THE SIMULATION OF THIRD ORDER SYSTEMS WITH A LEADING TIME-CONSTANT BY A SINGLE OPERATIONAL AMPLIFIER—IV

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In the previous papers (Wadhwa 1964*a*, *b*, *c*) on the same topic eight basic circuits, each employing three capacitors and six resistors and capable of simulating, under certain conditions, third order systems with a leading time-constant, were presented. In this paper two more basic circuits for simulating the same type of third order systems are presented. The circuits are analysed, the design formulae obtained and the conditions of their physical realizability discussed.

Introduction

In previous communications (Wadhwa 1964*a*, *b*, *c*) on this topic eight basic circuits, each capable of simulating, under certain conditions, a particular type of the general third order linear systems, that is systems with a leading time-constant, had been presented. The purpose of this paper is to present the remaining two basic circuits, discuss and obtain the design formulae and the conditions of physical realizability.

Third Order System Simulation

Two basic circuits, each employing three capacitors and six resistors (with a certain arbitrary choice of resistor values) and capable of simulating, under certain conditions, third order systems with a leading time-constant, that is

$$F(S) = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}, \qquad \ldots \qquad (1)$$

are shown in Fig. 1.

Transfer function for the circuit of Fig. 1(*a*) can be shown to be

$$\frac{E_0}{E_1} = -\frac{\left(\frac{\alpha\beta}{5\alpha+\alpha\beta+3}\right)(RC_5S+1)}{\frac{\alpha\beta}{(5\alpha+\alpha\beta+3)}R^3C_2C_5C_7S^3+\left[\frac{\alpha}{(5\alpha+\alpha\beta+3)}R^2C_2C_5+\frac{\alpha(\beta+1)}{(5\alpha+\alpha\beta+3)}R^2C_2C_7+\frac{3\alpha\beta}{(5\alpha+\alpha\beta+3)}R^2C_5C_7\right]S^2+\left[\frac{(2\alpha+1)}{(5\alpha+\alpha\beta+3)}RC_2+\frac{\alpha(\beta+3)}{(5\alpha+\alpha\beta+3)}RC_5+\frac{3\alpha(\beta+1)}{(5\alpha+\alpha\beta+3)}RC_7\right]S+1} \qquad \ldots \qquad (2)$$

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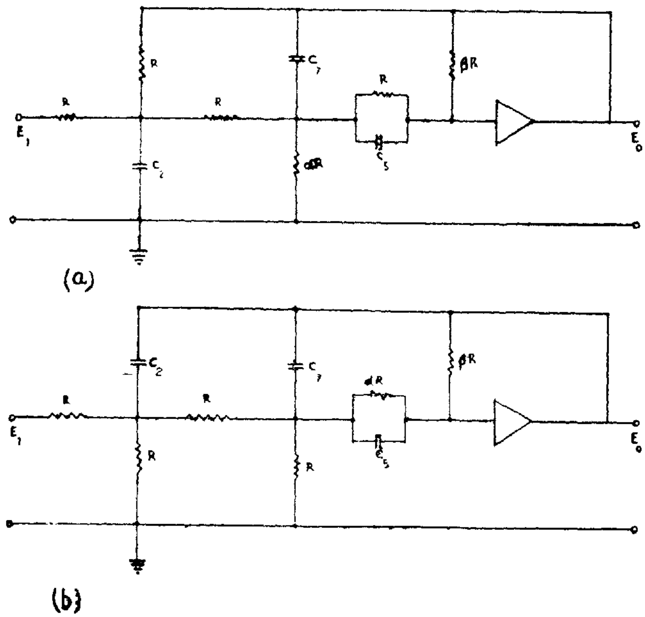

FIG. 1 Networks for the simulation of $\frac{E_0}{E_1} = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}$.

(1) and (2) will be identical if

$$b_0 = \frac{\alpha\beta}{(5\alpha+\alpha\beta+3)} \quad \ldots \quad (3)$$

$$b_1 = T_5 \quad \ldots \quad (4)$$

$$a_1 = \frac{(2\alpha+1)}{(5\alpha+\alpha\beta+3)}T_2 + \frac{\alpha(\beta+3)}{(5\alpha+\alpha\beta+3)}T_5 + \frac{3\alpha(\beta+1)}{(5\alpha+\alpha\beta+3)}T_7 \quad \ldots \quad (5)$$

$$a_2 = \frac{\alpha}{(5\alpha+\alpha\beta+3)}T_2T_5 + \frac{\alpha(\beta+1)}{(5\alpha+\alpha\beta+3)}T_2T_7 + \frac{3\alpha\beta}{(5\alpha+\alpha\beta+3)}T_5T_7 \quad \ldots \quad (6)$$

$$a_3 = \frac{\alpha\beta}{(5\alpha+\alpha\beta+3)}T_2T_5T_7 \quad \ldots \quad (7)$$

where

$$T_n = RC_n. \quad \ldots \quad (8)$$

Elimination of α, β, T_5 and T_7 from (3) to (7) gives a quartic

$$b_0b_1^2(1-b_0)T_2^4 - [3b_0b_1^2(a_1-b_0b_1) - a_3(1-2b_0) - a_2b_0b_1]T_2^3 + 3(3a_2b_0b_1^2 - a_1a_3)T_2^2 - 9a_3(3b_0b_1^2 - a_2)T_2 - 27a_3^2 = 0 \quad \ldots \quad (9)$$

which, as shown in Appendix I, will have at least one positive real root corresponding to which a set of positive real α, β, T_5 and T_7 exists, provided that

$$\left.\begin{array}{l} (1-b_0) > 0 \\ (a_1-b_0b_1) > 0 \\ b_1(a_1-b_0b_1)^2 > 4a_3(1-b_0) \\ (a_2b_1-a_3) > 0 \\ OB > OP > OA \end{array}\right\} \qquad (10)$$

and

$$\beta > \frac{5b_0}{(1-b_0)} \qquad (11)$$

where

$$\left.\begin{array}{l} OA = \dfrac{3b_1(a_1-b_0b_1)-\sqrt{\left\{9b_1^2(a_1-b_0b_1)^2-36a_3b_1(1-b_0)\right\}}}{2b_1(1-b_0)} \\ OB = \dfrac{3b_1(a_1-b_0b_1)+\sqrt{\left\{9b_1^2(a_1-b_0b_1)^2-36a_3b_1(1-b_0)\right\}}}{2b_1(1-b_0)} \\ OP = \left(\dfrac{3a_3b_1}{a_2b_1-a_3}\right) \end{array}\right\} \qquad (12)$$

and β can be shown to be given by

$$\beta = \frac{T_2\left(a_3+b_0b_1^2T_2\right)}{(a_2b_1-a_3)T_2-3a_3b_1}. \qquad (13)$$

The proper procedure for design of the network would be first to check and see if the inequalities of (10) are satisfied. Then solve (Turnbull 1947) (9) for T_2, and substitute the positive real value(s) of T_2 into (13) and see if the inequality of (11) is satisfied. Satisfaction of the conditions of (10) and (11) signify that a positive real set of α, β, T_2, T_5 and T_7 exists for which the circuit of Fig. 1(*a*) is physically realizable.

The circuit component values may be then conveniently determined by solving (9), (13), (3), (4) and (7).

Referring to the circuit of Fig. 1(*b*), its transfer function can be shown to be

$$\frac{E_0}{E_1} = -\frac{\left(\dfrac{\beta}{5\alpha+3}\right)(\alpha RC_5S+1)}{\dfrac{\alpha\beta}{(5\alpha+3)}R^3C_5C_7C_8S^3+\left[\dfrac{3\alpha\beta}{(5\alpha+3)}R^2C_5C_7+\dfrac{\alpha(\beta+1)}{(5\alpha+3)}R^2C_5C_8+\dfrac{(\alpha+\beta)}{(5\alpha+3)}R^2C_7C_8\right]S^2+\left[\dfrac{3\alpha}{(5\alpha+3)}RC_5+\dfrac{3(\alpha+\beta)}{(5\alpha+3)}RC_7+\dfrac{(2\alpha+\beta+1)}{(5\alpha+3)}RC_8\right]S+1} \qquad (14)$$

(1) and (14) will be identical if

$$b_0 = \frac{\beta}{(5\alpha+3)} \qquad (15)$$

$$b_1 = \alpha T_5 \quad \ldots \quad (16)$$

$$a_1 = \frac{3\alpha}{(5\alpha+3)} T_5 + \frac{3(\alpha+\beta)}{(5\alpha+3)} T_7 + \frac{(2\alpha+\beta+1)}{(5\alpha+3)} T_8 \quad \ldots \quad (17)$$

$$a_2 = \frac{3\alpha\beta}{(5\alpha+3)} T_5 T_7 + \frac{\alpha(\beta+1)}{(5\alpha+3)} T_5 T_8 + \frac{(\alpha+\beta)}{(5\alpha+3)} T_7 T_8 \quad \ldots \quad (18)$$

$$a_3 = \frac{\alpha\beta}{(5\alpha+3)} T_5 T_7 T_8 \quad \ldots \quad (19)$$

where

$$T_n = RC_n. \quad \ldots \quad (20)$$

Elimination of α, β, T_5 and T_8 from (15) to (19) gives a quartic

$$A_0 T_7^4 - A_1 T_7^3 + A_2 T_7^2 - A_3 T_7 + A_4 = 0 \quad \ldots \quad (21)$$

where

$$\left.\begin{aligned} A_0 &= 135 b_0^3 b_1^3 \\ A_1 &= 45 b_0^2 b_1^2 (a_2 + 5 b_0 b_1^2) \\ A_2 &= 15 b_0 b_1 (a_1 a_3 + 5 a_2 b_0 b_1^2 + 4 a_3 b_0 b_1) \\ A_3 &= 5 a_3 [a_3(1+2b_0) + b_0 b_1 (5 a_1 b_1 + a_2 + 15 b_0 b_1^2)] \\ A_4 &= 10 a_3^2 b_1 (1+3b_0) \end{aligned}\right\} \quad \ldots \quad (22)$$

which can have no negative roots and will have two positive real roots if its discriminant Δ is negative.

As shown in Appendix II, at least one set of positive real α, β, T_5, T_7 and T_8 exists provided a set of conditions listed in any one of the undermentioned seven cases is satisfied together with

$$\beta > 3b_0. \quad \ldots \quad (23)$$

(i) $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$; $OP > ON$

Either $OL > OB$ .. (24)

or $OM > OB > OA > OL$ .. (25)

or
$$\left.\begin{aligned} & OA > OM \\ \text{and either } & OA > OP \\ \text{or } & OQ > OB \\ \text{or } & OP > OB \end{aligned}\right\} \quad \ldots \quad (26)$$

or $OM > OB > OL > OA$ .. (27)

or
$$\left.\begin{aligned} & OB > OM > OL > OA \\ \text{and either } & OQ > OB \\ \text{or } & OP > OB \end{aligned}\right\} \quad \ldots \quad (28)$$

or $OB > OM > OA > OL$
and either $OQ > OB$
or $OP > OB$ .. .. .. (29)

(ii) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0;\ OQ > ON > OP$

Either $OL > OB$ .. .. (30)

or $OM > OB > OA > OL$ .. .. (31)

or $OB > OA > OM > OL$
and either $OA > OQ$
or $OQ > OB$ .. .. .. (32)

or $OM > OB > OL > OA$ .. .. (33)

or $OB > OM > OL > OA$
and $OQ > OB$ .. .. (34)

or $OB > OM > OA > OL$
and $OQ > OB$ .. .. (35)

(iii) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0;\ ON > OQ > OP$

Either $OL > OB$
and either $OQ > OA > OP$
or $OQ > OB > OP$ .. .. (36)

or $OM > OB > OA > OL$
and either $OQ > OB > OP > OA$
or $OB > OQ > OA > OP$ .. .. .. (37)

or $OB > OA > OM > OL$
and either $OQ > OA > OP$
or $OQ > OB > OA > OP$
or $OQ > OB > OP > OA$
or $OB > OQ > OA > OP$ .. .. .. (38)

or $OM > OB > OL > OA$
and either $OQ > OA > OP$
or $OQ > OB > OP$ .. .. .. (39)

or $OB > OM > OL > OA$
and either $OQ > OA > OP$
or $OQ > OB > OP$ .. .. .. (40)

or $OB > OM > OA > OL$
and either $OQ > OA > OP$
or $OQ > OB > OP$ .. .. (41)

(iv) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 > 0$

$$\left.\begin{array}{lll} \text{Either} & & OQ > OP > ON \\ & \text{and either} & OQ > OA > OP \\ & \text{or} & OQ > OB > OP \\ & \text{or} & OQ > OB \end{array}\right\} \quad \ldots \quad (42)$$

$$\left.\begin{array}{lll} \text{or} & & OQ > ON > OP \\ & \text{and either} & OA > OP \\ & \text{or} & OQ > OB \end{array}\right\} \quad \ldots \quad (43)$$

$$\left.\begin{array}{lll} \text{or} & & ON > OQ > OP \\ & \text{and either} & OQ > OA > OP \\ & \text{or} & OQ > OB > OP \end{array}\right\} \quad \ldots \quad (44)$$

(v) $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 > 0$

Either $\quad OQ > OP > ON \quad \ldots \quad (45)$

or $\quad OQ > ON > OP \quad \ldots \quad (46)$

(vi) $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 > 0$

Either $\quad OQ > OP > ON \quad \ldots \quad (47)$

or $\quad OQ > ON > OP \quad \ldots \quad (48)$

$$\left.\begin{array}{ll} \text{or} & ON > OQ > OP \\ \text{and} & \Delta = I^3 - 27J^2 < 0 \end{array}\right\} \quad \ldots \quad (49)$$

(vii) $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 < 0$

where

$$\Delta = I^3 - 27J^2 < 0 \quad \ldots \quad (50)$$

$$\left.\begin{array}{l}
\Delta_1 = 25a_1^2b_0^2b_1^2 - 12a_3b_0b_1(1+5b_0)(2+5b_0)\,;\quad \Delta_2 = 225b_0^2b_1^4 - 36a_3b_0b_1 \\
\Delta_3 = B_3^2 - 300a_3b_0^3b_1^3\,;\quad B_3 = 5a_2b_0b_1 - a_3(1+5b_0) \\
OA = \dfrac{5a_1b_0b_1 - \sqrt{\Delta_1}}{6b_0b_1(1+5b_0)}\,;\quad OB = \dfrac{5a_1b_0b_1 + \sqrt{\Delta_1}}{6b_0b_1(1+5b_0)} \\
OL = \dfrac{15b_0b_1^2 - \sqrt{\Delta_2}}{18b_0b_1}\,;\quad OM = \dfrac{15b_0b_1^2 + \sqrt{\Delta_2}}{18b_0b_1}\,;\quad OP = \dfrac{B_3 - \sqrt{\Delta_3}}{30b_0^2b_1^2} \\
OQ = \dfrac{B_3 + \sqrt{\Delta_3}}{30b_0^2b_1^2}\,;\quad ON = \dfrac{5b_1}{3}\,;\quad I = A_0A_4 - \dfrac{A_1A_3}{4} + \dfrac{A_2^2}{12} \\
J = \dfrac{A_0}{2}\left(\dfrac{A_2A_4}{3} - \dfrac{A_3^2}{8}\right) - \dfrac{A_1}{16}\left(A_1A_4 - \dfrac{A_2A_3}{3}\right) - \left(\dfrac{A_2}{6}\right)^3
\end{array}\right\} \quad (51)$$

The network design procedure would be to first compute the values of Δ_1, Δ_2, Δ_3, B_3, OP, OQ, ON with the aid of (51) and see if any one of the seven cases mentioned above is applicable and then check to see if any one set of conditions listed under the relevant case together with (23) is satisfied. The satisfaction of these conditions signifies that the circuit of Fig. 1(*b*) is physically realizable. The circuit component values may be then determined by solving the quartic of (21) for T_7. Substituting the positive real value(s) of T_7 into (2.7) gives β; α, T_5 and T_8 may then be obtained with the aid of (15), (16) and (19).

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APPENDIX I

CONDITIONS UNDER WHICH THE CIRCUIT OF FIG. 1(*a*) IS PHYSICALLY REALIZABLE

Simulation of the system represented by (1) with the circuit of Fig. 1(*a*) is possible only if values of α, β, T_2, T_5 and T_7 obtained as the solution of equations

$$b_0 = \frac{\alpha\beta}{(5\alpha+\alpha\beta+3)} \qquad (1.1)$$

$$b_1 = T_5 \qquad (1.2)$$

$$a_1 = \frac{(2\alpha+1)}{(5\alpha+\alpha\beta+3)}T_2+\frac{\alpha(\beta+3)}{(5\alpha+\alpha\beta+3)}T_5+\frac{3\alpha(\beta+1)}{(5\alpha+\alpha\beta+3)}T_7 \qquad (1.3)$$

$$a_2 = \frac{\alpha}{(5\alpha+\alpha\beta+3)}T_2T_5+\frac{\alpha(\beta+1)}{(5\alpha+\alpha\beta+3)}T_2T_7+\frac{3\alpha\beta}{(5\alpha+\alpha\beta+3)}T_5T_7 \qquad (1.4)$$

$$a_3 = \frac{\alpha\beta}{(5\alpha+\alpha\beta+3)}T_2T_5T_7 \qquad (1.5)$$

are positive real, where *a*'s and *b*'s are positive real constants.

Elimination of α, T_5 and T_7 from (1.1), (1.2), (1.3), (1.5) and (1.1), (1.2), (1.4), (1.5) give the following two equations:

$$\frac{1}{\beta} = \frac{3b_1(a_1-b_0b_1)T_2-b_1(1-b_0)T_2^2-9a_3}{b_0b_1T_2^2+9b_0b_1^2T_2+9a_3}. \qquad (1.6)$$

$$\frac{1}{\beta}=\frac{(a_2b_1-a_3)T_2-3a_3b_1}{T_2(a_3+b_0b_1^2T_2)}. \quad \cdots \quad (1.7)$$

The intersection of the curves of (1.6) and (1.7) in the first quadrant of the $T_2-\frac{1}{\beta}$ plane will give T_2 and β both positive real. It is evident from (1.1) that

$$\alpha=\frac{3b_0}{(1-b_0)\beta-5b_0} \quad \cdots \quad (1.8)$$

will be also real and positive if

$$\left.\begin{aligned}(1-b_0)&>0\\ \text{and}\quad \beta&>\left(\frac{5b_0}{1-b_0}\right)\end{aligned}\right\}. \quad \cdots \quad (1.9)$$

The corresponding T_5 and T_7 as seen from (1.2) and (1.7) will be also positive real.

Now, the curve of (1.6) will cut the T_2-axis $\left(\text{i.e. } \frac{1}{\beta}=0\right)$ at two points A, B whose T_2-coordinates may be obtained by equating to zero the right-hand side of (1.6) and solving the resulting quadratic

$$b_1(1-b_0)T_2^2-3b_1(a_1-b_0b_1)T_2+9a_3=0. \quad \cdots \quad (1.10)$$

The roots of (1.10) are

$$T_{2(A,\,B)}=\frac{3b_1(a_1-b_0b_1)\pm\sqrt{\left\{9b_1^2(a_1-b_0b_1)^2-36a_3b_1(1-b_0)\right\}}}{2b_1(1-b_0)} \quad \cdots \quad (1.11)$$

which will be positive, if

$$(a_1-b_0b_1)>0 \quad \cdots \quad (1.12)$$

and real, if

$$b_1(a_1-b_0b_1)^2>4a_3(1-b_0). \quad \cdots \quad (1.13)$$

Similarly, the curve of (1.7) will cut the T_2-axis at a point P whose T_2-coordinate is given by

$$OP=\left(\frac{3a_3b_1}{a_2b_1-a_3}\right) \quad \cdots \quad (1.14)$$

which is real and will be positive, if

$$(a_2b_1-a_3)>0. \quad \cdots \quad (1.15)$$

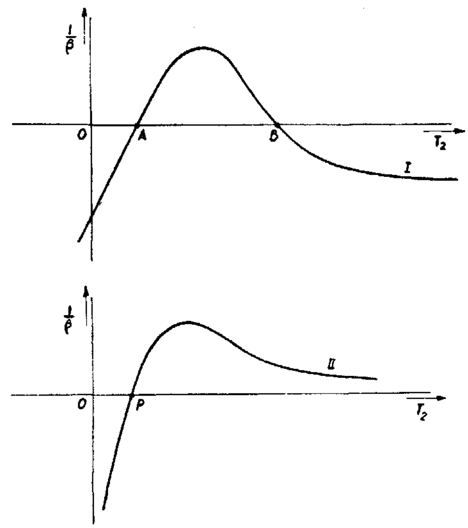
Hence, if the inequalities of (1.9), (1.12), (1.13) and (1.15) are satisfied then a portion of the curves of (1.6) and (1.7) can exist in the first quadrant of the $T_2-\frac{1}{\beta}$ plane, and the curves for positive T_2 are sketched in Fig. 1.1.

The curves will intersect each other in the first quadrant provided that

$$OB>OP>OA \quad \cdots \quad (1.16)$$

and that the conditions of (1.9), (1.12), (1.13) and (1.15) are satisfied.

FIG. 1.1. Sketches of curves for positive T_2.



$$\text{I} \quad \frac{1}{\beta} = \frac{3b_1(a_1-b_0b_1)T_2 - b_1(1-b_0)T_2^2 - 9a_3}{b_0b_1T_2^2 + 9b_0b_1^2T_2 + 9a_3}$$

$$\text{II} \quad \frac{1}{\beta} = \frac{(a_2b_1-a_3)T_2 - 3a_3b_1}{T_2\left(b_0b_1^2T_2 + a_3\right)}$$

To summarize, if the inequalities of (1.9), (1.12), (1.13), (1.15) and (1.16) are satisfied, then a positive real set of α, β, T_2, T_5 and T_7 exists for which the circuit of Fig. 1(a) is physically realizable.

APPENDIX II

CONDITIONS UNDER WHICH THE CIRCUIT OF FIG. 1(b) IS PHYSICALLY REALIZABLE

It is possible to simulate the system represented by (1) with the circuit of Fig. 1(b) only if the values of α, β, T_5, T_7 and T_8 obtained as the solution of equations

$$b_0 = \frac{\beta}{(5\alpha+3)} \qquad \text{(2.1)}$$

$$b_1 = \alpha T_5 \qquad \text{(2.2)}$$

$$a_1 = \frac{3\alpha}{(5\alpha+3)}T_5 + \frac{3(\alpha+\beta)}{(5\alpha+3)}T_7 + \frac{(2\alpha+\beta+1)}{(5\alpha+3)}T_8 \qquad \text{(2.3)}$$

$$a_2 = \frac{3\alpha\beta}{(5\alpha+3)}T_5T_7 + \frac{\alpha(\beta+1)}{(5\alpha+3)}T_5T_8 + \frac{(\alpha+\beta)}{(5\alpha+3)}T_7T_8 \qquad \text{(2.4)}$$

$$a_3 = \frac{\alpha\beta}{(5\alpha+3)} T_5 T_7 T_8 \quad \cdots \quad (2.5)$$

are positive and real.

Elimination of α, T_5 and T_8 from (2.1), (2.2), (2.3), (2.5) and (2.1), (2.2), (2.4), (2.5) give the following two equations:

$$\frac{1}{\beta} = \frac{3b_0b_1(1+5b_0)T_7^2 - 5a_1b_0b_1T_7 + a_3(2+5b_0)}{b_0(9b_0b_1T_7^2 - 15b_0b_1^2T_7 + a_3)}. \quad \cdots \quad (2.6)$$

$$\frac{1}{\beta} = \frac{15b_0^2b_1^2T_7^2 - \{5a_2b_0b_1 - a_3(1+5b_0)\}T_7 + 5a_3b_0b_1}{a_3b_0(3T_7 - 5b_1)}. \quad \cdots \quad (2.7)$$

The intersection of the curves of (2.6) and (2.7) in the first quadrant of the $T_7 - \frac{1}{\beta}$ plane will give both T_7 and β as positive real. It is evident from (2.1) that

$$\alpha = \left(\frac{\beta - 3b_0}{5b_0}\right) \quad \cdots \quad (2.8)$$

will be also real and positive, if

$$\beta > 3b_0. \quad \cdots \quad (2.9)$$

The corresponding T_5 and T_8, as seen from (2.2) and (2.5), will be also positive real.

Now (2.6) will have a pair of zeros and a pair of poles which are respectively given by

$$T_{7(A,\,B)} = \frac{5a_1b_0b_1 \pm \sqrt{\Delta_1}}{6b_0b_1(1+5b_0)} \quad \cdots \quad (2.10)$$

$$T_{7(L,\,M)} = \frac{15b_0b_1^2 \pm \sqrt{\Delta_2}}{18b_0b_1}. \quad \cdots \quad (2.11)$$

The zeros and poles will be real or complex depending on whether Δ_1 and Δ_2 are positive or negative.

Similarly, (2.7) has a real pole given by

$$T_{7(N)} = ON = \frac{5b_1}{3} \quad \cdots \quad (2.12)$$

and a pair of zeros given by

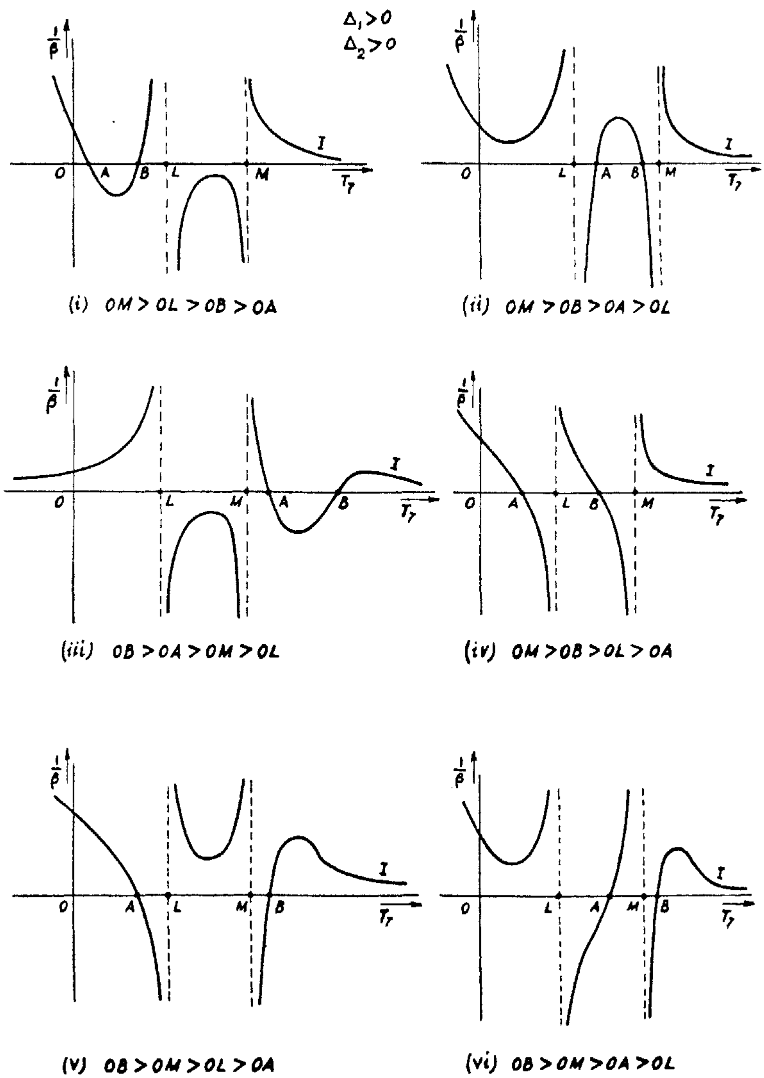
$$T_{7(P,\,Q)} = \frac{B_3 \pm \sqrt{\Delta_3}}{30b_0^2b_1^2} \quad \cdots \quad (2.13)$$

which will be real or complex depending on whether Δ_3 is positive or negative, where

$$\left.\begin{aligned} \Delta_1 &= \left[25a_1^2b_0^2b_1^2 - 12a_3b_0b_1(1+5b_0)(2+5b_0)\right] \\ \Delta_2 &= \left[225b_0^2b_1^4 - 36a_3b_0b_1\right] \\ \Delta_3 &= \left[B_3^2 - 300a_3b_0^3b_1^3\right] \\ B_3 &= \left[5a_2b_0b_1 - a_3(1+5b_0)\right] \end{aligned}\right\} \quad \cdots \quad (2.14)$$

The branches of the curves for positive T_7 and various possible cases of interest wherein a portion of the curves can exist in the first quadrant are sketched in Figs. 2.1(a), (b) and (c).

FIG. 2.1(a). Sketches of curves for positive T_7.



It is obvious from (2.11) and (2.12) that

$$ON > OM > OL. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.15)$$

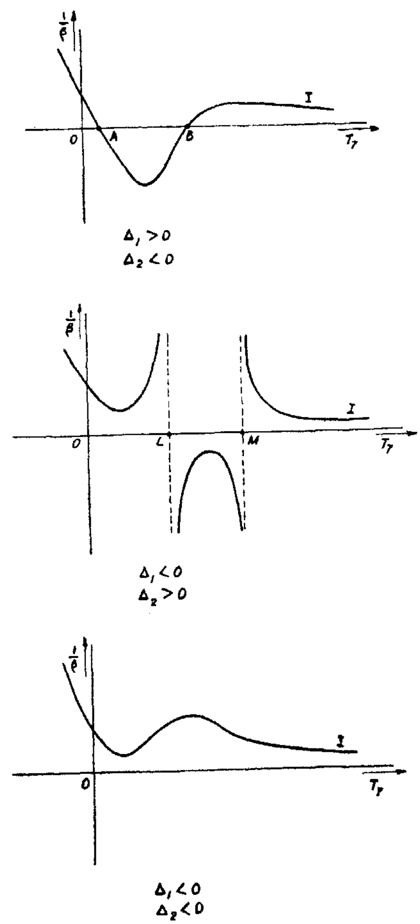
Elimination of β from (2.6) and (2.7) gives a quartic

$$A_0T_7^4 - A_1T_7^3 + A_2T_7^2 - A_3T_7 + A_4 = 0 \quad \ldots \quad \ldots \quad (2.16)$$

where

$$\left.\begin{aligned} A_0 &= 135b_0^3b_1^3 \\ A_1 &= 45b_0^2b_1^2(a_2+5b_0b_1^2) \\ A_2 &= 15b_0b_1(a_1a_3+5a_2b_0b_1^2+4a_3b_0b_1) \\ A_3 &= 5a_3[a_3(1+2b_0)+b_0b_1(5a_1b_1+a_2+15b_0b_1^2)] \\ A_4 &= 10a_3^2b_1(1+3b_0) \end{aligned}\right\} \quad .. \quad (2.17)$$

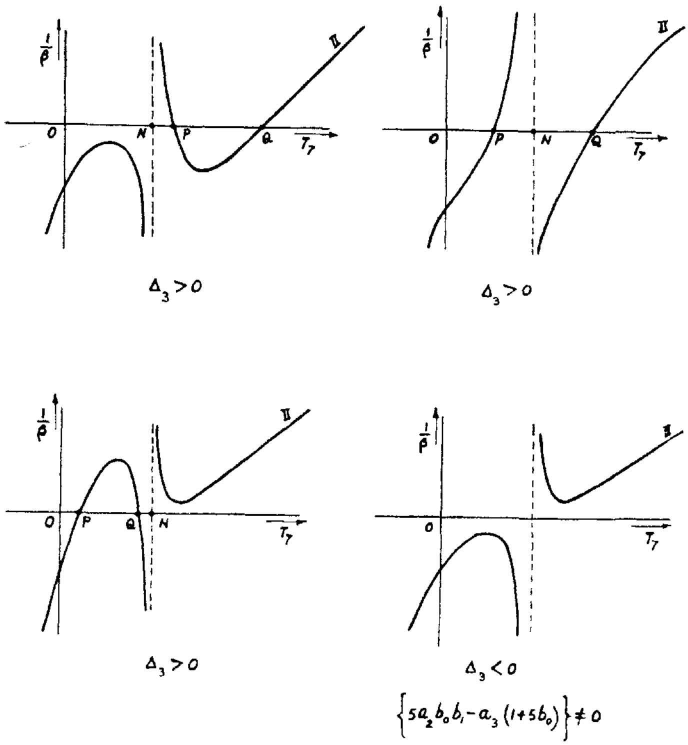
FIG. 2.1(b). Sketches of curves for positive T_7.



which can have no negative roots and will have two positive real roots if its discriminant Δ is negative. The real roots of (2.16) signify real points of intersection of the curves of (2.6) and (2.7), and the positive roots mean that the points of intersection lie on the right of the $\frac{1}{\beta}$-axis.

In view of expression (2.15), it should be evident that the curves of (2.6) and (2.7) will intersect each other at at least one point in the first quadrant

FIG. 2.1(*c*). Sketches of curves for positive T_7.



$$\text{I}\quad \frac{1}{\beta} = \frac{3b_0b_1(1+5b_0)T_7^2 - 5a_1b_0b_1T_7 + a_3(2+5b_0)}{b_0\left(9b_0b_1T_7^2 - 15b_0b_1^2T_7 + a_3\right)}$$

$$\text{II}\quad \frac{1}{\beta} = \frac{15b_0^2b_1^2T_7^2 - \{5a_2b_0b_1 - a_3(1+5b_0)\}T_7 + 5a_3b_0b_1}{a_3b_0(3T_7 - 5b_1)}$$

if a set of conditions listed under any one of the following seven cases is satisfied:

(i) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0;\ OP > ON.$

Either $OL > OB$ .. .. (2.18)

or $OM > OB > OA > OL$ .. .. .. (2.19)

or $OA > OM$
and either $OA > OP$
or $OQ > OB$
or $OP > OB$ .. .. (2.20)

or $OM > OB > OL > OA$ .. .. .. (2.21)

$$\text{or} \quad \left.\begin{array}{ll} & OB > OM > OL > OA \\ \text{and either} & OQ > OB \\ \text{or} & OP > OB \end{array}\right\} \qquad (2.22)$$

$$\text{or} \quad \left.\begin{array}{ll} & OB > OM > OA > OL \\ \text{and either} & OQ > OB \\ \text{or} & OP > OB \end{array}\right\} \qquad (2.23)$$

(ii) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0;\ OQ > ON > OP.$

$$\text{Either} \quad OL > OB \qquad (2.24)$$

$$\text{or} \quad OM > OB > OA > OL \qquad (2.25)$$

$$\text{or} \quad \left.\begin{array}{ll} & OB > OA > OM > OL \\ \text{and either} & OA > OQ \\ \text{or} & OQ > OB \end{array}\right\} \qquad (2.26)$$

$$\text{or} \quad OM > OB > OL > OA \qquad (2.27)$$

$$\text{or} \quad \left.\begin{array}{ll} & OB > OM > OL > OA \\ \text{and} & OQ > OB \end{array}\right\} \qquad (2.28)$$

$$\text{or} \quad \left.\begin{array}{ll} & OB > OM > OA > OL \\ \text{and} & OQ > OB \end{array}\right\} \qquad (2.29)$$

(iii) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0;\ ON > OQ > OP.$

$$\text{Either} \quad \left.\begin{array}{ll} & OL > OB \\ \text{and either} & OQ > OA > OP \\ \text{or} & OQ > OB > OP \end{array}\right\} \qquad (2.30)$$

$$\text{or} \quad \left.\begin{array}{ll} & OM > OB > OA > OL \\ \text{and either} & OQ > OB > OP > OA \\ \text{or} & OB > OQ > OA > OP \end{array}\right\} \qquad (2.31)$$

$$\text{or} \quad \left.\begin{array}{ll} & OB > OA > OM > OL \\ \text{and either} & OQ > OA > OP \\ \text{or} & OQ > OB > OA > OP \\ \text{or} & OQ > OB > OP > OA \\ \text{or} & OB > OQ > OA > OP \end{array}\right\} \qquad (2.32)$$

$$\text{or} \quad \left.\begin{array}{ll} & OM > OB > OL > OA \\ \text{and either} & OQ > OA > OP \\ \text{or} & OQ > OB > OP \end{array}\right\} \qquad (2.33)$$

$$\text{or} \quad \left.\begin{array}{ll} & OB > OM > OL > OA \\ \text{and either} & OQ > OA > OP \\ \text{or} & OQ > OB > OP \end{array}\right\} \qquad (2.34)$$

or $OB > OM > OA > OL$
and either $OQ > OA > OP$
or $OQ > OB > OP$ .. .. .. (2.35)

(iv) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 > 0.$

Either $OQ > OP > ON$
and either $OQ > OA > OP$
or $OQ > OB > OP$
or $OQ > OB$ .. .. (2.36)

or $OQ > ON > OP$
and either $OA > OP$
or $OQ > OB$ .. .. (2.37)

or $ON > OQ > OP$
and either $OQ > OA > OP$
or $OQ > OB > OP$ .. .. (2.38)

(v) $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 > 0.$

Either $OQ > OP > ON$ .. .. (2.39)

or $OQ > ON > OP$ .. .. (2.40)

(vi) $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 > 0.$

Either $OQ > OP > ON$ .. .. (2.41)

or $OQ > ON > OP$ .. .. (2.42)

or $ON > OQ > OP$
and $\Delta = I^3 - 27J^2 < 0$ .. .. (2.43)

(vii) $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 < 0.$

$$\Delta = I^3 - 27J^2 < 0 \quad \text{.. .. .. ..} \quad (2.44)$$

To summarize, if any one of the seven cases mentioned above is applicable and any one set of conditions listed under the relevant case together with the inequality of (2.9) are satisfied then a set of positive real α, β, T_5, T_7 and T_8 exists for which the circuit of Fig. 1(*b*) is physically realizable.