 > 0$, $\epsilon > 0$ that for $|\mu| \leqslant \mu_0$, $\epsilon \leqslant \epsilon_0$ the above equations have got a simple root in the region $|\mu| \leqslant \mu_0$, $\sqrt{a_2} - \epsilon < \rho < \sqrt{a_2}$. Combining the above results, we get that for $|\mu| \leqslant \min\{\mu_0, \mu^{(0)}(\epsilon)\}$ the transformation T will be continuous everywhere inside the ring, i.e. in the region $-\sqrt{a_1} < \rho < \sqrt{a_2}$.

It is evident that along with the transformation T, a transformation T^{-1} inside the ring is defined which is continuous with T as if $K_1 = T(K)$, then $K = T^{-1}(K_1)$, for K and K_1 both lie inside the ring.

Now let K be an arbitrary point of the boundary of the ring. Consider such a sequence of points K' inside the ring that $K' \to K$. Then by the continuity of the transformation T we have $T(K') \to T(K)$. Evidently $T(K)$ will lie on the boundary of the ring for if $T(K)$ does not lie on the ring, then $K = T^{-1}\, T(K)$ will be inside the ring.

Property 2.

The transformation T for $\mu = 0$ given by (2.2) also shows its monotonic character as we have marked earlier. This character will be preserved for the transformation T corresponding to $\mu \neq 0$ by virtue of its being a continuous function of μ.

Property 3.

$T = RU$ where R and U are the transformations of the ring into itself, R is the reflexive mapping on the x-axis and U possesses the property that $U^2 = I$ (I being identical transformation).

Proof:

We have seen in § (3, I) that if the equations of motion be of the form (3.2, I) and if

$$x_i = x_i(t),\ y_i = y_i(t) \quad (i = 1, 2, 3)$$

and as well as

$$x_i = x_i(-t),\ y_i = -y_i(-t) \quad (i = 1, 2, 3)$$

are solutions, then the former solution is said to be symmetric (a simple analysis of these conditions show that the solution is symmetrical about the x-axis).

From these equations one can say that two solutions (x_i, y_i) and $(\bar{x}_i, \bar{y}_i)$ will be symmetrical about the x-axis if $x_i(0) = \bar{x}_i(0)$, $y_i(0) = -\bar{y}_i(0)$.

Let K be an arbitrary point of the ring. Let $L = T(K)$, $K' = R(K)$, $L' = R(L)$, then we shall prove that $K' = T(L')$. Let $\bar{K}, \bar{L}, \bar{K}', \bar{L}'$ be the points on the trajectory corresponding to the points K, L, K', L' on the ring.

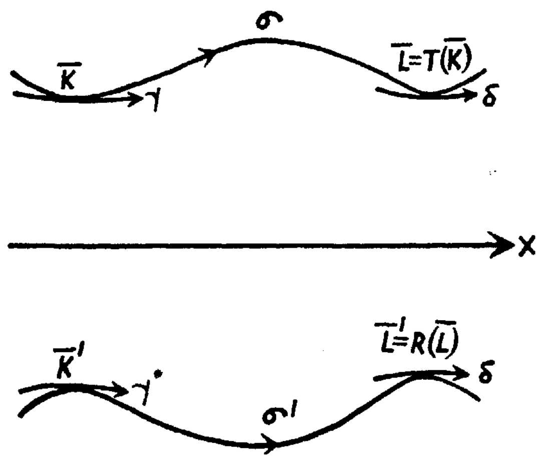

FIG. 4.

By the definition of the transformation T, the trajectory σ passing through $\bar{K}$ and touching some auxiliary curve γ will necessarily pass through $T(\bar{K}) = \bar{L}$ and will touch some auxiliary curve δ at it. γ and δ both of them are symmetrical about the x-axis. Again, it is clear that the trajectory σ' through $\bar{K}'$ will necessarily touch the reflection of γ into the x-axis, i.e. γ itself.

Now as $\bar{K}$ and $\bar{K}'$ lie on σ and σ' which satisfy the initial conditions for symmetry and thus if $\bar{L}$ corresponds to the moment $t = \tau$, then a point $\bar{L}'$ must correspond to $t = -\tau$ on the trajectory σ'. But $\bar{L}$ is a point of contact of the trajectory with the analytical continuation δ for the moment $t = \tau$, so $\bar{L}'$ also must be the point of contact of the curve σ' with the analytical continuation δ. This shows that $\bar{L}' = T^{-1}(\bar{K}')$, i.e. $T(\bar{L}') = \bar{K}'$ and this is equivalent to the statement that $K' = T(L')$.

Hence we have $K = R(K') = RT(L') = RTR(L) = RTRT(K)$. Introducing $U = RT$, we find $K = U^2(K)$ and thus $U^2 = I$.

Property 4.

Transformations R, T, U satisfy the relations $U^{-1} = U$, $TU = R$.

Denoting $U^{(k)} = RT^{(k)}$, it is found that $\{U^{(n)}\}^2 = 1$ for all integral n.
DEFINITION: T is said to be equivalent to T^{-1} if a transformation A exists such that $T^{-1} = ATA^{-1}$.

5. Transformation defining the coefficient of rotation or the rotational numbers (Merman 1961)

Let us consider the closed curve, the points P of which are defined by the angular coordinate τ (τ being different from τ used in § 3, II), i.e. to each value of τ from the range $o \leqslant \tau \leqslant 2\pi$, there corresponds one definite point $P(\tau)$ of the curve and conversely to each point P of the closed curve let there correspond a number $\tau(P)$ of the range stated above so that for the variation of τ from $\tau = 0$ to $\tau = 2\pi$, the whole curve can be obtained or for the variation of τ we can also take the range $-\infty < \tau < +\infty$, but then we shall have $P(\tau+2k\pi) \equiv P(\tau)$.

Let a continuous and monotonic transformation of this curve into itself defined by $f(\tau)$ be given such that to each point $P(\tau)$ corresponds another point $P_1(\tau_1)$ of the same curve where $\tau_1 = f(\tau)$. Let us note the following properties of the transformation $f(\tau)$ which were established by Poincare (1885).

Property 1.

The transformation is such that if P precedes Q, i.e. $\tau(P) < \tau(Q)$, then $P_1(\tau_1)$ precedes $Q_1(\tau_1)$.

(This property will be used for the definition of the transformation $f(\tau)$ in the further treatment).

Property 2.

$$f(\tau+2l\pi) \equiv f(\tau)+2l\pi \quad (l = 1, 2, 3 \ldots) \qquad \ldots \qquad (5.1)$$

Property 3.

If $\tau_k = f(\tau_{k-1})$ $(k = 1, 2, 3 \ldots)$, $\tau_0 = \tau$, then the variation of τ by $2l\pi$ leaves $\tau_k - \tau$ unchanged.

For $k = 1$, the statement follows from (5.1) and assumes the property to be true for $k = k-1$. Let

$$\tau_{k-1}(\tau+2l\pi) = \tau_{k-1}(\tau)+2l\pi.$$

Then

$$\begin{aligned}\tau_k(\tau+2l\pi) &= f[\tau_{k-1}(\tau+2l\pi)]\\ &= f[\tau_{k-1}(\tau)+2l\pi]\\ &= f[\tau_{k-1}(\tau)]+2l\pi\\ &= \tau_k(\tau)+2l\pi.\end{aligned}$$

Thus the property is established. By the definition of the transformation we find that it is sufficient to consider the points $P(\tau)$ in the region $o \leqslant \tau < 2\pi$ and by virtue of the property 3, we find that the sequence $\{\tau_k-\tau\}$ is bounded both above and below. So there will exist $a^{(k)}$ and $b^{(k)}$ such that

$$a^{(k)} \leqslant \tau_k-\tau \leqslant b^{(k)}.$$

Property 4.

If l and k are two integers other than zero, then

$$\frac{a^{(l)}}{l} \leqslant \frac{b^{(k)}}{k}$$

for all l and k.

Property 5.

Any two intervals $\left(\frac{a^{(k)}}{k}, \frac{b^{(k)}}{k}\right)$ and $\left(\frac{a^{(l)}}{l}, \frac{b^{(l)}}{l}\right)$ have got common parts.

6. Rotation numbers and their properties

Since $b^{(k)}-a^{(k)} < 2\pi$, then $\lim\limits_{k\to\infty}\left(\frac{b^{(k)}}{k}-\frac{a^{(k)}}{k}\right) = 0$. This shows that from the sequence $\left\{\frac{a^{(k)}}{k}, \frac{b^{(k)}}{k}\right\}$ one can always choose a subsequence $\left\{\frac{a^{(k_r)}}{k_r}, \frac{b^{(k_r)}}{k_r}\right\}$ whose length tends to zero and one encloses the other. Then both the ends of the intervals will tend to a common limit α and this number has been called by Poincare the coefficient of rotation or the rotational number by Merman. α lies inside all the intervals of the sequence

$$\left(\frac{a^{(k)}}{k}, \frac{b^{(k)}}{k}\right) \quad (k = 1, 2 \ldots), \text{ i.e.}$$

$$\frac{a^{(k)}}{k} \leqslant \alpha \leqslant \frac{b^{(k)}}{k} \quad (k = 1, 2 \ldots)$$

whence $a^{(k)} \leqslant k\alpha \leqslant b^{(k)}$. Comparing this inequality with the inequality

$$a^{(k)} \leqslant \tau_k-\tau \leqslant b^{(k)}$$

we shall get

$$a^{(k)}-b^{(k)} \leqslant \tau_k-\tau-k\alpha \leqslant b^{(k)}-a^{(k)}.$$

From the inequality $a^{(k)} \leqslant k\alpha \leqslant b^{(k)}$ it follows that for some of the points of the closed curve

$$\tau_k - \tau \leqslant k\alpha$$

(in particular, for those points which attain the least value $a^{(k)}$ and for some of the points $\tau_k - \tau \geqslant k\alpha$)
(in particular, for those points which attain the greatest value $b^{(k)}$).

Hence by virtue of the continuity of $f(\tau)$ there exists one point such that $\tau_k - \tau = k\alpha$. It shows that the kth application of the transformation of the curve into itself rotates the point by the amount $k\alpha$. This statement signifies the name of rotational number.

Let the rotational number be commensurable with 2π, i.e. $\frac{\alpha}{2\pi} = \frac{p}{q}$. It shows that there exists at least one point which is an invariant point for each repetition of q number of times of the transformation of the closed curve into itself.

Theorem.

The transformations of the boundaries of the ring possess all the properties stated above and thus they define two rotational numbers corresponding to the two boundaries.

To prove the theorem we shall try to find out the explicit form of these transformations where we shall use the property that these transformations are the limits of the transformations over the circles of the ring when the circles tend to the boundaries.

We have already seen in § 4 that if τ and τ_1 are two consecutive roots of the equations

$$n(\tau) = m(\tau), \frac{dn(\tau)}{d\tau} = \frac{dm(\tau)}{d\tau} \quad \text{.. .. ..} \quad (6.1)$$

then the transformation, we are interested in, leads $n(\tau)$ to $n(\tau_1)$, i.e. if T be our transformation, then $Tn(\tau) = n(\tau_1)$. Here $m(\tau)$ and $n(\tau)$ are the normals from a point on the boundary to the auxiliary curve and the trajectory in the neighbourhood of the auxiliary curve corresponding to the common point of contact.

In general the length of a normal will depend on seven constants (six constants of integration and one the small parameter μ). As our intermediate orbit lies on the z-plane for $\mu = 0$ which shows that two constants of integration corresponding to z and $\dot{z}$ are merely functions of μ and they will vanish with μ. Amongst the five constants of integration one constant can be taken to be Jacobi's constant and the second one $\tau - \tau_0$. Thus

$$n(\tau) = n(\tau - \tau_0, C, C_1, C_2, \mu)$$

where C_1 and C_2 are the two arbitrary constants of integration and C is fixed for the particular trajectory. As in Batrakov (1955) we can take the quantities C_1 and C_2 as constants for the analytical continuation of the circular orbit for $\mu = 0$ and they may be determined by the initial values $n(0)$ and $n'(0)$. Thus there will be no loss of generality if we assume $C_1 = C_2 = 0$ for the analytical continuation. By virtue of the above property $n(\tau-\tau_0, C, C_1, C_2, \mu)$ can be expanded in the neighbourhood of the analytical continuation in powers of C_1, C_2 in the form

$$n(\tau-\tau_0, C, C_1, C_2, \mu) = \phi_0(\tau-\tau_0, C, \mu) + \sum_{m_1+m_2 \geqslant 1} \phi^{(m_1, m_2)}(\tau-\tau_0, C, \mu) C_1^{m_1} C_2^{m_2}$$

whose convergence for the sufficiently small values of $|C_1|$, $|C_2|$ follows from the fact that the solution is a holomorphic function of the initial values.

The quantity $m(\tau)$, for fixed C and μ, defined by the leaves of the ring in the region $-\sqrt{a_1} \leqslant \rho \leqslant \sqrt{a_2}$ depends on ρ and so

$$m(\tau) = m(\tau, C, \mu, \rho)$$

and for the point of contact

$$n(\tau-\tau_0, C, 0, 0, \mu) = m(\tau, C, \mu, \theta) \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

In the neighbourhood of the boundaries of the ring we can expand

$$m(\tau, C, \mu, \rho) = \psi^{(0)}(\tau, C, \mu) + \sum_{m=1}^{\infty} \psi_{\pm}^{(m)}(\tau, C, \mu)(\rho \pm \sqrt{a_{2,1}})^m .$$

Substituting these values in (6.1) we shall get two equations for the determination of τ and ρ for the fixed values for τ_0, C, μ, C_1, C_2. Let τ and ρ be given by

$$\tau_k = \tau_k(\tau_0, C, \mu, C_1, C_2)$$

$$\rho = \rho(\tau_0, C, \mu, C_1, C_2).$$

To find the transformation in the neighbourhood of the boundary of the ring we shall have to find out the limit of the transformation when $n(\tau) \to 0$. This shows that for the boundary each of the functions in (6.2) must be equal to zero. This limit corresponding to $n(\tau) \to 0$ is equivalent to the statement $C_1 \to 0$, $C_2 \to 0$. So far we are concerned with ρ we have $\lim_{\substack{C_1 \to 0 \\ C_2 \to 0}} \rho = \sqrt{a_{1,2}}$.

Then it remains to find the $\lim_{\substack{C_1 \to 0 \\ C_2 \to 0}} \tau_k$.

Writing the two eqns. (6.1) in the form

$$n(\tau)\frac{dm(\tau)}{d\tau} - m(\tau)\frac{dn(\tau)}{d\tau} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6.3)$$

and for limit, let us put $C_1 = C_2 = A$, then (6.3) can be put in the form

$$\sum_{m+n \geqslant 0} \left[f^{(m+1)}\psi_\tau^{(n+1)} - \psi^{(n+1)}f_\tau^{(m+1)}\right] A^m(\rho-\sqrt{a_2})^n = 0$$

where $f^m = \sum_{m_1+m_2} \phi^{(m_1, m_2)}$. Putting $A = 0 = \rho-\sqrt{a_{1,2}}$, the above equation reduces to

$$f^{(1)}(\tau-\tau_0, C, \mu)\frac{d}{d\tau}\psi^{(1)}(\tau, C, \mu)-\psi^{(1)}(\tau, C, \mu)\frac{d}{d\tau}f^{(1)}(\tau-\tau_0, C, \mu) = 0 \quad .. \quad (6.4)$$

which shows that

$$f^{(1)}(\tau-\tau_0, C, \mu) = \psi^{(1)}(\tau, C, \mu) \quad .. \quad .. \quad .. \quad (6.5)$$

is the solution of the equation where the constant of integration is included in $\psi^{(1)}(\tau, C, \mu)$. (6.5) completely determines the value of τ_k and also that τ_k is a continuous function of μ. For $\mu = 0$, (6.5) gives the solution

$$\tau = \tau_k = \text{const}+\frac{k\pi\sqrt{a_{1,2}}}{2}$$

as has been obtained in (4.2) and so for $\mu \neq 0$ the solution can be obtained for small μ as near as one chooses to the above solution. For $\mu = 0$ the transformations for the two boundaries are given in the form

$$\theta_1 = f_e(\theta) = \theta-2\pi a_2^{3/2}$$

$$= f_i(\theta) = \theta-2\pi a_1^{3/2}$$

where e denotes the external boundary and i, the internal boundary. Again as

$$\theta_1'-\theta_1 = f_{e,i}(\theta')-f_{e,i}(\theta) = \theta'-\theta$$

so if $P(\theta)$ precedes $Q(\theta')$, then $P_1(\theta_1)$ precedes $Q_1(\theta')$. For $\mu \neq 0$ also this property of precedence and consequence of the transformation is preserved for $\tau_{k+1}-\tau_k$ is bounded for all k. This property defines the coefficient of rotation and thus corresponding to the external and the internal boundaries we shall have two rotational numbers $\sigma_1(\mu)$ and $\sigma_2(\mu)$. Evidently $\sigma_i(0) = 2\pi a_i^{3/2} (i = 1, 2)$ and so $\sigma_i(\mu) = 2\pi a_i^{3/2}(\mu, C)$.

Lastly, one can notice that the analytical transformation T^k can be written in the form

$$T^k(\rho, \theta) = (\rho_k, \theta_k)$$

where

$$\rho_k = f_k(\rho, \theta), \theta_k = \phi_k(\rho, \theta)$$

and

$$f_k(\rho, \theta+2m\pi) = f_k(\rho, \theta), \phi_k(\rho, \theta+2m\pi) = \phi_k(\rho, \theta)+2m\pi$$

where m is an integer. The first property follows from the fact that geometrically the two points are coincident. Since function θ_1 is a monotonic function of θ and it proves the monotonic nature of $\phi_k(\rho, \theta)$ for a constant ρ

which goes to prove that $\phi_k(\rho, \theta)$ possesses all the properties possessed by τ_k and thus for small μ, $\phi_k(\rho, \theta)$ must satisfy the second property.

Hence the theorem is completely established.

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