AN APPROXIMATE SOLUTION OF DYNAMICAL EQUATIONS OF THE MEAN-TWISTED TRAJECTORY OF A SPINNING SHELL

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The small oscillations of the axis of a spinning shell on the mean-twisted trajectory are studied qualitatively. The characteristic lines of oscillations show a stable node at the position of equilibrium. On the quantitative side we have introduced a small correction to Otto-Lardillon equations which are solved in approximate form. The correction arises due to the effect of the cross-wind force on the shell.

Introduction

Fowler (H.M.S.C. 1925) points out that when the initial vibrations of a shell damp out, the shell-axis settles down to a sort of equilibrium position on a path which he terms as mean-twisted trajectory, and the shell-axis executes a small motion about the equilibrium position. The subsequent angular motion is then due to the precessional angular velocity 'ω' about the instantaneous direction of motion of the centre of mass of the shell (eqn. 1.9). In this paper the nature of small oscillations of the shell-axis about its equilibrium position on the mean-twisted trajectory is studied qualitatively. The problem thus leads to a set of non-linear differential equations and they are solved for the restricted case of low-angle fire. As can be expected, the characteristic lines of oscillations show a stable node at the position of equilibrium. On the quantitative side a small correction to Otto-Lardillon equations (War Office, U.K. 1951) is given. This correction arises due to the effect of the cross-wind force on the shell.

1. Dynamical Equations of the Mean-twisted Trajectory

The dynamical equations of the mean-twisted trajectory are obtained by neglecting the force of 'inertia' as compared to the 'gyroscopic' force and neglecting all such aerodynamic forces that arise due to the spin of the projectile (H.M.S.C. 1925, p. 620).

Let Q be the centre of mass of the projectile whose co-ordinates are (x, y, z) at any time 't' relative to a right-handed orthogonal Cartesian co-ordinate system, Q-XYZ (*see* Fig. 1). We choose the fixed axis QX horizontal in the plane of shooting, QY vertical and QZ perpendicular to the plane of departure, its positive direction being to the right of the gunner. Let P and A be points where the directions of the centre of mass and the axis of the shell cut the surface of the unit sphere with Q as centre. Let (l, m, n) be the direction-cosines of the axis of the shell (QA). Let θ and ψ be the altitude and azimuth of the point P and δ be the angle of yaw $(= A\hat{Q}P)$. Now taking Q-xyz as a rotating frame of reference Qx along QP, Qy along the upward-drawn normal and Qz to the right of the gunner as viewed from the gun position. Let ϕ be the angle of orientation of the plane of yaw PAQ from the vertical plane YQP; then we have

$$l = \cos\delta, \qquad \text{(1.1)}$$

$$m = \sin\delta\cos\phi, \qquad \text{(1.2)}$$

$$n = \sin\delta\sin\phi. \qquad \text{(1.3)}$$

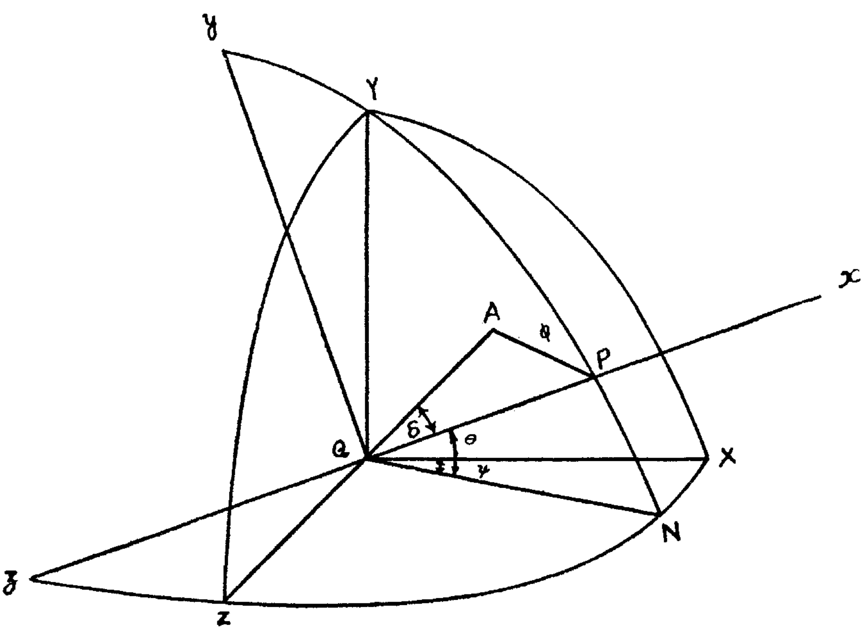

FIG. 1. The frame of reference Q-XYZ.

Then the equations of motion of the centre of gravity are given by(*)

$$v' = -R\tilde{m} - g\sin\theta, \qquad \text{(1.4)}$$

$$\theta' = -g\frac{\cos\theta}{v} + km, \qquad \text{(1.5)}$$

$$\psi'\cos\theta = kn \qquad \text{(1.6)}$$

and the equations of the angular motion are

$$m' = -(\omega + \psi'\sin\theta)n - l\theta', \qquad \text{(1.7)}$$

$$n' = (\omega + \psi'\sin\theta)m - l\psi'\cos\theta. \qquad \text{(1.8)}$$

* An overhead dash signifies differentiation with respect to time.

These equations are obtained by ignoring the spin effects from the β type equations of Fowler *et al.* (1920).

In the above equations it may be noted,

$$\tilde{m} = \text{mass of the shell,}$$
$$R = \text{the retardation force,}$$
$$\omega = \frac{\mu}{AN},$$

and

$$k = \frac{L}{\tilde{m}v},$$

where L is the lift force, μ is the moment factor, N is the axial spin and A the axial moment of inertia of the shell. The values of k and ω can also be written as

$$\omega = \frac{\rho_0 r^3}{AN} \times \frac{v^2}{f} f_M\left(\frac{v}{a}, \delta\right) \quad \ldots \quad (1.9)$$

$$k = \frac{\rho_0 r^2}{\tilde{m}} \times \frac{v}{f} f_L\left(\frac{v}{a}, \delta\right), \quad \ldots \quad (1.10)$$

where f is the tenuity factor for height given by

$$\rho = \rho_0/f. \quad \ldots \quad (1.11)$$

The factor $\lambda = k/\omega$ is small and is independent of air density. It can be taken as a slowly varying parameter with an average value λ (say) (*refer* tables in H.M.S.C. 1925). It is hence treated as a constant in further analysis.

2. The Solution of Second Ballistic Problem for the Case of Low-angle Fire

Writing $\sigma = g\dfrac{\cos\theta}{v}$ the equations of angular motion will reduce to

$$m' = -(\omega + nk\tan\theta)n + l(\sigma - km) \quad \ldots \quad (2.1)$$

$$n' = (\omega + nk\tan\theta)m - nkl. \quad \ldots \quad (2.2)$$

Now, $nk \tan\theta$ can be neglected in comparison with ω; for the ratio of the two terms $\left(\dfrac{k}{\omega}\right)\delta\tan\theta$ is of the second order.

Now, since

$$mm' + nn' = \delta' \sin\delta\cos\delta$$
$$mn' - m'n = \phi' \sin^2\delta$$

we have

$$\delta' = \sigma\cos\phi - k\sin\delta \quad \ldots \quad (2.3)$$

$$\phi' = \omega - \sigma\cot\delta\sin\phi. \quad \ldots \quad (2.4)$$

For small yawing motion $\sin\delta$ can be replaced by δ and $\cos\delta$ by 1, with a fractional error of $\frac{1}{2}\delta^2$.

To an order of first approximation we have in the mean trajectory $\sigma = \omega \sin \delta$ (for small yawing motion)

$$\therefore \qquad \delta' = \delta(\omega \cos \phi - k) \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

$$\phi' = \omega(1 - \sin \phi). \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

So long as the parameters ω and k remain steady along the mean-twisted trajectory the following may be taken as a solution of the second ballistic problem :—

$$\delta = \frac{C_1}{1 - \sin \phi} \exp \left(+2\lambda \Big/ 1 - \tan \frac{\phi}{2} \right) \quad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

$$\phi = 2 \tan^{-1} \left\{ 1 - \frac{1}{\frac{\omega t}{2} + C_2} \right\}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

This solution will fail when in course of motion the shell enters the sound barrier. Just before attaining the sonic speed, ω falls abruptly until it becomes steady again when the motion of the shell becomes subsonic. Initially we are considering the vibrations of the shell axis as small; ω and k remain fairly steady and probably our solution will work.

From the above solutions it is clear that the equilibrium position $\delta = 0$, $\phi = \frac{\gamma}{2}$ is never reached until $t \to +\infty$, for due to the asymmetric action of the muzzle-blast the shell starts out with some initial yaw which goes on increasing till the point of fall, while the yaw remains small throughout the trajectory.

It is not difficult to see that the second ballistic problem can still be solved, even if we do not neglect the small term 'km' in eqn. (1.5), and as before a similar conclusion regarding the solution holds. In the present case

$$\sigma = \omega\delta + k\delta \cos \phi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

and hence the equations of angular motion are

$$\delta' = \delta(\omega \cos - k \sin^2\phi) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

$$\phi' = \omega(1 - \sin \phi) - k \sin \phi \cos \phi. \quad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

Remembering that $\lambda = k/\omega$ is small, the integrals of these equations are clearly

$$\delta = \frac{C_3}{(1 - \sin \phi)} \left[\exp - \lambda \left(\cos \phi - \frac{2}{1 - \tan \phi/2} \right) \right] \quad \ldots \quad \ldots \quad (2.12)$$

$$\omega t = \frac{2}{1 - \tan \frac{\phi}{2}} + \frac{\lambda}{1 - \sin \phi} + C_4. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

3. Stability of the Equilibrium Position ($\delta = 0$, $\phi = \pi/2$)

It is a well-known result in the principle of non-linear mechanics (Minorsky 1947) that the trajectories of the type represented above characterize motion in the neighbourhood of equilibrium and the problem of stability is one of that of this equilibrium position. That this position is unstable in this case has been pointed out by Pougachev (1942), but he asserts that a conditional stability exists. Since we have started with a stable shell, we shall show that the origin ($m = o = n$) exhibits a stable node but this is a critical point where the phase trajectories have a tendency to be spiralic, thus showing that instability may develop.

Consider

$$m' = -n\omega+\sigma-km \quad \ldots \quad (3.1)$$

$$n' = m\omega-kn \quad \ldots \quad (3.2)$$

obtained by neglecting non-linear terms. Writing $m+in = \zeta$ we get

$$\zeta' = i(\omega+ik)\zeta+\sigma. \quad \ldots \quad (3.3)$$

This being a linear equation of the first order, we write the particular integral as

$$\left[\omega+i\left(k+\frac{d}{dt}\right)\right]\zeta = i\sigma,$$

i.e.

$$(1+iP)\zeta = i\sigma/\omega$$

where

$$\omega P = k+\frac{d}{dt}.$$

The complete solution of the differential equation can be written as

$$\zeta = (\text{const})\, e^{-\int kdt}e^{i\int\omega dt}\,(1-iP+i^2P^2+i^3P^3\ldots)i\sigma/\omega$$

which gives

$$m \simeq P(\sigma/\omega)$$

$$n \simeq \sigma/\omega$$

$$\therefore \quad m \simeq \lambda\sigma/\omega+\frac{1}{\omega}\frac{d}{dt}(\sigma/\omega) \quad \ldots \quad (3.4)$$

$$n \simeq \sigma/\omega. \quad \ldots \quad (3.5)$$

Since $\sigma \simeq n\omega$, it is sufficient to consider

$$m' = -km \quad \ldots \quad (3.6)$$

$$n' = m\omega-kn \quad \ldots \quad (3.2)$$

for discussion of stability.

The phase trajectories of the system are clearly

$$m = (\text{const})\, e^{-\lambda\frac{n}{m}}. \quad \ldots \quad (3.7)$$

It is not difficult to see that all the characteristic lines of this family approach the origin along On. The origin is thus a stable node but it is on the boundaries between the regions of nodes and spirals (Hayashi 1953).

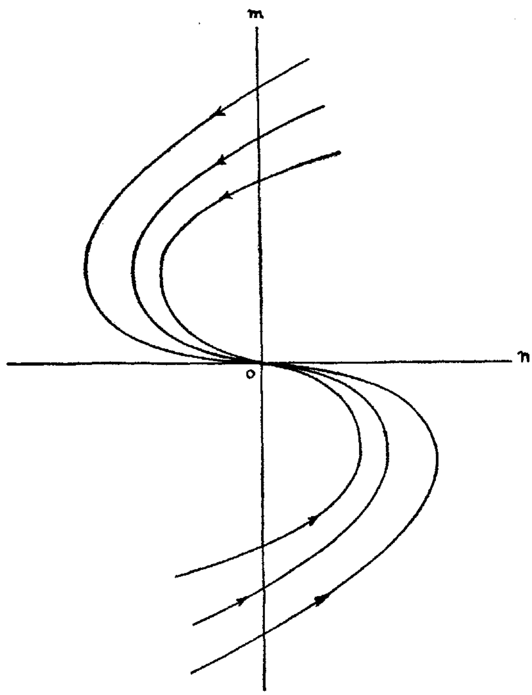

FIG. 2. The phase trajectories.

4. MODIFICATION OF OTTO-LARDILLON EQUATIONS IN PRESENCE OF THE CROSS-WIND FORCE

As the Otto-Lardillon equation is obtained with quadratic law of resistance, the velocity of such projectiles cannot exceed 800 ft./sec. and at such velocities the lift and the moment coefficients are practically constant. Therefore k can be assumed to be proportional to v, the constant of proportionality α (say) being a constant.

The equations of motion of the C.G. are

$$v' = -cv^2 - g \sin\theta \qquad (4.1)$$

$$\theta' = -\sigma(1-\lambda^2)+\lambda \frac{d}{dt}(\sigma/\omega), \qquad (4.2)$$

i.e.

$$\theta' = -\sigma(1-\lambda^2)+\lambda^2 \frac{d}{dt}(\sigma/k)$$

where $c = \frac{R}{\bar{m}}$ and $\lambda = \frac{k}{\omega}$ are assumed to be constants.

This on simplification leads to

$$\theta' = -\sigma(1-\lambda^2\beta), \quad \ldots \quad (4.3)$$

where

$$\beta = 1+\frac{2c}{\alpha}+\frac{3g \sin\theta}{\alpha v^2}, \quad \ldots \quad (4.4)$$

i.e.

$$\beta = A+\frac{B \sin\theta}{v^2}, \text{ say.} \quad \ldots \quad (4.5)$$

From (4.1) and (4.3) the hodograph equation can be written. By putting

$$q = v\cos\theta$$

$$\frac{dq}{d\theta} = \frac{dv}{d\theta}\cos\theta - v\sin\theta,$$

which on simplification leads to

$$\frac{dp}{d\theta} = -\frac{2c}{g}\sec^3\theta$$

$$-2\lambda^2\left[\left\{\left(1+\frac{5c}{\alpha}\right)\tan\theta\right\}p+p^2B\sin^2\theta\cos\theta+\frac{c}{g}A\sec^3\theta\right], \quad \ldots \quad (4.6)$$

where

$$p = 1/q^2.$$

The value of p can be found from the solution of this differential equation which is of Riccati's type.* But as our parameter λ^2 is small, we shall assume an expansion for $p(\theta)$ as a power series in λ^2 and neglect higher powers of λ^2. Thus we write

$$p(\theta) = -\frac{2c}{g}\left[\xi(\theta)-\lambda^2\eta(\theta)\right]. \quad \ldots \quad (4.7)$$

Here $\xi(\theta) = \int \sec^3\theta d\theta$ is the same as the Otto-Lardillon function.

Putting (4.7) in (4.6) and equating coefficients of λ^2 we have

$$\frac{d\eta}{d\theta} = -\left[A\sec^3\theta - 2\left(A+\frac{c}{g}B\right)\xi(\theta)\tan\theta + 4B\frac{c}{g}\xi^2(\theta)\sin^2\theta\cos\theta\right]$$

which on integration becomes

$$\eta(\theta) = -\log(\sec\theta+\tan\theta)\left[\left(1+\frac{c}{\alpha}\right)-\frac{2c}{\alpha}\sin^2\theta+\frac{c}{\alpha}\sin^3\theta\log(\sec\theta+\tan\theta)\right]$$

$$-\frac{c}{\alpha}\sin\theta + I(\theta), \quad \ldots \quad (4.8)$$

where

$$I(\theta) = \int\left(1+\frac{c}{\alpha}\right)\tan\theta\log(\sec\theta+\tan\theta)d\theta. \quad \ldots \quad (4.9)$$

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* Signorini (*see* Leimans and Minorsky 1958) also obtains a similar hodograph equation for the twisted trajectory when the projectile is influenced by cross-forces. In fact the determination of a space curve from its intrinsic equation requires the integration of an equation of Riccati's type.

From (4.7) we get

$$\frac{1}{V^2\cos^2\theta_0} - \frac{1}{v^2\cos^2\theta} = \frac{2c}{g}[\xi(\theta)-\xi(\theta_0)-\lambda^2\{\eta(\theta)-\eta(\theta_0)\}],$$

i.e.

$$\frac{cv^2}{g} = \frac{\sec^2\theta}{\frac{g\sec^2\theta_0}{cV^2} - 2[\xi(\theta)-\xi(\theta_0)-\lambda^2\{\eta(\theta)-\eta(\theta_0)\}]} \quad \ldots \quad \ldots \quad (4.10)$$

Thus having obtained v^2 as a function of θ, we proceed to find the elements of the trajectory in terms of θ. Thus

$$cx = -\int_{\theta_0}^{\theta} \frac{cv^2}{g(1-\lambda^2\beta)}\, d\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.11)$$

$$cy = -\int_{\theta_0}^{\theta} \frac{cv^2}{g(1-\lambda^2\beta)} \tan\theta\, d\theta \quad \ldots \quad \ldots \quad \ldots \quad (4.12)$$

$$cz = -\int_{\theta_0}^{\theta} \frac{\psi cv^2}{g(1-\lambda^2\beta)}\, d\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.13)$$

$$\sqrt{c}t = -\int_{\theta_0}^{\theta} \frac{\sqrt{cv}\sec\theta}{g(1-\lambda^2\beta)}\, d\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.14)$$

where

$$\psi' = \lambda g/v. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.15)$$

Example:

A 3 in. 16 lb. shell, whose C.G. is at a distance of 4·88 in. from the base, is fired with a velocity of 800 ft./sec. The range, angle of fall and the final velocity are obtained in both the cases, that is, with and without cross-wind forces. The results are tabulated in Table I.

TABLE I

Details	Otto's values	Values in M.T. trajectory
Range	2,886 yd.	2907·8 yd.
Final velocity	658·4 ft. sec.	660 ft./sec.
Angle of fall	16° 52′	17°

CONCLUSION

From this it is observed that the range, angle of fall and the final velocity are all increasing due to the cross-wind. The drift can also be found from the eqn. (4.13) by numerical integrations as it is done by Fowler (H.M.S.C. 1925).

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