EFFECT OF VISCOUS DAMPING ON THE FLEXURAL VIBRATIONS OF A ROD

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The effect of introducing viscous damping terms in Timoshenko's equation for flexural vibrations of a rod is considered. For a rod with internal damping, we get different responses for vibrations which are purely harmonic in time and those which are purely harmonic in distance. Explicit solutions have been obtained for a rod with Kelvin type damping both in shear and extension.

1. Introduction

The flexural vibrations of a thin elastic rod are given by the well-known Bernoulli-Euler equations. When the thickness of the rod increases, shear and rotatory inertia effects become appreciable. An equation taking these effects into consideration was given by Timoshenko (1921). The effect of viscous damping on the flexural vibrations has been considered by various authors (Mindlin, Stubner and Cooper 1948; Newman 1959; H. C. Lee 1960; Crandal and Yildiz 1962) and more recently by the present author (1963).

The flexural vibrations of an elastic rod give rise to harmonic waves of constant amplitude travelling with a velocity which depends on the wavelength. To investigate the dependence of the velocity of wave propagation on the wavelength, we consider a solution of the vibration equation of the type $\exp i\,(\omega t-\mu x)$. For a purely elastic rod the frequency ω and the wave number μ are both real numbers, and it is immaterial whether we investigate the dependence of ω on μ or vice versa. For a rod with internal damping this is no longer true. Now ω and μ cannot be both real numbers, and the behaviour of a beam vibrating with a real frequency and complex wave number will be different from one vibrating with a complex frequency and a real wave number. The difference will be appreciable for large damping.

This point was touched upon in the discussion on a paper by Newman (1959, 1960). Newman considered a material with thermal damping in axial strain but no damping in shear. It will be more natural to take a visco-elastic material with damping both in axial strain and shear. The choice of thermoelastic material by Newman was due to his belief that a single equation for the transverse displacement could not be obtained for other cases. In the present paper we have derived the equation for the flexural vibrations of the general viscoelastic beam, taking shear and rotatory inertia effects into consideration. This equation has been applied to a viscoelastic beam having Kelvin-Voigt type of damping, both in shear and extension. It is possible in this case to obtain explicit solutions for beams vibrating either with a real frequency or with a real wave number. The natural vibrations of a cantilever have also been considered. It appears that when the damping in shear and extension are of the same nature, the cantilever will vibrate with a complex frequency but of the wave numbers one will be real and the other pure imaginary. When the damping in shear and extension are of different natures, the frequency and wave number of the solution will be both complex numbers.

2. Equations of Motion

In the derivation of the equation of flexural vibration of elastic rods, it is assumed that the stress σ is proportional to the strain ϵ, that is

$$\sigma = E\epsilon, \qquad (1)$$

where E is a constant. When internal damping is present the relation between the stress and strain becomes time dependent and may be written (E. H. Lee 1960)

$$P\sigma = Q\epsilon, \qquad (2)$$

where P and Q are linear operators with respect to the time t. If the material of the rod is represented by a viscoelastic model made up of a finite number of springs and dashpots, P and Q are differential operators of the form

$$P = \sum_{r=0}^{n} p_r \frac{\partial^r}{\partial t^r}, \quad Q = \sum_{r=0}^{n} q_r \frac{\partial^r}{\partial t^r}, \qquad (3)$$

where p_r, q_r are constants (some of which may be zero) depending on the model chosen, and n depends on the number of elements in the model.

A relation of type (2) will hold for viscoelastic materials whether the stress and strain are in elongation, shear or dilatation. Of course, the operators P and Q will be different in each case. We shall distinguish between the three cases by the suffixes e, s and d respectively.

Let y denote the transverse displacement and ψ the angular rotation of a beam element at distance x from the origin at time t, then the shearing strain ϕ on the element is

$$\frac{\partial y}{\partial x} - \psi. \qquad (4)$$

In fact, y and ψ vary slowly over the cross-section of the rod, and so does ϕ. Following Timoshenko we neglect the variation of y and ψ over the cross-section, taking simply their central values in our analysis. So (4) gives the central shear ϕ_c. To compensate for the variation in ϕ, we introduce an averaging factor k' as shown below.

The shearing force S on the cross-section α is given by

$$P_s S = \int Q_s \phi \, d\alpha = k'\alpha Q_s \phi_c,$$

or

$$P_s S = k'\alpha Q_s\left(\frac{\partial y}{\partial x} - \psi\right). \qquad (5)$$

The extensional strain of an element distant z from the neutral plane of the rod is $z\partial\psi/\partial x$. The corresponding stress σ is given by

$$P_e \sigma = Q_e(z\partial\psi/\partial x). \qquad (6)$$

The bending moment M on a cross-section is therefore given by

$$P_e M = \int (P_e \sigma) z \, d\alpha = \int Q_e(\partial\psi/\partial x) z^2 \, d\alpha,$$

or

$$P_e M = \alpha k^2 Q_e(\partial\psi/\partial x), \qquad (7)$$

where k is the radius of gyration of the section.

The equations for transverse and rotatory motions of an element of the rod are respectively

$$\rho\alpha \frac{\partial^2 y}{\partial t^2} - \frac{\partial S}{\partial x} = 0 \qquad (8)$$

and

$$\rho\alpha k^2 \frac{\partial^2 \psi}{\partial t^2} - S - \frac{\partial M}{\partial x} = 0. \qquad (9)$$

Operating on equations (8) and (9) by P_s and $P_s P_e$ respectively and substituting for S and M from (5) and (7), we get

$$\rho\alpha P_s \frac{\partial^2 y}{\partial t^2} - k'\alpha Q_s\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = 0 \qquad (10)$$

and

$$\rho\alpha k^2 P_s P_e \frac{\partial^2 \psi}{\partial t^2} - k'\alpha P_e Q_s\left(\frac{\partial y}{\partial x} - \psi\right) - \alpha k^2 P_s Q_e \frac{\partial^2 \psi}{\partial x^2} = 0. \qquad (11)$$

Operating on (11) by $Q_s(\partial/\partial x)$ and then substituting for $Q_s(\partial\psi/\partial x)$ from (10), we can get rid of ψ and obtain the equation of vibration

$$k' Q_s Q_e \frac{\partial^4 y}{\partial x^4} + \frac{k'\rho}{k^2} P_e Q_s \frac{\partial^2 y}{\partial t^2} - \rho(P_s Q_e + k' Q_s P_e)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \rho^2 P_s P_e \frac{\partial^4 y}{\partial t^4} = 0. \qquad (12)$$

Keeping in mind that P_s, Q_s, P_e, Q_e are linear differential operators, we see that this is a linear partial differential equation of order $2n+4$. Moreover, if we define the inverse operators Q_s^{-1}, Q_e^{-1} in the usual way, and further define the operators

$$G^{-1} = P_s Q_s^{-1} \text{ and } E^{-1} = P_e Q_e^{-1}, \qquad (13)$$

we can write eqn. (12) in the more convenient form

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{k^2} E^{-1} \frac{\partial^2 y}{\partial t^2} - \rho\left(E^{-1} + \frac{1}{k'} G^{-1}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2}{k'} E^{-1} G^{-1} \frac{\partial^4 y}{\partial t^4} = 0. \qquad (14)$$

This is the same as Timoshenko's equation, but with the difference that E^{-1} and G^{-1} are now operators instead of constants.

3. Forms of Solution

It may not be always possible to obtain the general solution of eqn. (12), but we can examine some of the interesting forms of solution. Being a linear equation, a solution of the form

$$y = e^{pt+qx} \qquad (15)$$

is always possible. It will be, however, of greater interest to consider a solution of the form

$$y = e^{i\omega t+qx}, \qquad (16)$$

where ω is real. Here we are seeking a periodic solution which is undamped in time. Such a case could arise for a rod undergoing a forced periodic vibration of one end. This form also appears suitable for the study of propagation of a periodic disturbance in a rod.

Substitution of (16) in eqn. (14) gives

$$q^4+\rho\omega^2\left(\frac{1}{E}+\frac{1}{k'G}\right)q^2-\frac{\rho\omega^2}{k^2E}+\frac{\rho^2\omega^4}{k'GE}=0, \qquad (17)$$

where now

$$E \equiv E(i\omega)=\frac{Q_e(i\omega)}{P_e(i\omega)}=E_1(\omega)+iE_2(\omega), \qquad (18)$$

E_1+iE_2 being the complex modulus of elasticity in extension for the visco-elastic material. Similarly

$$G \equiv G(i\omega)=\frac{Q_s(i\omega)}{P_s(i\omega)}=G_1(\omega)+iG_2(\omega). \qquad (19)$$

The complex moduli for materials can often be measured directly. For this reason this form of solution is most convenient for actual materials. We can solve (17) as

$$q^2=-\frac{\rho\omega^2}{2}\left(\frac{1}{E}+\frac{1}{k'G}\right)\mp\sqrt{\left\{\frac{\rho^2\omega^4}{4}\left(\frac{1}{k'G}-\frac{1}{E}\right)^2+\frac{\rho\omega^2}{k^2E}\right\}}. \qquad (20)$$

This gives q explicitly as a function of ω. If E and G are known (by measurement or otherwise) we can easily compute q for various values of ω.

For large values of ω the second term inside the radical sign in (20) is negligible compared to the first, and we get

$$\frac{q^2}{\omega^2}=-\frac{\rho}{k'G} \text{ or } -\frac{\rho}{E}$$

approximately. So we see that for large frequencies one of the modes of vibration depends mainly on G and the other on E.

Another interesting form of solution is

$$y = e^{pt+i\mu x}, \qquad (21)$$

where μ is real. Here we are seeking a solution for which the displacements are sinusoidal. Such a case will arise for a rod simply supported at both ends. In this case eqn. (14) gives on substitution

$$\frac{\rho^2 p^4}{k'G(p)E(p)}+\frac{\rho p^2}{k^2E(p)}+\left\{\frac{\rho p^2}{E(p)}+\frac{\rho p^2}{k'G(p)}\right\}\mu^2+\mu^4 = 0, \quad \ldots \quad \ldots \quad (22)$$

where now

$$E(p) \equiv \frac{Q_e(p)}{P_e(p)} \text{ and } G(p) \equiv \frac{Q_s(p)}{P_s(p)}. \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

This equation will be, in general, much more difficult to solve than eqn. (17), since it is not a simple quadratic in p^2. Moreover, for actual materials $E(p)$ and $G(p)$ will not be easy to evaluate for complex values of p.

In some cases the solution may be of the form (15) where p and q are both complex numbers, neither of them pure imaginary. A case in point would be the free vibrations of a cantilever rod of general viscoelastic material. The relation between p and q will be the same as (22) with $-q^2$ replacing μ^2. Here, again, a numerical evaluation of the solution will be cumbersome.

For these vibrations the imaginary part of p will give the frequency of vibration, while its real part will give the coefficient of damping in time. The imaginary part of q will give the wave number, while its real part will give the coefficient of damping with distance.

4. Solution for Voigt Material

To illustrate the above forms of solution we shall take a rod of Kelvin or Voigt material for which the relation between extensional stress and strain is of the form

$$\sigma = \left(E_0+\eta_e\frac{\partial}{\partial t}\right)\epsilon. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

The viscoelastic model for this material consists of a spring of elasticity E_0 in parallel with a dashpot of viscosity η_e. The damping for this model increases with the frequency of vibration.

We shall have relations of type (24) for shear and dilatation too, with different values G_0, K_0, η_s, η_d of the elastic and viscous constants. But, similar to the elastic case, the three operators E, G and K will be connected by the relation

$$E^{-1} = \tfrac{1}{3}G^{-1}+\tfrac{1}{9}K^{-1}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

We can make use of this relation in calculating E from K and G. It is known experimentally that the damping in bulk elasticity is much less than that in shear. A quite useful approximation is obtained by taking an elastic bulk modulus (Kolsky and Shi 1958; Prasad 1963), *i.e.* $\eta_d = 0$. As the other extreme we can take identical behaviour in dilatation and shear (Kolsky and Shi 1958), that is

$$\eta_d/K_0 = \eta_s/G_0 = \beta, \text{ say}. \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

This, by (25), also gives

$$\eta_e/E_0 = \beta. \qquad \qquad (27)$$

This relation introduces a great simplification in our analysis. Since K_0 is roughly three times G_0, the difference in results for the two extreme assumptions is not large.

We now make the assumption (26), and take further

$$E_0 = \frac{8}{3}G_0 \text{ and } k' = \frac{8}{9},$$

to suit roughly the actual materials. Then, putting $y = e^{pt+qx}$, eqn. (14) gives

$$q^4 - \frac{3\rho p^2}{2G_0(1+\beta p)}q^2 + \frac{3\rho p^2}{8k^2G_0(1+\beta p)} + \frac{27\rho^2 p^4}{64G_0^2(1+\beta p)^2} = 0. \qquad \qquad (28)$$

It is convenient to introduce non-dimensional variables q_1, p_1, β_1 given by

$$q_1 = kq,\ p_1 = kp/c_0,\ \beta_1 = \beta c_0/k, \qquad \qquad (29)$$

where

$$c_0 = \sqrt{(G_0/\rho)}.$$

Then (28) becomes

$$q_1^4 - \frac{3p_1^2 q_1^2}{2(1+\beta_1 p_1)} + \frac{3p_1^2}{8(1+\beta_1 p_1)} + \frac{27p_1^4}{64(1+\beta_1 p_1)^2} = 0. \qquad \qquad (30)$$

5. Vibrations with a Real Frequency

Considering first vibrations with a real frequency, we put $p_1 = i\omega_1$, where ω_1 is real. Then eqn. (30) gives

$$\frac{q_1^2}{\omega_1^2} = -\frac{3(1-i\beta_1\omega_1)}{4(1+\beta_1^2\omega_1^2)}\left[1 \pm \sqrt{\left\{\frac{1}{4}+\frac{2(1+i\beta_1\omega_1)}{3\omega_1^2}\right\}}\right], \qquad \qquad (31)$$

which can be used to evaluate q_1 for different values of ω_1. This was done for four different values of damping: $\beta_1 = 10^{-3}$, 10^{-1}, $\frac{1}{2}$ and 1.

There are two modes of vibration. The first mode, which corresponds to the vibrations excited in a thin rod, gives a solution of the form

$$q_1/\omega_1 = qc_0/\omega = \pm(iA+B), \qquad \qquad (32)$$

where A and B are positive. Taking the lower sign, we obtain for the solution of eqn. (14)

$$y = e^{i\omega(t-Ax/c_0)} \cdot e^{-B\omega x/c_0}. \qquad \qquad (33)$$

This shows that the periodic disturbance travels forward in the rod with a velocity c_0/A. The amplitude of the wave decreases as it travels, the damping factor being $e^{-B\omega x/c_0}$. Taking the other sign in (32) will give a similar wave travelling backwards. For the second mode also the solution is of the same form, except that for frequencies less than a certain value B is negative, and the wave form travels with an increasing amplitude.

If the velocity c_0/A is plotted on a graph against the frequency ω or wave number μ (the imaginary part of q), we get two branches of the velocity curve corresponding to the two modes of vibration. For small or moderate damping ($\beta_1 \leqslant 0{\cdot}1$) the curves do not differ much from those for the undamped Timoshenko beam. For large damping the velocity increases much above that of the undamped beam with the increase in frequency. This is due to the increase in modulus $\surd(G_1^2+G_2^2)$ of the elasticity in shear. Theoretically the increase in velocity will occur for every case of damping, but for small and moderate damping the increase becomes appreciable only for frequencies so large that our equation of vibration remains no longer valid.

The damping coefficient B is roughly proportional to $\beta_1\omega_1$ for the first mode. For the second mode B decreases to a minimum and then increases as the frequency increases. For $\beta_1 \geqslant 0{\cdot}866$, however, B continually decreases as ω_1 increases. We also find that for sufficiently large frequencies ($\omega_1 > 1{\cdot}15$ roughly), the damping coefficient B for the first mode is larger than the corresponding coefficient for the second mode of equal frequency. So we get the paradoxical result that in travelling equal distances the first mode is damped more than the second. The reason is that the velocity of the second mode, and consequently its wavelength, is much larger than that of the first. When we consider the two modes for the same wavelength, then the damping for the first mode is smaller.

The solution (33) shows that for a particular point the displacement is strictly periodic in time. But (33) can also be written as

$$y = e^{(i\omega+B\omega/A)(t-Ax/c_0)} \,.\, e^{-B\omega t/A}. \qquad \cdots \quad (34)$$

This shows that we can consider $e^{-B\omega t/A}$ as the time damping factor of the travelling wave. The coefficient $-B\omega/A$ will now bear a convenient comparison with the damping coefficient of the next section.

6. Vibrations with a Real Wave Number

To consider the sinusoidal vibrations of a rod of Voigt material, we put in eqn. (30): $q_1 = i\mu_1$, where μ_1 is real. Then we get, on rearrangement,

$$p_1^4+\frac{8}{9}\beta_1(1+4\mu_1^2)p_1^3+\frac{8}{9}\left(1+4\mu_1^2+\frac{8}{3}\beta_1^2\mu_1^4\right)p_1^2+\frac{128}{27}\beta_1\mu_1^4p_1+\frac{64}{27}\mu_1^4 = 0. \qquad \cdots \quad (35)$$

This can be factorized into quadratic factors:

$$(p_1^2+2\beta_1K_1p_1+2K_1)(p_1^2+2\beta_1K_2p_1+2K_2) = 0, \qquad \cdots \quad (36)$$

where

$$K_1,\, K_2 = \frac{2}{9}[1+4\mu_1^2 \mp \surd(1+8\mu_1^2+4\mu_1^4)]. \qquad \cdots \quad (37)$$

Hence we get the solution

$$p_1 = -\beta_1K \pm i\surd(2K-\beta_1^2K^2), \qquad \cdots \quad (38)$$

where we take either K_1 or K_2 for K in (38). These will give the solutions for the two modes of vibration.

The displacement will be given by

$$y = e^{-iA_1(t-\mu x/A_1)} \cdot e^{-B_1 t}, \quad \ldots \quad (39)$$

where

$$A_1 = (c_0/k)\sqrt{(2K-\beta_1^2 K^2)} \text{ and } B_1 = c_0\beta_1 K/k.$$

We see that the motion is periodic with time damping as long as

$$2K-\beta_1^2 K^2 > 0.$$

When $2K-\beta_1^2K^2 < 0$, the motion is no longer periodic. It is so heavily damped that it dies down exponentially. This can happen for both the modes of vibration. For $\beta_1 = 1$, the critical damping occurs at $\mu_1 = 2{\cdot}222$ for the first mode and at $\mu_1 = 1{\cdot}031$ for the second mode.

Rod vibrations of the above nature will generally be produced by standing waves. The wave velocity is given by the ratio A_1/μ. A graph of this velocity against the wave number μ will not be much different from the velocity curve of the previous section (or of an undamped beam) for cases of small or moderate damping. For large damping the velocities for both the modes are considerably less than the velocities for the undamped case, and decrease further with an increase of the wave number.

The time damping depends on $\beta_1 K$. Since

$$K_1 \simeq \frac{4}{3}\mu_1^4 \text{ and } K_2 \simeq \frac{4}{9}(1+4\mu_1^2)$$

for $\mu_1 \ll 1$, the damping of the first mode is far less than that of the second for long waves. Both increase with the wave number but the former always remains less than the latter.

Figs. 1 and 2 give respectively the graphs of velocity and damping coefficient plotted against the real wave number. The dotted curves give the values for the present section, while the continuous curves give those for the preceding section. In both the figures the curves nearer the bottom are for the first mode and those nearer the top for the second mode.

7. Vibrations of a Cantilever

To obtain the natural frequencies of a freely vibrating cantilever, we shall apply the appropriate boundary conditions to the general solution

$$y = e^{pt}(A_1 \cosh q_1x+B_1 \sinh q_1x+A_2 \cosh q_2x+B_2 \sinh q_2x) \quad \ldots \quad (40)$$

of eqn. (14). Here q_1 and q_2 are the two roots, distinct in magnitude, of the equation

$$q^4-\left\{\frac{\rho}{E(p)}+\frac{\rho}{k'G(p)}\right\}p^2q^2+\frac{\rho p^2}{k^2E(p)}+\frac{\rho^2p^4}{k'E(p)G(p)} = 0. \quad \ldots \quad (41)$$

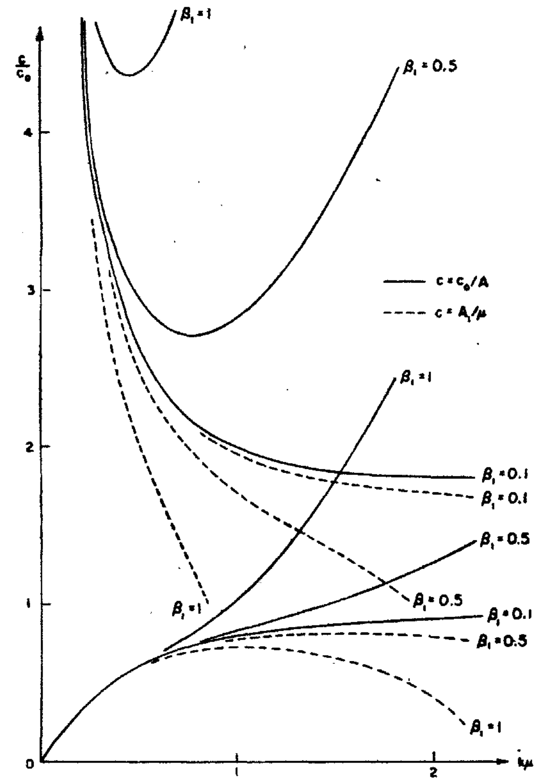

FIG. 1. Dispersion curves for a viscoelastic rod.

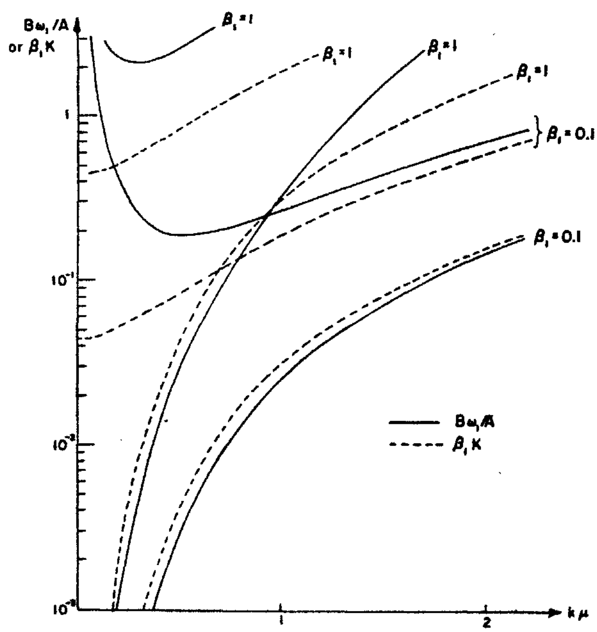

FIG. 2. Damping coefficient.

The boundary conditions at the fixed end $x = 0$ are

$$y = 0 \quad \ldots \quad (42)$$

and

$$\psi = 0.$$

With the help of eqns. (10) and (11), the latter condition may be replaced by

$$\frac{\partial y}{\partial x}+\frac{k^2}{k'}EG^{-1}\left(\frac{\partial^3 y}{\partial x^3}-\frac{\rho}{k'}G^{-1}\frac{\partial^3 y}{\partial x\partial t^2}\right)=0. \quad \ldots \quad (43)$$

The boundary conditions at the free end $x = l$ are $S = 0$ and $M = 0$. These give, by (5) and (7),

$$\frac{\partial y}{\partial x}-\psi=0 \text{ and } \frac{\partial \psi}{\partial x}=0.$$

Reducing these conditions also in terms of y alone, we get

$$\frac{\partial^3 y}{\partial x^3}-\rho\left(E^{-1}+\frac{1}{k'}G^{-1}\right)\frac{\partial^3 y}{\partial x\partial t^2}=0 \quad \ldots \quad (44)$$

and

$$\frac{\partial^2 y}{\partial x^2}-\frac{\rho}{k'}G^{-1}\frac{\partial^2 y}{\partial t^2}=0 \quad \ldots \quad (45)$$

respectively.

Applying these boundary conditions to solution (40), eliminating A_1, B_1, A_2, B_2 from the four resulting equations and simplifying, we get the following equation for the natural frequencies of vibration

$$2+\left\{2-\frac{k^2\rho p^2}{E}\left(1-\frac{E}{k'G}\right)^2\right\}\cosh q_1 l \cosh q_2 l$$
$$+\frac{\rho p^2}{E}\left(1+\frac{E}{k'G}\right)\frac{1}{q_1 q_2}\sinh q_1 l \sinh q_2 l=0. \quad \ldots \quad (46)$$

Here E and G stand for $E(p)$ and $G(p)$ respectively, and q_1, q_2 are the two distinct roots of eqn. (41). Equations (41) and (46) are a set of simultaneous equations in p and q which can be solved, by successive approximation or otherwise, for the characteristic frequencies.

For the model under discussion, these two equations reduce to

$$q_1^4-\frac{3p_1^2 q_1^2}{2(1+\beta_1 p_1)}+\frac{3p_1^2}{8(1+\beta_1 p_1)}+\frac{27p_1^4}{64(1+\beta_1 p_1)^2}=0 \quad \ldots \quad (47)$$

and

$$2+\left\{2-\frac{3p_1^2}{2(1+\beta_1 p_1)}\right\}\cosh\frac{q_{11}l}{k}\cosh\frac{q_{12}l}{k}$$
$$+\frac{3p_1^2}{2q_{11}q_{12}(1+\beta_1 p_1)}\sinh\frac{q_{11}l}{k}\sinh\frac{q_{12}l}{k}=0. \quad \ldots \quad (48)$$

in terms of the dimensionless variables used earlier. For this particular model, we can simplify these equations further by putting

$$\frac{p_1^2}{1+\beta_1 p_1}=-\omega_1^2, \quad \ldots \quad (49)$$

where ω_1 is real. Then (47) and (48) reduce to

$$q_1^4+\frac{3}{2}\omega_1^2q_1^2-\frac{3}{8}\omega_1^2+\frac{27}{64}\omega_1^4=0 \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

and

$$2+\left(2+\frac{3}{2}\omega_1^2\right)\cosh\frac{q_{11}l}{k}\cosh\frac{q_{12}l}{k}-\frac{3\omega_1^2}{2q_{11}q_{12}}\sinh\frac{q_{11}l}{k}\sinh\frac{q_{12}l}{k}=0. \quad \ldots \quad (51)$$

Equation (50) gives, on solution,

$$q_1^2=-\frac{3}{4}\omega_1^2\mp\sqrt{\left(\frac{3}{8}\omega_1^2+\frac{9}{64}\omega_1^4\right)}. \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

So the values of q_1 are of the form $\pm i\alpha_1$, $\pm\alpha_2$, where α_1 and α_2 are real. In terms of α_1 and α_2, (51) may be written

$$2+\left(2+\frac{3}{2}\omega_1^2\right)\cos\frac{\alpha_1 l}{k}\cosh\frac{\alpha_2 l}{k}-\frac{3\omega_1^2}{2\alpha_1\alpha_2}\sin\frac{\alpha_1 l}{k}\sinh\frac{\alpha_2 l}{k}=0. \quad \ldots \quad (53)$$

Equations (52) and (53) can now be solved for the real number ω_1. Then p_1 can be obtained from (49). We get

$$p_1=-\tfrac{1}{2}\beta_1\omega_1^2\pm i\omega_1\sqrt{(1-\tfrac{1}{4}\beta_1^2\omega_1^2)}. \quad \ldots \quad \ldots \quad \ldots \quad (54)$$

It should be noted that the damping parameter β_1 enters into the time coefficient p_1 only. The damping does not affect the variation of y with x. Of the four independent terms in (40), two are pure sinusoidal and the other two hyperbolic. It is believed that a similar solution will be possible for all models for which $E(p)/G(p)$ = constant or, in other words, whose visco-elastic behaviour in shear and elongation is similar.

However, the solution will not give a real or purely imaginary value for q_1 when $E(p)/G(p)$ is not a constant. Consider, for example, the model taken by Newman (1959). For this model the eqns. (50) and (51) will be replaced by

$$q_1^4+\frac{3}{2}\omega_1^2\left(1+\frac{3}{4}\beta_1p_1\right)q_1^2-\frac{3}{8}\omega_1^2+\frac{27}{64}\omega_1^4(1+\beta_1p_1)=0 \quad \ldots \quad \ldots \quad (55)$$

and

$$2+\left\{2+\frac{3}{2}\omega_1^2\left(1+\frac{3}{2}\beta_1p_1\right)^2\right\}\cosh\frac{q_{11}l}{k}\cosh\frac{q_{12}l}{k}$$
$$-\frac{3\omega_1^2(1+\frac{3}{4}\beta_1p_1)}{2q_{11}q_{12}}\sinh\frac{q_{11}l}{k}\sinh\frac{q_{12}l}{k}=0, \quad \ldots \quad (56)$$

where

$$p_1=i\omega_1\sqrt{(1-\tfrac{1}{4}\beta_1^2\omega_1^2)}-\tfrac{1}{2}\beta_1\omega_1^2$$

and ω_1 is not necessarily real.

For the undamped case $\beta_1=0$, we get the same solution as before. For $\beta_1\neq 0$, the deviation from this solution will depend on the magnitude of β_1. Considering β_1p_1 as a small quantity and applying perturbation methods, we can show that both q_{11} and q_{12} will now be complex numbers. A comparative set of characteristic values for the two models for the first four modes of vibration of the cantilever are given in Table I for the case $\beta_1 = 0{\cdot}1$ and $l/k = 30$.

TABLE I

Characteristic values for the cantilever $l/k = 30$; $\beta_1 = 0{\cdot}1$

p_1	$\alpha_1 l/k$	$\alpha_2 l/k$
(a) Present Model		
0·00631 40*i*–0·00002	1·873	1·858
0·03739 *i*–0·00007	4·643	4·435
0·09696 *i*–0·00047	7·738	6·869
0·17336 *i*–0·00150	10·787	8·706
(b) Newman's Model		
0·00631 4*i*–0·000002	1·873–0·00000 1*i*	1·858–0·00000 8*i*
0·03739 *i*–0·00006	4·643–0·00013 *i*	4·435–0·00066 *i*
0·09696 *i*–0·00038	7·738–0·00072 *i*	6·869–0·00586 *i*
0·17336 *i*–0·00108	10·787–0·00072 *i*	8·706–0·02142 *i*

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