# SUPERPOSABILITY AND SELF-SUPERPOSABILITY IN FLUID DYNAMICS—II

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We have (i) modified the definition of superposability in order that it may be applicable to all types of fluids and studied its implications for a class of non-Newtonian fluids, (ii) introduced the idea of homogeneous additivity of fluid flows and compared it with the concept of self-additivity, (iii) discussed the existence of steady viscous Beltrami flows, (iv) discussed the characterization of steady axially-symmetric self-superposable viscous flows, (v) discussed the characterization of steady viscous pseudo-planar Beltrami and self-superposable flows, (vi) introduced new definition of superposability wherein we require only one of the flows to satisfy the condition of integrability and studied its implications for both Newtonian and non-Newtonian fluids, (vii) subsequently obtained three-dimensional general rotational fluid flows from two-dimensional flows due to (*a*) two-dimensional vortex, (*b*) two-dimensional spiral vortex and (*c*) two-dimensional vortex doublet, (viii) obtained axi-symmetric solutions by considering conditions of integrability in poloidal and toroidal velocity fields respectively and (ix) discussed new two-dimensional solutions and axi-symmetric irrotational flows.

## 1. INTRODUCTION

The present paper is to be regarded as continuation of Kapur (1961*a*) where one of us (J. N. K.) had (i) established some general theorems about steady self-additive (or self-superposable) flows, (ii) discussed the decay of vorticity for non-steady self-additive flows, (iii) introduced the concept of transitivity of additivity of fluid flows and stated some general theorems about the same, (iv) discussed the most general two-dimensional rotational flows superposable on a two-dimensional irrotational flow, (v) obtained the general axi-symmetric rotational flows superposable on axi-symmetric irrotational flows, (vi) discussed the axially-symmetric unsteady superposable flows, (vii) discussed the superposability conditions of Gold and Krzywoblocki (1958) for incompressible non-Newtonian flows, (viii) compared the conditions of superposability in magnetohydrodynamics obtained by Gold and Krzywoblocki (1958), Kapur (1959, 1960*a*) and Bhatnagar (1960) and generalized two theorems of Ram Moorty (1960) about these, and (ix) extended the concept of superposability to inviscid and viscous compressible flows.

In § 2 of the present paper we have slightly modified the definition of superposability so that it is applicable to all types of fluids—compressible or incompressible, Newtonian or non-Newtonian, elastic or inelastic. We have obtained the condition of superposability in the general case and studied its implications for a class of non-Newtonian fluids.

In § 3, we introduce the concept of homogeneous additivity of fluid flows and distinguish it from the concept of self-additivity. While for Newtonian and non-Newtonian fluids with constant coefficients of viscosity and cross-viscosity, the conditions of self-additivity and homogeneous additivity are identical; for more general fluids, the conditions are quite different.

In § 4, we discuss the existence of steady viscous Beltrami flows. These flows had been characterized in Kapur (1960*b*). Based on this characterization, Rathna and Rajeshwari (1961*a*) tried to prove the non-existence of such flows. We give an alternative discussion of the same result.

In § 5, we discuss the characterization of steady axially-symmetric self-superposable viscous flows given in Kapur (1960*b*). In § 6, the characterization of steady viscous pseudo-planar Beltrami and self-superposable flows have been discussed.

In § 7, we introduce a new definition of superposability wherein we require only one of the flows to satisfy the condition of integrability. We discuss the implications of this new definition for both Newtonian and non-Newtonian fluids.

In subsequent sections by making use of the new principle of superposability we have constructed new solutions of equations of Fluid Dynamics. In § 7.1, we have constructed most general three-dimensional rotational fluid flows from two-dimensional flows due to (i) two-dimensional vortex, (ii) two-dimensional spiral vortex and (iii) two-dimensional vortex doublet and radial flow in first three cases only.

In § 7.2, axi-symmetric solutions have been investigated by considering the conditions of integrability in poloidal and toroidal velocity fields respectively and compared with corresponding steady poloidal and toroidal flows superposable on each other obtained in Kapur (1960*b*) by considering the conditions of integrability in both the flows.

In §§ (7.3) and (7.4), we discuss the construction of new two-dimensional solutions from flows due to a (i) two-dimensional vortex, (ii) two-dimensional spiral vortex, (iii) two-dimensional vortex doublet and (iv) radial flow and construction of axi-symmetric rotational flows from axi-symmetric irrotational flows.

## 2. Superposability of Motions of a General Fluid

### 2.1. *Definition of superposability*

Ram Ballabh (1940, 1952) defined superposability of two flows as follows:

'Two flows of an incompressible fluid with uniform density $\rho$, kinematic viscosity $\nu$, velocity vectors $\vec{q}_1$, $\vec{q}_2$, pressures $p_1$, $p_2$, and force potentials

$\Omega_1$, $\Omega_2$ are said to be superposable or additive, if it is possible to determine a pressure $p_1+p_2+\pi$ such that $(\vec{q}_1+\vec{q}_2,\ p_1+p_2+\pi,\ \Omega_1+\Omega_2)$ is also a solution of Navier-Stokes equations with the necessary modifications in initial and boundary conditions.'

We modify the definition as follows:

'If $\vec{q}_1$, $\vec{q}_2$ are two possible velocity vector fields (*i.e.* vector fields satisfying the basic equations determining the motion of the system), then these will be said to be superposable or additive if $\vec{q}_1+\vec{q}_2$ is also a possible velocity vector field.'

## 2.2. *Some remarks on the definition*

We make the following remarks about the definition: (i) It is not restricted to incompressible fluids (Kapur 1961*a*), *i.e.* it can be applied equally well to compressible fluids. (ii) It is not explicitly restricted to Newtonian fluids; it can be applied to all classes of non-Newtonian fluids. (iii) It is implicit in the definition that initial and boundary conditions may have to be modified. It is also implicit that pressure field may have to be modified as suggested in the earlier definition. It is also implicit that in compressible fluids, density, temperature and entropy fields may have to be modified. In viscous compressible fluids, even the viscosity field has to be modified (Kapur 1961*a*). (iv) Nothing is said in this definition about the external force system. This is not a restriction since in almost all discussions in Fluid Dynamics, we do not consider external force systems. Moreover since a conservative external force influences only the pressure field, we need not insist on the force system for the superposed motion to be given by force potential $\Omega_1+\Omega_2$. In fact if we assume that it is given by $\Omega_1+\Omega_2+\Omega$ then we have for the two fluid motions and for the superposed motion

$$\frac{\partial \vec{q}_1}{\partial t}+\left(\vec{\omega}_1\times\vec{q}_1\right)+\text{grad}\left(\tfrac{1}{2}\vec{q}_1^{\,2}+\frac{p_1}{\rho}+\Omega_1\right)=-\nu\,\text{curl}\,\vec{\omega}_1 \quad \ldots \quad \ldots \quad (1)$$

$$\frac{\partial \vec{q}_2}{\partial t}+\left(\vec{\omega}_2\times\vec{q}_2\right)+\text{grad}\left(\tfrac{1}{2}\vec{q}_2^{\,2}+\frac{p_2}{\rho}+\Omega_2\right)=-\nu\,\text{curl}\,\vec{\omega}_2 \quad \ldots \quad \ldots \quad (2)$$

$$\frac{\partial\left(\vec{q}_1+\vec{q}_2\right)}{\partial t}+\left(\vec{\omega}_1+\vec{\omega}_2\right)\times\left(\vec{q}_1+\vec{q}_2\right)+\text{grad}\left\{\tfrac{1}{2}\left(\vec{q}_1+\vec{q}_2\right)^2 + \frac{p_1+p_2+\pi}{\rho}+\Omega_1+\Omega_2+\Omega\right\}=-\nu\,\text{curl}\left(\vec{\omega}_1+\vec{\omega}_2\right) \quad \ldots \quad (3)$$

where $\vec{\omega}_1$, $\vec{\omega}_2$ are vorticity vectors and $\nu$ is the kinematic coefficient of viscosity. From (1), (2) and (3)

$$\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\text{grad}\left(\vec{q}_1\cdot\vec{q}_2+\frac{\pi}{\rho}+\Omega\right)=0. \quad \ldots \quad \ldots \quad (4)$$

By taking curl of both sides of (4), we get the condition of superposability

$$\operatorname{curl}\left(\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1\right)=0 \qquad (5)$$

which is the same as obtained by Bhatnagar and Verma (1957) with the earlier definition. Equation (4) shows that only the adjustment in pressure is to be considered.

Our not mentioning the force system in the definition helps in clearing a misunderstanding which the earlier definition sometimes may create. It appears from the earlier definition that if two motions are possible under two different systems of forces, then we can talk of superposed motion only under a force system which is the vector sum of the two force systems. Thus if both the motions are possible under gravity $g$, then the superposed motion is possible under gravity $2g$. The above discussion shows that our modification of the original definition removes this difficulty. The difficulty could also be overcome by replacing $(\vec{q}_1+\vec{q}_2,\ p_1+p_2+\pi,\ \Omega_1+\Omega_2)$ in the original definition by $(\vec{q}_1+\vec{q}_2,\ p_1+p_2+\pi,\ \Omega_1+\Omega_2+\Omega)$.

2.3. *Superposability condition for incompressible fluids*

For a general incompressible fluid, the equations of continuity and motion are:

$$\nabla\cdot\vec{q}=0 \qquad (6)$$

$$\frac{\partial\vec{q}}{\partial t}+\vec{\omega}\times\vec{q}+\operatorname{grad}\left(\tfrac{1}{2}\vec{q}^2+\frac{p}{\rho}+\Omega\right)=\frac{1}{\rho}\nabla\cdot T \qquad (7)$$

where $T$ is the deviatoric stress tensor.

If $T_1$, $T_2$, $T_{12}$ are the deviatoric stress tensors for the motions $\vec{q}_1$, $\vec{q}_2$, $\vec{q}_1+\vec{q}_2$ respectively, then the condition for superposability is

$$\operatorname{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1\right]=\frac{1}{\rho}\operatorname{curl}\left[\nabla\cdot(T_{12}-T_1-T_2)\right]. \qquad (8)$$

If the relation between stress tensor $T$ and strain rate tensor $D$ is represented by

$$T=f(D) \qquad (9)$$

then condition (8) becomes

$$\operatorname{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1\right]=\frac{1}{\rho}\operatorname{curl}\left[\nabla\cdot\{f(D_1+D_2)-f(D_1)-f(D_2)\}\right] \qquad (10)$$

For Newtonian fluids

$$T=f(D)=\mu D \qquad (11)$$

where $\mu$ is the constant coefficient of viscosity.

Thus

$$\begin{aligned}\text{curl}\,[\nabla\,.\,(T_{12}-T_1-T_2)] &= \frac{1}{\rho}\,\text{curl}\,[\nabla\,.\,\{f(D_1+D_2)-f(D_1)-f(D_2)\}] \\ &= \frac{1}{\rho}\,\text{curl}\,[\nabla\,.\,\{\mu(D_1+D_2)-\mu D_1-\mu D_2\}] \\ &= 0\end{aligned}$$

and so the condition of superposability reduces to (5).

We consider the non-Newtonian fluid for which

$$T = \sum_{n=1}^{N} \mu_n D^n = \mu_1 D+\mu_2 D^2+\ldots+\mu_N D^N. \quad \ldots \quad (12)$$

The condition of superposability becomes

$$\begin{aligned}\text{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1\right] &= \text{curl}\left[\nabla\,.\left\{\sum_{n=1}^{N}\left(\nu_n(D_1+D_2)^n-\nu_n D_1^n-\nu_n D_2^n\right)\right\}\right] \\ &= \text{curl div}\,\{\nu_2(D_1D_2+D_2D_1)+\nu_3\left(D_1^2D_2+D_2D_1^2\right. \\ &\quad \left.+D_2^2D_1+D_1D_2^2+D_1D_2D_1+D_2D_1D_2\right)+\ldots\} \quad \ldots \quad (13)\end{aligned}$$

Thus the condition of superposability is independent of the first coefficient of viscosity, but it depends on all the other coefficients of viscosity.

The ordinary non-Newtonian fluids with constant coefficients of viscosity and cross-viscosity is a particular case of (12) when $N = 2$.

### 2.4. *Simultaneous additivity and transitivity of additivity of fluid motions*

The condition that $m$ motions $\vec{q}_1, \vec{q}_2, \ldots\ldots, \vec{q}_m$ are simultaneously additive is (Kapur 1961*a*)

$$\begin{aligned}&\text{curl}\left[\left(\vec{\omega}_1+\vec{\omega}_2+\ldots+\vec{\omega}_m\right)\times\left(\vec{q}_1+\vec{q}_2+\ldots+\vec{q}_m\right)-\left(\vec{\omega}_1\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2+\ldots\right.\right. \\ &\left.\left.\ldots+\vec{\omega}_m\times\vec{q}_m\right)\right] = \text{curl div}\left[\sum_{n=1}^{N}\left\{\nu_n(D_1+D_2+\ldots+D_m)^n-\nu_n(D_1^n+D_2^n+\ldots+D_m^n)\right\}\right] \quad (14)\end{aligned}$$

For $N = 2$, *i.e.* for non-Newtonian fluids with constant coefficients of viscosity and cross-viscosity the following theorems, which are true for Newtonian fluids, are true.

*Theorem 1:* If every pair of $m$ fluid flows is additive, then the $m$ fluids are simultaneously additive, but the converse of this in general is not true.

*Theorem 2:* If the flows I and II are additive and flows II and III are additive, then flows I and III will be additive if and only if the flows I, II and III are simultaneously additive.

It is easily seen that neither of the theorems is true for $N > 2$.

2.5. *Condition of integrability*

By taking curl of both sides of (7), we get the condition of integrability, *viz.*

$$\frac{\partial \vec{\omega}}{\partial t}+\text{curl}\left(\vec{\omega}\times\vec{q}\right)=\frac{1}{\rho}\text{ curl div }T \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

or

$$\frac{\partial \vec{\omega}}{\partial t}+\text{curl}\left(\vec{\omega}\times\vec{q}\right)=\frac{1}{\rho}\text{ curl div }f(D). \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

## 3. Self-additive and homogeneously additive Flows

A motion is said to be self-additive if it is additive to itself, *i.e.* a vector $\vec{q}$ will be said to be self-additive if when $\vec{q}$ is a possible vector field, $2\vec{q}$ is also a possible vector field.

A motion will be said to be homogeneously additive if whenever $\vec{q}$ is a possible vector field, $K\vec{q}$ is also a possible vector field for all constant values of $K$.

It is obvious that if a motion is homogeneously additive, it will be self-additive, but the converse need not be true.

From (16), a motion will be homogeneously additive if

$$K\frac{\partial \vec{\omega}}{\partial t}+K^2\text{ curl}\left(\vec{\omega}\times\vec{q}\right)=\frac{1}{\rho}\text{curl div}f(KD) \quad \ldots \quad \ldots \quad (17)$$

is true for all $K$.

A motion will be self-additive if both (16) and

$$\frac{\partial \vec{\omega}}{\partial t}+2\text{ curl}\left(\vec{\omega}\times\vec{q}\right)=\frac{1}{2\rho}\text{curl div}f(2D) \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

are simultaneously true.

For fluid given by (12), eqn. (17) becomes

$$K\frac{\partial \vec{\omega}}{\partial t}+K^2\text{ curl}\left(\vec{\omega}\times\vec{q}\right)=\text{curl div}\left[\sum_{n=1}^{N}\nu_n K^n D^n\right]. \quad \ldots \quad \ldots \quad (19)$$

If it is to be true for all $K$, we get

$$\frac{\partial \vec{\omega}}{\partial t}=\nu_1\text{ curl div }D=\nu_1\nabla^2\vec{\omega} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$\text{curl}\left(\vec{\omega}\times\vec{q}\right)=\nu_2\text{ curl div }D^2 \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

$$0=\nu_3\text{ curl div }D^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

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$$0=\nu_N\text{ curl div }D^N. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

For a vector $\vec{q}$ to be homogeneously additive, $N$ equations have to be simultaneously satisfied. For non-Newtonian fluids with constant coefficients of viscosity and cross-viscosity there are only two conditions, *viz.* (20) and (21).

For self-additivity, we get from (16) and (18)

$$\frac{\partial \vec{\omega}}{\partial t}+\text{curl}\left(\vec{\omega}\times\vec{q}\right) = \text{curl div}\left\{\sum_{n=1}^{N} \nu_n D^n\right\} \quad \ldots \quad (24)$$

and

$$\frac{\partial \vec{\omega}}{\partial t}+2\,\text{curl}\left(\vec{\omega}\times\vec{q}\right) = \text{curl div}\left\{\sum_{n=1}^{N} 2^{n-1}\nu_n D^n\right\}. \quad \ldots \quad (25)$$

These are equivalent to

$$\frac{\partial \vec{\omega}}{\partial t} = \text{curl div}\left\{\sum_{n=1}^{N} (2-2^{n-1})\nu_n D^n\right\} \quad \ldots \quad (26)$$

and

$$\text{curl}\left(\vec{\omega}\times\vec{q}\right) = \text{curl div}\left\{\sum_{n=1}^{N} (2^{n-1}-1)\nu_n D^n\right\}. \quad \ldots \quad (27)$$

For $N = 2$, these become

$$\frac{\partial \vec{\omega}}{\partial t} = \nu_1\,\text{curl div}\, D = \nu_1 \nabla^2 \vec{\omega} \quad \ldots \quad (28)$$

$$\text{curl}\left(\vec{\omega}\times\vec{q}\right) = \nu_2\,\text{curl div}\, D^2 \quad \ldots \quad (29)$$

which are the same as (20) and (21).

Thus for Newtonian fluids and non-Newtonian fluids with constant coefficients of viscosity and cross-viscosity, the two concepts of homogeneous additivity and self-additivity are equivalent, but for more general non-Newtonian fluids, every homogeneously additive flow is self-additive, but not vice versa.

To get solutions of Navier-Stokes equations, we can try to obtain

(i) solutions of the simultaneous set (20) to (23), or
(ii) solutions of (24) and (25), or
(iii) solutions of (26) and (27), or
(iv) solutions of

$$\frac{\partial \vec{\omega}}{\partial t} = \nu_1 \nabla^2 \vec{\omega} \quad \ldots \quad (30)$$

and

$$\text{curl}\left(\vec{\omega}\times\vec{q}\right) = \text{curl div}\left\{\sum_{n=2}^{N} \nu_n D^n\right\}. \quad \ldots \quad (31)$$

The last set will not give self-additive or homogeneously additive flows, but still may give useful solutions.

From (14), (28) and (29) we deduce the following theorems for non-Newtonian fluids with constant coefficients of viscosity and cross-viscosity:

*Theorem 3:* If $m$ self-additive motions are simultaneously additive, then the superposed motion obtained by superposing them is also self-additive.

*Theorem 4:* If each of $m$ motions is self-additive and the motions are additive in pairs, then the superposed motion is self-additive.

*Theorem 5:* If $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_m$ are possible vector fields, the necessary and sufficient conditions that

$$c_1\vec{q}_1 + c_2\vec{q}_2 + \ldots + c_m\vec{q}_m \quad \ldots \quad (32)$$

should represent possible vector fields for all $c_1, c_2, \ldots, c_m$ are

$$\text{curl}\left[\vec{\omega}_i \times \vec{q}_j + \vec{\omega}_j \times \vec{q}_i\right] = \nu_2 \text{ curl div } \{D_iD_j + D_jD_i\} \quad \ldots \quad (33)$$

$$[i, j = 1, 2, \ldots, m]$$

*i.e.* all the motions should be self-additive and the motions should be additive in pairs. Further, if these $\frac{m(m+1)}{2}$ conditions are satisfied, all motions given by (32) would be self-additive.

*Theorem 6:* If $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_m$ are each self-superposable and if $\vec{q}_i + \vec{q}_j$ are also self-superposable for all $i, j$, then every pair of them is superposable.

*Theorem 7:* If $\vec{q}$ be superposable on each of $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_m$, then $\vec{q}$ is superposable on any linear combination of $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_m$.

These theorems generalize the theorems of Kapur (1961*a*), Srivastava (1956) and Rathna and Rajeshwari (1961*b*).

## 4. On the Existence of Steady Viscous Beltrami Flows

Truesdell (1954) remarked: 'It would be valuable to investigate in general the question of the consistency of a Beltrami motion with the adherence condition at a rigid boundary.' Motivated by this remark Rathna and Rajeshwari (1961*a*) carried out an investigation and concluded that 'a steady Beltrami flow in viscous liquids under conservative forces is not possible'. Their investigation was based on the characterization of such flows as given by Kapur (1960*b*).

We note that the proof given by them does not establish the non-existence of Beltrami flows given by the general integral of the partial differential equation concerned, so that the authors are led to 'surmise' the final result. What the proof achieves is to make the existence of steady axially-symmetric viscous Beltrami flows very implausible.

Moreover the characterization in Kapur (1960*b*) referred to only axially-symmetric flows and, as such, the discussion in Rathna and Rajeshwari (1961*a*) refers only to such flows. We know that two-dimensional Beltrami motions are not possible, but even then we have the vast range of three-dimensional Beltrami flows which may not be axially-symmetric and which are characterized by

$$\operatorname{div} \vec{q} = 0, \ \nabla^2(\lambda \vec{q}) = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (34)$$

The discussion given in Rathna and Rajeshwari (1961*a*) does not, of course, throw any light on the remark of Truesdell (1954) referred to above, since it does not investigate the consistency of a Beltrami motion with the adherence condition at a rigid boundary. This interesting question still remains open.

It is stated in Rathna and Rajeshwari (1961*a*) that the characterization in Kapur (1960*b*) had left out the following equation for the stream function $\psi$,

$$E^4\psi \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \psi = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (35)$$

In point of fact, this equation occurs in (18) and (19) of that paper. In fact the eqn. (24) of that paper could not have been deduced without making use of the above equation. It is also obvious that eqns. (2.12) and (2.13) of Rathna and Rajeshwari (1961*a*) have been taken from eqns. (19) and (24) of Kapur (1960*b*).

The characterization (25) of Kapur (1960*b*), *viz.* that if

$$\vec{q} = -\frac{1}{r}\frac{\partial \psi}{\partial z}\vec{i_r} + \frac{\Omega}{r}\vec{i_\theta} + \frac{1}{r}\frac{\partial \psi}{\partial r}\vec{i_z} \qquad \ldots \quad \ldots \quad \ldots \quad (36)$$

where $\vec{i_r}$, $\vec{i_\theta}$, $\vec{i_z}$ are the unit vectors in cylindrical polar coordinates $r$, $\theta$, $z$, then for steady viscous axially-symmetric Beltrami flows

$$\Omega = f(\psi); \ E^2\psi + b[f(\psi) - a] = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (37)$$

where

$$f(\psi) + a \log(f(\psi) - a) = b\psi + c \qquad \ldots \quad \ldots \quad \ldots \quad (38)$$

is certainly subject to the consideration that $\psi$ determined from these equations has to be consistent with (20) and (21) of that paper which reduce to

$$\left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \frac{f^2(\psi)[f(\psi) - a]}{a}. \qquad \ldots \quad \ldots \quad \ldots \quad (39)$$

Incidentally, this provides an alternative proof of the result of Rathna and Rajeshwari's paper. The complete integral of (39) is

$$\phi(\psi) = r \cos \alpha + z \sin \alpha + c \qquad \ldots \quad \ldots \quad \ldots \quad (40)$$

where $\alpha$ and $c$ are constants and

$$\frac{1}{\phi'^2(\psi)} = \frac{f^2(\psi)[f(\psi)-a]}{a}. \qquad (41)$$

Solution (40) does not satisfy

$$E^2\psi + b[f(\psi)-a] = 0. \qquad (42)$$

The general integral is obtained by eliminating $\alpha$ between

$$\phi(\psi) = r\cos\alpha + z\sin\alpha + h(\alpha)$$

and

$$0 = -r\sin\alpha + z\cos\alpha + h'(\alpha). \qquad (43)$$

Substituting in (42), we get

$$b[f(\psi)-a] + \frac{1}{\phi'}\left\{\frac{1}{\phi(\psi)-h(\alpha)} - \frac{\phi''}{\phi'^2} - \frac{\cos\alpha}{r}\right\} = 0 \qquad (44)$$

which will not, in general, be true and thus we find that steady viscous axially-symmetric Beltrami flows are highly unlikely to exist.

In Kapur (1960*b*) the particular case suggested, *viz.*

$$a = 0,\ c = 0,\ f(\psi) = b\psi \qquad (45)$$

does not satisfy (39) and will not thus give viscous Beltrami flows.

## 5. Characterization of steady axially-symmetric viscous Flows

In Kapur (1960*b*), it has been shown that for such flows $\psi$ and $\Omega$ must satisfy the equations

$$\Omega = f(\psi) \qquad (46)$$

$$E^2(\psi) + f(\psi)f'(\psi) = r^2 g'(\psi) \qquad (47)$$

$$f'(\psi)[-f(\psi)f'(\psi) + r^2 g'(\psi)] + f''(\psi)\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = 0 \qquad (48)$$

$$[f(\psi)f''(\psi) + f'^2(\psi)][-f(\psi)f'(\psi) + r^2 g'(\psi)]$$
$$+ [f(\psi)f'''(\psi) + 3f'(\psi)f''(\psi)]\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = 0. \qquad (49)$$

The first three equations are all right, but the fourth must be replaced by

$$[ff''' + 3f'f'' - r^2 g'''] \left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] + [ff'' - f'^2 - r^2 g''][-ff' + r^2 g']$$
$$= 4rg''\left(\frac{\partial\psi}{\partial r}\right). \qquad (50)$$

This also shows that the characterization (38) of that paper must be replaced by (46), (47), (48) and (50) above except in the case when $g'(\psi)$ is a constant.

When $g'(\psi) = 0$, we get axially-symmetric Beltrami flows.

For a purely poloidal steady axially-symmetric self-superposable viscous flow $f(\psi) = 0$ and the characterization becomes

$$E^2\psi = r^2h(\psi) \qquad \ldots \quad (51)$$

$$h''\left[\left(\frac{\partial\psi}{\partial r}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2\right]+hh'r^2 = -\frac{4}{r}h'\frac{\partial\psi}{\partial r}. \qquad \ldots \quad (52)$$

We find that

(i) $h(\psi) = 0$ gives irrotational flows characterized by

$$E^2\psi = 0 \qquad \ldots \quad (53)$$

which are all self-superposable.

(ii) If $h(\psi) =$ a constant $c$, the second equation is identically satisfied and self-superposable flows are given by

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2} = cr^2. \qquad \ldots \quad (54)$$

If $\psi$ is a function of $r$ only, we get

$$\psi = A+Br^2+\frac{cr^4}{8} \qquad \ldots \quad (55)$$

$$\vec{q} = \left(2B+\frac{c}{2}r^2\right)\vec{i_z}. \qquad \ldots \quad (56)$$

This gives us the well-known Poiseuille flow in a circular tube showing that for such flows the vorticity is proportional to $r$. (iii) The only steady axially-symmetric flows which are possible for Newtonian viscous or non-Newtonian viscous fluids with constant coefficients of viscosity and cross-viscosity and in which $\Omega$ and $\psi$ are functions of $r$ only are given by

$$\Omega = Ar^2+B \qquad \ldots \quad (57)$$

$$\psi = C_1+D_1r^2+E_1r^2\log r^2+F_1r^4 \qquad \ldots \quad (58)$$

$$\vec{q} = \left(Ar+\frac{B}{r}\right)\vec{i_\theta}+[2D_1+2E_1(\log r^2+1)+4F_1r^2]\vec{i_z} \qquad \ldots \quad (59)$$

$$\text{curl}\,\vec{q} = -\left(\frac{4E_1}{r}+8F_1r\right)\vec{i_\theta}+2A\vec{i_z} \qquad \ldots \quad (60)$$

$$\vec{q}\times\text{curl}\,\vec{q} = \left[2A\left(Ar+\frac{B}{r}\right)+\left(\frac{4E_1}{r}+8F_1r\right)\{2D_1+2E_1(\log r^2+1)+4F_1r^2\}\right]\vec{i_z}. \qquad (61)$$

It is easily verified that

$$\text{curl}\left[\vec{q}\times\text{curl}\,\vec{q}\right] = 0.$$

Thus all motions in which $\psi$ and $\Omega$ are functions of $r$ only are self-superposable.

It is also verified that $\vec{q}\times\text{curl}\ \vec{q}=0$ if and only if $A=0$, $E_1=0$, $F_1=0$ in which case curl $\vec{q}=0$, so that no Beltrami motions of this kind are possible.

The present motions include:

(i) Poiseuille flow in a circular pipe.

(ii) Poiseuille flow in an annulus.

(iii) Flow between two rotating cylinders.

(iv) We consider steady flows in which all velocity components are functions of $r$ only. These are given by

$$\Omega = Ar^{2-R}+B \qquad \text{(62)}$$

$$E^2\psi = -\frac{Cr^2}{R+2}(r^2+C^2)^{-\left(\frac{R}{2}+1\right)}+Fr^2 \qquad \text{(63)}$$

where

$$R=\frac{K}{\nu},\ C^2=\frac{2K\nu_c}{\nu}. \qquad \text{(64)}$$

This will satisfy the condition of self-superposability if

$$E^2\Omega = 0,\ E^4\psi = 0. \qquad \text{(65)}$$

If

$$A=0,\ C=0$$

giving

$$\Omega = B,\ E^2\psi = Fr^2 \qquad \text{(66)}$$

$$\vec{q} = -\frac{K}{r}\vec{i_r}+\frac{B}{r}\vec{i_\theta}+\left(\frac{Fr^2}{2}+G\right)\vec{i_z} \qquad \text{(67)}$$

$$\text{curl}\ \vec{q} = -Fr\vec{i_\theta} \qquad \text{(68)}$$

$$\vec{q}\times\text{curl}\ \vec{q} = KF\vec{i_z}+Fr\left(\frac{Fr^2}{2}+G\right)\vec{i_r}. \qquad \text{(69)}$$

No rotational axially-symmetric flow of the type considered can be a Beltrami flow, but all such flows are self-superposable.

(v) The axially-symmetric self-superposable steady non-Newtonian flows are characterized by

$$E^2\Omega = 0 \qquad \text{(70)}$$

$$\frac{1}{r}\frac{\partial(\psi,\Omega)}{\partial(r,z)} = r\nu_c\left[\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)D_{12}+\frac{\partial}{\partial z}D_{23}\right] \qquad \text{(71)}$$

$$E^4\psi = 0 \qquad \text{(72)}$$

$$\frac{2\Omega\Omega_z}{r^2}+\frac{1}{r}\frac{\partial(\psi, E^2\psi)}{\partial(r,z)}+\frac{2}{r^2}\psi_z E^2\psi = r\nu_c\left[\frac{\partial^2}{\partial r^2}(D_{13})+\frac{\partial}{\partial r}\left(\frac{D_{13}}{r}\right)\right.$$
$$\left.+\frac{\partial^2}{\partial r\partial z}(D_{33}-D_{11})-\frac{\partial}{\partial z}\left(\frac{D_{11}-D_{22}}{r}\right)-\frac{\partial^2 D_{13}}{\partial z^2}\right] \qquad \text{(73)}$$

$$[D_{ij}] = [d_{ij}]^2. \qquad \text{(74)}$$

For purely poloidal fields these reduce to

$$E^4\psi = 0 \quad \ldots \quad (75)$$

$$\frac{1}{r}\frac{\partial(\psi, E^2\psi)}{\partial(r, z)} + \frac{2}{r^2}\psi_z E^2\psi = r\nu_c\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right)\left\{2\frac{\psi_z}{r^3}\left(\psi_{rr} - \psi_{zz} - \frac{\psi_r}{r}\right)\right\} + 4\frac{\partial^2}{\partial r\partial z}\left\{\frac{2\psi_z\psi_{rz}}{r^3} - \frac{\psi_z^2}{r^4}\right\} - \frac{\partial}{\partial z}\frac{1}{r}\left\{4\left(\frac{\psi_z}{r^2} - \frac{\psi_{rz}}{r}\right)^2 + \left(\frac{\psi_{rr}}{r} - \frac{\psi_{zz}}{r} - \frac{\psi_r}{r^2}\right)^2 - \frac{4\psi_z^2}{r^4}\right\}\right]. \quad \ldots \quad (76)$$

For purely toroidal field, we get just

$$E^2\Omega = 0. \quad \ldots \quad (77)$$

If $\Omega$ and $\psi$ are functions of $r$ only, we get

$$\Omega_{rr} - \frac{\Omega_r}{r} = 0 \quad \ldots \quad (78)$$

$$\psi_{rrrr} - \frac{2}{r}\psi_{rrr} + \frac{3}{r^2}\psi_{rr} - \frac{3}{r^3}\psi_r = 0. \quad \ldots \quad (79)$$

This is the case discussed in (iii).

If all velocity components are functions of $r$ only, we get

$$\frac{d\Omega}{dr} = 0 \quad \ldots \quad (80)$$

$$\frac{d^2\Omega}{dr^2} - \frac{1}{r}\frac{d\Omega}{dr} = 0 \quad \ldots \quad (81)$$

$$E^4\psi = 0 \quad \ldots \quad (82)$$

$$-\frac{K}{r}\frac{d}{dr}(E^2\psi) + \frac{2K}{r^2}E^2\psi = r\nu_c\left\{\frac{d^2}{dr^2}\left(\frac{2K}{r^3}E^2\psi\right) + \frac{d}{dr}\left(\frac{2K}{r^4}E^2\psi\right)\right\} \quad \ldots \quad (83)$$

the case discussed in (iv).

## 6. Pseudo-planar steady viscous Beltrami and self-superposable Flows

(i) *Beltrami flows*

We consider a two-dimensional flow $(u, v, o)$ with stream function $\psi$ so that

$$u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x},\ \vec{\omega} = -\nabla^2\psi\vec{i_z}. \quad \ldots \quad (84)$$

In two-dimensional flow, condition of integrability is

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = -\nu\nabla^4\psi. \quad \ldots \quad (85)$$

If the flow $(u, v, w)$ is a Beltrami flow, we get

$$\frac{\dfrac{\partial\psi}{\partial y}}{\dfrac{\partial w}{\partial y}}=\frac{-\dfrac{\partial\psi}{\partial x}}{-\dfrac{\partial w}{\partial x}}=\frac{w}{-\nabla^2\psi} \qquad (86)$$

from where

$$w=f(\psi) \qquad (87)$$

$$\nabla^2\psi=-f(\psi)f'(\psi)=\phi(\psi)\,(\text{say}). \qquad (88)$$

Substituting in (85)

$$\nabla^4\psi=0 \qquad (89)$$

or

$$\nabla^2(\phi(\psi))=0 \quad \text{or} \quad \left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2=-\frac{\phi\phi'}{\phi''}, \qquad (90)$$

so that the velocity magnitude is constant along streamlines. Thus the streamlines for the two-dimensional flows will be either circles or straight lines.

For steady Beltrami flow to be possible, the conditions of integrability are

$$\nabla^2\left[\frac{\partial w}{\partial y},\,-\frac{\partial w}{\partial x},\,-\nabla^2\psi\right]=0 \qquad (91)$$

*i.e.*

$$\nabla^2(f(\psi))=C,\ \nabla^4\psi=0 \qquad (92)$$

$$f''\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]+f'\,\nabla^2\psi=C \qquad (93)$$

From (88), (90) and (93) we get

$$f''\left[-\frac{\phi\phi'}{\phi''}\right]+f'\phi=C$$

or

$$f^2f'f''^2-f^2f'^2f'''-2ff'^3f''=C[ff'''+3f'f'']. \qquad (94)$$

This is the equation to determine $f(\psi)$. The common solution of the eqns. (88) and (90), if that exist, will determine pseudo-planar steady viscous Beltrami flows.

(ii) *Self-superposable flows*

In three-dimensional flow $(u, v, w)$ respective conditions of self-superposability and integrability are

$$\text{curl}\left(\vec{\omega}\times\vec{q}\right)=0 \qquad (95)$$

and

$$\nabla^2\vec{\omega}=0$$

*i.e.*

$$\nabla^2 w=c,\ \nabla^4\psi=0. \qquad (96)$$

From (95) we get

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0, \quad i.e.\ \nabla^2\psi = g(\psi) \quad .. \quad .. \quad .. \quad (97)$$

where $g(\psi)$ is an arbitrary function of $\psi$.
From (96) we get

$$w = F_1(x+iy)+F_2(x-iy)+\frac{c}{4}(x^2+y^2) \quad .. \quad .. \quad .. \quad (98)$$

and

$$\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2 = -\frac{gg'}{g''}. \quad .. \quad .. \quad .. \quad (99)$$

If for any arbitrary function $g(\psi)$ the eqns. (97) and (99) happen to have common solutions, they will determine steady viscous self-superposable flows.

Equations determining pseudo-planar steady viscous Beltrami and self-superposable flows happen to be identical except that in the former case $f(\psi)$ has to satisfy eqn. (94) while in the latter case $g(\psi)$ is an arbitrary function.

## 7. Superposability Conditions for incompressible Fluids when only one Flow satisfies Condition of Integrability

One of the important uses to which the concept of superposability has been recently put is in the construction of new solutions of the basic equations of incompressible viscous fluid dynamics, gas dynamics and magnetogas dynamics (Kapur 1961*b*, 1961*c*; Kapur and Bhatia 1963). Knowing a velocity distribution to be a solution of the basic equations, it is desired to find a more general velocity distribution by adding velocity components to the original distribution. It is necessary that the original and the final velocity distributions satisfy the basic equations. It is, however, not necessary for the added velocity components to satisfy the equations. It is this idea which is made use of in this section.

In this case, condition of superposability is

$$\frac{\partial\vec{\omega}_2}{\partial t}+\text{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2\right] = \frac{1}{\rho}\,\text{curl}\,[\nabla\,.\,(T_{12}-T_1)] \quad .. \quad (100)$$

where $T_{12}$ and $T_1$ are deviatoric stress tensors for motions $\vec{q}_1+\vec{q}_2$ and $\vec{q}_1$ respectively. The condition of integrability for the motion $\vec{q}_1$ has only been taken into account.

If the relation between stress tensor $T$ and strain rate tensor $D$ is given by eqn. (9), the condition of superposability (100) becomes

$$\frac{\partial\vec{\omega}_2}{\partial t}+\text{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2\right] = \frac{1}{\rho}\,\text{curl}\,[\nabla\,.\,\{f(D_1+D_2)-f(D_1)\}]. \quad .. \quad (101)$$

For steady flows, the eqn. (101) becomes

$$\operatorname{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2\right]=\frac{1}{\rho}\operatorname{curl}\left[\nabla\cdot\{f(D_1+D_2)-f(D_1)\}\right]. \quad ..\ (102)$$

For Newtonian fluids from (11) and (102) we get the condition of superposability as follows:

$$\operatorname{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2\right]=\frac{1}{\rho}\operatorname{curl}\left[\nabla\cdot\mu D_2\right]. \quad ..\quad ..\ (103)$$

For non-Newtonian fluids from (12) and (102) we get the condition of superposability as follows:

$$\operatorname{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2\right]=\operatorname{curl}\left[\nabla\cdot\left\{\sum_{n=1}^{N}\left(\nu_n(D_1+D_2)^n-\nu_n D_1^n\right)\right\}\right]$$

$$=\operatorname{curl}\operatorname{div}\left[\nu_1 D_2+\nu_2\left(D_1D_2+D_2D_1+D_2^2\right)\right.$$

$$\left.+\nu_3\left(D_2^3+D_1^2D_2+D_2D_1^2+D_2^2D_1+D_1D_2^2+D_1D_2D_1+D_2D_1D_2\right)+\ldots\right]. \quad (104)$$

Thus condition of superposability is not even independent of the first coefficient of viscosity when the condition of integrability in flow $\vec{q}_1$ has only been taken into account unlike the case when condition of integrability of both the flows is considered.

### 7.1. *Construction of three-dimensional solutions of equations of fluid dynamics from two-dimensional solutions by using new principle of superposability*

In non-Newtonian incompressible steady fluid flow $(u, v, w)$ the conditions of integrability of equations of motion from eqns. (9) and (10) of Kapur and Bhatia (1963)

$$-\frac{uu_\theta}{r}+uu_{r\theta}+\frac{vu_{\theta\theta}}{r}-\frac{2vv_\theta}{r}+vu_r+rvu_{rr}-uv_r-ruv_{rr}$$

$$=\nu\left[\frac{u_{\theta\theta\theta}}{r^2}+2u_{rr\theta}-\frac{2v_{\theta\theta}}{r^2}-2v_{rr}-rv_{rrr}+\frac{v_r}{r}-\frac{v}{r^2}\right]$$

$$+\nu_c\left[w_r\frac{\partial}{\partial\theta}\left(w_{rr}+\frac{w_{\theta\theta}}{r^2}+\frac{w_r}{r}\right)-w_\theta\frac{\partial}{\partial r}\left(w_{rr}+\frac{w_{\theta\theta}}{r^2}+\frac{w_r}{r}\right)\right] \quad ..\ (105)$$

and

$$uw_r+\frac{v}{r}w_\theta=\nu\left(w_{rr}+\frac{1}{r^2}w_{\theta\theta}+\frac{w_r}{r}\right)+\nu_c\left[w_r\left(u_{rr}+\frac{u_{\theta\theta}}{r^2}-\frac{v_\theta}{r^2}\right)\right.$$

$$+2u_rw_{rr}-\frac{2}{r^2}u_rw_{\theta\theta}+w_\theta\left(\frac{v_{\theta\theta}}{r^3}+\frac{v_{rr}}{r}-\frac{v_r}{r^2}+\frac{v}{r^3}\right)$$

$$\left.+2w_{r\theta}\left(\frac{u_\theta}{r^2}+\frac{v_r}{r}-\frac{v}{r^2}\right)\right] \quad ..\ (106)$$

where $u$, $v$ and $w$ are functions of $r$ and $\theta$ only.

Let us consider the condition of integrability in Newtonian flow $(u, v, o)$ only.

The eqn. (106) is satisfied identically and eqn. (105) reduces to

$$-\frac{uu_\theta}{r}+uu_{r\theta}+v\frac{u_{\theta\theta}}{r}-\frac{2vv_\theta}{r}+vu_r+rvu_{rr}-uv_r-ruv_{rr}$$

$$=\nu\left[\frac{u_{\theta\theta\theta}}{r^2}+2u_{rr\theta}-\frac{2v_{\theta\theta}}{r^2}-2v_{rr}-rv_{rrr}+\frac{v_r}{r}-\frac{v}{r^2}\right]. \quad \text{.. (107)}$$

Two-dimensional flow $(u, v, o)$ and $(o, o, w)$ are superposable if

$$uw_r+\frac{v}{r}w_\theta=\nu\left(w_{rr}+\frac{1}{r^2}w_{\theta\theta}+\frac{w_r}{r}\right). \quad \text{.. .. .. (108)}$$

We now consider some special cases.

(A) *Two-dimensional vortex*

Let a two-dimensional vortex of strength $K$ be placed at the origin. Then its complex potential is

$$W=iK\log r-K\theta, \quad \text{.. .. .. .. (109)}$$

so that its velocity potential is

$$\phi=-K\theta. \quad \text{.. .. .. .. (110)}$$

In Newtonian flows let us investigate the $z$-velocity component of three-dimensional flow $\left(0, \frac{K}{r}, w\right)$.

The superposability condition (108) becomes

$$\frac{K}{r^2}w_\theta=\nu\left((w_{rr}+\frac{1}{r^2}w_{\theta\theta}+\frac{w_r}{r}\right). \quad \text{.. .. .. (111)}$$

Integrating we get

$$w=\sum\left[C_t r^{\sqrt{\frac{K}{\nu}k-k^2}}e^{k\theta}+D_t r^{-\sqrt{\frac{K}{\nu}k-k^2}}e^{k\theta}\right] \quad \text{.. .. (112)}$$

where $C_t$ and $D_t$ are arbitrary constants.

Three-dimensional rotational motion

$$\left[0, \frac{K}{r}, \sum\left\{C_t r^{\sqrt{\frac{K}{\nu}k-k^2}}e^{k\theta}+D_t r^{-\sqrt{\frac{K}{\nu}k-k^2}}e^{k\theta}\right\}\right]$$

has been obtained from two-dimensional irrotational motion $\left(0, \frac{K}{r}\right)$ due to vortex.

Three-dimensional solution constructed here does not continue to hold in Reiner-Rivlin fluid flows with constant coefficients of viscosity and cross-viscosity unlike the usual case when the conditions of integrability are taken into account of both the flows.

(B) *Two-dimensional spiral vortex*

Let a two-dimensional spiral vortex, *i.e.* a source and a vortex, be situated at the origin, then the complex potential is given by (Milne-Thomson 1949)

$$W = -(m \log r+K\theta)+i(K \log r-m\theta), \qquad \ldots (113)$$

so that its velocity potential is given by

$$\phi = -(m \log r+K\theta). \qquad \ldots (114)$$

Let us investigate velocity component $w$ of three-dimensional Newtonian flow $\left(\frac{m}{r}, \frac{K}{r}, w\right)$.

Here the condition of superposability (108) becomes

$$r^2 w_{rr}+w_{\theta\theta}+rw_r = \frac{m}{\nu} rw_r+\frac{K}{\nu} w_\theta. \qquad \ldots (115)$$

Integrating we get

$$w = \sum \left\{E_i r^{\frac{m}{2\nu}+\sqrt{\frac{m^2}{4\nu^2}+\frac{K}{\nu}n-n^2}}+F_i r^{\frac{m}{2\nu}-\sqrt{\frac{m^2}{4\nu^2}+\frac{K}{\nu}n-n^2}}\right\}e^{n\theta}, \qquad \ldots (116)$$

where $n$, $Ei$ and $Fi$ are arbitrary constants.

Three-dimensional rotational motion

$$\left[\frac{m}{r}, \frac{K}{r}, \sum \left\{E_i r^{\frac{m}{2\nu}+\sqrt{\frac{m^2}{4\nu^2}+\frac{K}{\nu}n-n^2}}+F_i r^{\frac{m}{2\nu}-\sqrt{\frac{m^2}{4\nu^2}+\frac{K}{\nu}n-n^2}}\right\}e^{n\theta}\right]$$

has been obtained from two-dimensional irrotational motion $\left(\frac{m}{r}, \frac{K}{r}\right)$ due to spiral vortex.

Three-dimensional solution constructed here does not continue to hold in Reiner-Rivlin fluid flows with constant coefficients of viscosity and cross-viscosity unlike the usual case when the conditions of integrability are taken into account of both the flows.

(C) *Radial flow*

A radial flow has velocity components (Durand 1935)

$$u = \frac{1}{r} G(v) \qquad \ldots (117)$$

$$v = 0 \qquad \ldots (118)$$

and vorticity

$$\vec{\omega} = -\frac{1}{r^2} G'(\theta)\vec{i_z}. \qquad \ldots (119)$$

The condition of integrability (107) of Stokes-Navier equation reduces to

$$2GG'+\nu(G'''+4G') = 0. \qquad \ldots (120)$$

Let us investigate velocity component $w$ of three-dimensional Newtonian flow $\left(\frac{1}{r} G(\theta), O, w\right)$.

Here the condition of superposability (108) becomes

$$r^2 w_{rr} + w_{\theta\theta} + r w_r = \frac{G}{\nu} r w_r. \quad \ldots \quad \ldots \quad \ldots (121)$$

Integrating we get

$$w = \sum \left[ L_i r^K e^{\sqrt{\left(\frac{G}{\nu} K - K^2\right)}\theta} + M_i r^K e^{-\sqrt{\left(\frac{G}{\nu} K - K^2\right)}\theta} \right] \quad \ldots \quad \ldots (122)$$

when $G$ is constant, $L_i$ and $M_i$ are arbitrary constants.

When $G$ is not a constant, from (121) we get

$$\left(D^2 + D'^2 - \frac{G}{\nu} D\right) w = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (123)$$

where

$$D \equiv \frac{\partial}{\partial R}, \; D' \equiv \frac{\partial}{\partial \theta}, \; r = e^R. \quad \ldots \quad \ldots \quad \ldots (124)$$

By assuming solution of the form $w = R'(R)\Theta(\theta)$, the eqn. (123) reduces to

$$\frac{d^2\Theta}{d\theta^2} = f(\theta)\Theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots (125)$$

where

$$f(\theta) = \left(\frac{G}{\nu} K - K^2\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots (126)$$

and

$$\frac{1}{R'} \frac{dR'}{dR} = K, \; i.e. \; R' = \alpha e^{KR} = \alpha r^K \quad \ldots \quad \ldots \quad \ldots (127)$$

where $\alpha$ and $K$ are constants.

The solution of eqn. (125) depends on form of $f(\theta)$ which itself depends on the solution of eqn. (120).

When $G$ is constant, three-dimensional rotational flow obtained from two-dimensional irrotational radial flow $\left(\frac{G}{r}, O\right)$ is

$$\left[\frac{G}{r}, O, \sum \left\{ L_i r^k e^{\sqrt{\left(\frac{G}{\nu} K - K^2\right)}\theta} + M_i r^k e^{-\sqrt{\left(\frac{G}{\nu} K - K^2\right)}\theta} \right\} \right].$$

(D) *Two-dimensional vortex doublet*

The velocity potential due to a vortex doublet situated at the origin along the axis of $y$ is given by

$$\phi = \frac{\lambda \cos \theta}{r}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (128)$$

Now let us investigate velocity component $w$ of three-dimensional superposed Newtonian motion $\left(\frac{\lambda \cos \theta}{r^2}, \frac{\lambda \sin \theta}{r^2}, w\right)$.

The condition of superposability (108) becomes

$$r^2 w_{rr}+w_{\theta\theta}+r w_r=\frac{\lambda}{\nu}\left(\cos\theta w_r+\frac{\sin\theta}{r}w_\theta\right). \qquad (129)$$

This determines $w$.

7.2. *Construction of axi-symmetric solutions of equations of fluid dynamics from* (A) *poloidal velocity fields and* (B) *toroidal velocity fields by using new principle of superposability*

(A) Let us consider the condition of integrability in poloidal flow $\vec{q}_1$ only where

$$\vec{q}_1=-\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r+o\,\vec{i}_\theta+\frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z. \qquad (130)$$

Let the superposed axi-symmetric flow be given by

$$\vec{q}=-\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r+\frac{\Omega}{r}\vec{i}_\theta+\frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z. \qquad (131)$$

The condition of superposability is

$$\text{curl}\left[\vec{\omega}_1\times\vec{q}_2+\vec{\omega}_2\times\vec{q}_1+\vec{\omega}_2\times\vec{q}_2\right]=\nu\nabla^2\vec{\omega}_2 \qquad (132)$$

*i.e.*

$$\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial(\psi,\Omega)}{\partial(r,z)}\right]=\nu\frac{\partial}{\partial z}(E^2\Omega) \qquad (133)$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(\psi,\Omega)}{\partial(r,z)}\right]=\nu\frac{\partial}{\partial r}(E^2\Omega) \qquad (134)$$

$$\frac{\partial\Omega}{\partial z}=0 \qquad (135)$$

where

$$E^2\equiv\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \qquad (136)$$

From (133) and (135) we have

$$\frac{1}{r}\frac{\partial(\psi,\Omega)}{\partial(r,z)}=f(r) \qquad (137)$$

where $f(r)$ is an arbitrary function of $r$ only.

From eqns. (134) and (137) we get

$$E^2\Omega=\frac{1}{\nu}f(r)+\frac{E_1}{\nu} \qquad (138)$$

where $E_1$ is an arbitrary constant.

On solving eqn. (138) we get

$$\Omega=F+Cr^2+\frac{1}{\nu}\int r\int r^{-1}f(r)\,dr^2+\frac{E_1}{2\nu}r^2\log r \qquad (139)$$

where $F$ and $C$ are arbitrary constants.

In poloidal flow $\vec{q}_1$ condition of integrability is

$$r\frac{\partial\left[\frac{1}{r^2}E^2\psi,\psi\right]}{\partial[z,r]}=\nu E^4\psi. \qquad \ldots \quad (140)$$

Equation (137) gives

$$-\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial r}=f(r) \qquad \ldots \quad (141)$$

*i.e.* $\frac{\partial\psi}{\partial z}$ is a function of $r$ only;

$$\psi=z\phi(r)+\delta(r) \qquad \ldots \quad (142)$$

where $\phi(r)$ and $\delta(r)$ are functions of $r$ only.

From (139) and (141) we get

$$\phi(r)=\frac{-f(r)}{2C+\frac{E_1}{2\nu}+\frac{E_1}{\nu}\log r+\frac{1}{\nu}\int r^{-1}f(r)dr}. \qquad \ldots \quad (143)$$

From (140) and (142) we have

$$\nu r^4\delta''''-(2\nu-\phi)r^3\delta'''+3(\nu-\phi)r^2\delta''-(3\nu+r^2\phi''-r\phi'-3\phi)r\delta'$$
$$=-\nu z(r^4\phi''''-2r^3\phi'''+3r^2\phi''-3r\phi')+zr\phi'(r^2\phi''-r\phi')-z\phi(r^3\phi'''-3r^2\phi''+3r\phi'). \qquad \ldots \quad (144)$$

This determines $\delta(r)$. So superposed axi-symmetric flow is given by

$$\left[-\frac{1}{r}\phi(r),\frac{\Omega}{r},\frac{1}{r}(z\phi'(r)+\delta'(r))\right]$$

where $\Omega$ and $\phi(r)$ are respectively given by (139) and (143).

Since $f(r)$ is an arbitrary function of $r$, let us choose the function such that $\phi$ comes out equal to some constant quantity $M$.

Then eqn. (143) gives such function as follows:

$$f(r)=Hr^{-\frac{M}{\nu}}-E_1 \qquad \ldots \quad (145)$$

where $H$ is an arbitrary constant.

Equation (144) reduces to

$$\nu r^4\delta''''-(2\nu-M)r^3\delta'''+3(\nu-M)r^2\delta''-3(\nu-M)r\delta'=0. \qquad \ldots \quad (146)$$

Integrating we have

$$\delta(r)=\alpha+\beta r^2+\gamma r^4+\eta r^{2-\frac{M}{\nu}}. \qquad \ldots \quad (147)$$

From (139) and (145), we get

$$\frac{\Omega}{r}=\frac{F}{r}+\left(C+\frac{E_1}{4\nu}\right)r-\frac{Hr^{1-\frac{M}{\nu}}}{M\left(2-\frac{M}{\nu}\right)} \qquad \ldots \quad (148)$$

so that

$$\vec{q} = -\frac{M}{r}\vec{i_r} + \left[\frac{F}{r} + \left(C + \frac{E_1}{4\nu}\right)r - \frac{Hr^{1-\frac{M}{\nu}}}{M\left(2-\frac{M}{\nu}\right)}\right]\vec{i_\theta} + \left[2\beta + 4\gamma r^2 + \eta\left(2-\frac{M}{\nu}\right)r^{-\frac{M}{\nu}}\right]\vec{i_z}. \quad \text{.. (149)}$$

Since we require the modified pressure also to be axially symmetric, this would require that $M = 0$, so in that case the most general superposed flow is given by

$$\vec{q} = \left(B_1 r + \frac{c_1}{r}\right)\vec{i_\theta} + [D_1 + E_2 r^2 + F_1 \log r]\vec{i_z}. \quad \text{.. .. (150)}$$

Thus by assuming the value of arbitrary function $f(r)$ given by (145), the superposed flow happens to be the same as obtained in Kapur (1960*b*) by taking into account condition of integrability in both toroidal and poloidal flows.

(B) Now let us consider the condition of integrability in toroidal flow $\vec{q_2}$ only where

$$\vec{q_2} = \frac{\Omega}{r}\vec{i_\theta}. \quad \text{.. .. .. .. (151)}$$

The condition of superposability is

$$\text{curl}\left[\vec{\omega_1}\times\vec{q_2} + \vec{\omega_2}\times\vec{q_1} + \vec{\omega_1}\times\vec{q_1}\right] = \nu\nabla^2\vec{\omega_1} \quad \text{.. .. (152)}$$

*i.e.*

$$\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial(\psi, \Omega)}{\partial(r, z)}\right] = 0, \quad \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(\psi, \Omega)}{\partial(r, z)}\right] = 0 \quad \text{.. .. (153)}$$

and

$$r\frac{\partial\left(\frac{1}{r^2}E^2\psi, \psi\right)}{\partial(z, r)} = \nu E^4\psi. \quad \text{.. .. .. (154)}$$

From eqn. (153) we have

$$\frac{\partial(\psi, \Omega)}{\partial(r, z)} = \lambda r, \quad \text{.. .. .. .. (155)}$$

where $\lambda$ is a constant.

In toroidal flow, condition of integrability reduces to

$$\frac{\partial}{\partial z}(E^2\Omega) = 0, \quad \frac{\partial}{\partial r}(E^2\Omega) = 0 \quad \text{.. .. .. (156)}$$

and

$$\frac{\partial\Omega}{\partial z} = 0. \quad \text{.. .. .. .. .. (157)}$$

The eqn. (154) happens to be the condition of integrability in poloidal flow $\vec{q_1}$. Therefore if the condition of integrability is satisfied in toroidal flow, then it must be satisfied in poloidal flow also.

The set of equations determining poloidal and toroidal flows being the same as in Kapur (1960*b*), therefore the most general poloidal and toroidal viscous fields superposable on each other are given by

$$\vec{q_1} = (D_1+E_1r^2+F_1 \log r)\,\vec{i_z} \quad \ldots \quad (158)$$

$$\vec{q_2} = \left(Br+\frac{C}{r}\right)\vec{i_\theta}. \quad \ldots \quad (159)$$

7.3. *Construction of two-dimensional solutions of equations of fluid dynamics from two-dimensional flows due to vortex, spiral vortex, vortex doublet and radial flow by using new principle of superposability*

Let us consider the condition of integrability in steady flows due to two-dimensional vortex, spiral vortex, vortex doublet and radial flow only.

The condition of integrability is

$$\left(\vec{q_1}\cdot\nabla\right)\vec{\omega_1}-\left(\vec{\omega_1}\cdot\nabla\right)\vec{q_1} = -\nu \text{ curl curl } \vec{\omega_1}, \quad \ldots \quad (160)$$

where $\vec{q_1}$ and $\vec{\omega_1}$ are velocity and vorticity vectors respectively.

The necessary and sufficient condition of superposability is

$$\text{curl}\left[\vec{\omega_1}\times\vec{q_2}+\vec{\omega_2}\times\vec{q_1}+\vec{\omega_2}\times\vec{q_2}\right] = -\nu \text{ curl curl } \vec{\omega_2}. \quad \ldots \quad (161)$$

(A) *Two-dimensional vortex*

Irrotational flow due to two-dimensional vortex of strength $K$ placed at the origin is given by

$$\vec{q_1} = \frac{K}{r}\vec{i_\theta}. \quad \ldots \quad (162)$$

The condition of integrability (160) is identically satisfied and condition of superposability (161) reduces to

$$\left(\frac{K}{r}+\frac{\partial\psi_2}{\partial r}\right)\frac{\partial D^2\psi_2}{\partial\theta}-\frac{\partial\psi_2}{\partial\theta}\frac{\partial}{\partial r}D^2\psi_2 = \nu r D^4\psi_2, \quad \ldots \quad (163)$$

where

$$D^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \quad \ldots \quad (164)$$

This equation determines $\psi_2$.

In case $D^2\psi_2$ happens to be independent of $\theta$, the solution obtained by Bhatnagar and Verma (1957) by considering the condition of integrability in both the flows continues to hold. In this case superposed flow, having vorticity constant on concentric circles with origin as their centre, is given by

$$\psi_2 = \theta[A \log r+B]+G(r) \quad \ldots \quad (165)$$

where $G(r)$ is given by

$$\omega_2 = F(r) = D_1 + C_1 \int \exp. [\tfrac{1}{2} A_1 (\log r)^2 + B_1 \log r] d (\log r) \quad .. \quad (166)$$

$$F(r) = G'' + \frac{G'}{r}. \quad .. \quad .. \quad .. \quad .. \quad (167)$$

(B) *Two-dimensional spiral vortex*

Irrotational flow due to two-dimensional vortex, *i.e.* a source and a vortex situated at the origin, is given by

$$\vec{q}_1 = \frac{m}{r} \vec{i}_r + \frac{K}{r} \vec{i}_\theta. \quad .. \quad .. \quad .. \quad .. \quad (168)$$

The condition of integrability (160) is identically satisfied and condition of superposability (161) reduces to

$$\left(m - \frac{\partial \psi_2}{\partial \theta}\right) \frac{\partial}{\partial r} D^2 \psi_2 + \left(\frac{K}{r} + \frac{\partial \psi_2}{\partial r}\right) \frac{\partial}{\partial \theta} D^2 \psi_2 = \nu r D^4 \psi_2. \quad .. \quad .. \quad (169)$$

This determines $\psi_2$.

(C) *Two-dimensional vortex doublet*

Irrotational flow due to two-dimensional vortex doublet situated at the origin along the axis of $y$ is given by

$$\vec{q}_1 = \frac{\lambda \cos \theta}{r^2} \vec{i}_r + \frac{\lambda \sin \theta}{r^2} \vec{i}_\theta. \quad .. \quad .. \quad .. \quad (170)$$

The condition of integrability (160) is identically satisfied and condition of superposability (161) reduces to

$$\left(\frac{\lambda \sin \theta}{r^2} + \frac{\partial \psi_2}{\partial r}\right) \frac{\partial}{\partial \theta} D^2 \psi_2 + \left(\frac{\lambda \cos \theta}{r} - \frac{\partial \psi_2}{\partial \theta}\right) \frac{\partial}{\partial r} D^2 \psi_2 = \nu r D^4 \psi_2. \quad .. \quad (171)$$

This equation determines $\psi_2$.

If $D^2 \psi_2$ happens to be constant, the eqns. (169) and (171) are satisfied. Then

$$r^2 \frac{\partial^2 \psi_2}{\partial r^2} + r \frac{\partial \psi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial \theta^2} = 4 M r^2. \quad .. \quad .. \quad .. \quad (172)$$

Integrating we have

$$\psi_2 = M r^2 + \Sigma r^E (H_t e^{iE\theta} + N_t e^{-iE\theta}), \quad .. \quad .. \quad .. \quad (173)$$

where $H_t$ and $N_t$ are arbitrary constants.

The new two-dimensional flow

$$\vec{q} = \left(\frac{\lambda \cos \theta}{r^2} - \frac{1}{r} \frac{\partial \psi_2}{\partial \theta}\right) \vec{i}_r + \left(\frac{\lambda \sin \theta}{r^2} + \frac{\partial \psi_2}{\partial r}\right) \vec{i}_\theta \quad .. \quad .. \quad (174)$$

is a rotational flow.

(D) *Radial flow*

Rotational radial flow is given by

$$\vec{q}_1 = \frac{1}{r} G(\theta)\vec{i}_r. \qquad \ldots \quad (175)$$

Vorticity and condition of integrability of this two-dimensional radial flow are given by (119) and (120) respectively.

The condition of superposability (161) reduces to

$$-\left(\frac{2}{r^3} G'(\theta) + \frac{\partial}{\partial r} D^2\psi_2\right)\frac{\partial\psi_2}{\partial\theta} + \left(\frac{\partial}{\partial\theta} D^2\psi_2 - \frac{1}{r^2} G''(\theta)\right)\frac{\partial\psi_2}{\partial r} + G(\theta)\frac{\partial}{\partial r} D^2\psi_2 = \nu r D^4\psi_2. \qquad \ldots \quad (176)$$

7.4. *Construction of axi-symmetric rotational flows from axi-symmetric irrotational flows by using new principle of superposability*

Let $u_1$, $v_1$ be axial and radial components of irrotational flow having $\psi_1$ as Stokes stream function, so that

$$u_1 = -\frac{1}{r}\frac{\partial\psi_1}{\partial r}, \quad v_1 = \frac{1}{r}\frac{\partial\psi_1}{\partial z}, \qquad \ldots \quad (177)$$

$$\vec{\omega}_1 = r^{-1}E^2\psi_1\vec{i}_\theta = 0. \qquad \ldots \quad (178)$$

From Bhatnagar and Verma (1957) the conditions of integrability for steady flows are

$$r\frac{\partial(\psi_i, r^{-2}E^2\psi_i)}{\partial(z, r)} = \nu E^4\psi_i. \qquad \ldots \quad (179)$$

Let us consider the condition of integrability in irrotational flow $\vec{q}_1$ only which is satisfied identically.

The condition of superposability is

$$\text{curl}\left[\vec{\omega}_1 \times \vec{q}_2 + \vec{\omega}_2 \times \vec{q}_1 + \vec{\omega}_2 \times \vec{q}_2\right] = -\nu \text{ curl curl } \vec{\omega}_2$$

or

$$\frac{\partial(\psi_1, r^{-2}E^2\psi_2)}{\partial(z, r)} + \frac{\partial(\psi_2, r^{-2}E^2\psi_2)}{\partial(z, r)} = \nu r^{-1}E^4\psi_2. \qquad \ldots \quad (180)$$

The most general solutions of eqns. (129), (144), (163), (169), (171) and (180) have not yet been obtained and the discovery of their general or even particular solutions will mean an important advance in the construction of solutions of basic equations of fluid dynamics.

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