ON UNIFORM HARMONIC SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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This paper deals with two theorems on the uniform Harmonic summability of the Fourier series and its conjugate series corresponding to a function $f(x)$, periodic and integrable (L) with period 2π.

1. Let the Fourier series corresponding to a function $f(x)$, periodic and integrable (L) with period 2π, be

$$(1/2)a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx). \qquad \ldots \quad \ldots \quad (1.1)$$

The conjugate series of (1.1) is

$$\sum_{n=1}^{\infty}(b_n \cos nx-a_n \sin nx). \qquad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

We denote by $S_n(x)$ and $\bar{S}_n(x)$ the partial sums of the series (1.1) and (1.2) respectively.

Let

$$\bar{f}_n \equiv \bar{f}_n(x) \equiv (1/\pi)\int_{1/n}^{\pi} \psi(t) \cot (1/2)t\, dt,$$

where

$$\psi(t)=\psi(t, x)=(1/2)\{f(x+t)-f(x-t)\},$$

and

$$\bar{f}(x)=\lim_{n\to\infty} \bar{f}_n(x) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

whenever the latter exists.

Also let

$$\phi(t)=f(x+t)+f(x-t)-2S. \qquad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

2. The object of this paper is to introduce a new concept of summability, i.e. *Uniform Harmonic Summability*, which we define as follows:

Let

$$u_0(x)+u_1(x)+u_2(x)+\ldots. \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

be any infinite series, and

$$U_\nu(x)=u_0(x)+u_1(x)+\ldots.+u_\nu(x).$$

If there exists a function $U = U(x)$ such that

$$(1/\log n)\sum_{k=0}^{n}(1/\overline{k+1})\{U_{n-k}(x)-U\} = o(1)$$

uniformly in a set E in which $U = U(x)$ is bounded, then we shall say that the series (2.1) is summable by Harmonic means uniformly in E to the sum U.

3. *Introduction.* In 1948, Siddiqi established the following theorem for the ordinary behaviour of Harmonic summability of Fourier series at a particular point:

Theorem. The Fourier series of the function $f(x)$ is summable by Harmonic means at a point x at which

$$g_1(t) = \int_0^t |g(u)|\,du = o(t/\log(1/t)),$$

where

$$g(t) = f(x+t)+f(x-t)-2f(x).$$

We propose to establish a more general theorem with the help of this theorem of Siddiqi. Our Theorem 1 concerns the uniform behaviour of Harmonic summability of Fourier series over a set E. If E contains only one point and further if $S(x) = f(x)$, then our Theorem 1 reduces to the theorem of Siddiqi. Our Theorem 2 concerns the uniform behaviour of Harmonic summability of conjugate series of a Fourier series over a set E.

4. We now establish the following theorems:

Theorem 1. If

$$\Phi(t) = \int_0^t |\phi(u)|\,du = o(t/\log(1/t)) \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

uniformly in a set E in which $S = S(x)$ is bounded, as $t \to +0$, then the series (1.1) is summable by Harmonic means uniformly in E to the sum S.

In order to prove Theorems 1 and 2 we shall require the following Lemmas:

Lemma I (Hardy and Rogosinski, 1947). *If $0 < t \leqslant \pi$, then*

$$\left|\sum_{k=0}^{n}(1/\overline{k+1})\cos(k+1)t\right| < A[1+\log^{+}(1/t)].$$

Lemma II (Titchmarsh, 1958). *For all values of n and t*

$$\left|\sum_{k=0}^{n}(1/\overline{k+1})\sin(k+1)t\right| \leqslant (\pi/2)+1.$$

Proof of Theorem 1. It is well known that

$$S_n(x)-S = (1/2\pi)\int_0^\pi \phi(t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt \qquad \dots \quad (4.2)$$

$$= (1/2\pi)\left(\int_0^\delta+\int_\delta^\pi\right)\phi(t)\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$= (1/2\pi)\int_0^\delta \phi(t)\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$+(1/2\pi)\int_\delta^\pi f(x+t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$+(1/2\pi)\int_\delta^\pi f(x-t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$-(S/\pi)\int_\delta^\pi \{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$= (1/2\pi)\int_0^\delta \phi(t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$+J_1+J_2+J_3,\text{ say.} \qquad \dots \quad \dots \quad (4.3)$$

Now,

$$J_1 = o(1),\text{ uniformly in } E \text{ (Hardy and Rogosinski, 1944)} \quad \dots \quad (4.4)$$

$$J_2 = (1/2\pi)\int_\delta^\pi f(x-t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$= (-1/2\pi)\int_{-\delta}^{-\pi} f(x+t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$= (1/2\pi)\int_{-\pi}^{-\delta} f(x+t)\,\{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt$$

$$= o(1),\text{ uniformly in } E \text{ (Hardy and Rogosinski, 1944).} \quad \dots \quad (4.5)$$

$$J_3 = (-S/\pi)\int_\delta^\pi \{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt \qquad \dots \quad \dots \quad (4.6)$$

or

$$|J_3| = |S|(1/\pi)\left|\int_\delta^\pi \{\sin{(n+1/2)t}/\sin{(1/2)t}\}\,dt\right|$$

$$\leqslant M(1/\pi)[o(1),\text{ uniformly in } E] \text{ (Hardy and Rogosinski, 1944).}$$

$$\therefore \quad |J_3| = o(1),\text{ uniformly in } E,$$

since $|S| = |S(x)| \leqslant M$ for every $x \in E$, where M is some constant.

Hence

$$J_3 = o(1), \text{ uniformly in } E. \quad .. \quad .. \quad .. \quad (4.7)$$

Now from (4.3), (4.4), (4.5) and (4.7), we have

$$S_n(x)-S = (1/2\pi)\int_0^\delta \phi(t)\{\sin(n+1/2)t/\sin(1/2)t\}\,dt+o(1), \text{ uniformly in } E,$$

and

$$(1/\log n)\sum_{k=0}^{n}(1/\overline{k+1})\{S_{n-k}(x)-S\}$$

$$= (1/\log n)\sum_{k=0}^{n}(1/\overline{k+1})(1/2\pi)\int_0^\delta \phi(t)\{\sin(n-k+1/2)t/\sin(1/2)t\}\,dt$$
$$+o(1), \text{ uniformly in } E$$

$$= \int_0^\delta \phi(t)\{(1/2\pi)(1/\log n)\sum_{k=0}^{n}(1/\overline{k+1})(\sin(n-k+1/2)t/\sin(1/2)t)\}\,dt$$
$$+o(1), \text{ uniformly in } E$$

$$= \int_0^\delta \phi(t)N_n(t)dt+o(1), \text{ uniformly in } E, \text{ say},$$

where

$$N_n(t) = \left\{(1/2\pi)(1/\log n)\sum_{k=0}^{n}(1/\overline{k+1})(\sin(n-k+1/2)t/\sin(1/2)t)\right\}$$

$$= \left(\int_0^{1/n}+\int_{1/n}^{\delta}\right)\phi(t)N_n(t)dt+o(1), \text{ uniformly in } E,$$

$$= L_1+L_2+o(1), \text{ uniformly in } E, \text{ say}. \quad .. \quad .. \quad .. \quad .. \quad (4.8)$$

Now

$$L_1 = \int_0^{1/n}\phi(t)N_n(t)\,dt$$
$$= O\left(\int_0^{1/n}|\phi(t)|\,n\,dt\right)$$
$$= O(\Phi(1/n)\,.\,n)$$
$$= o(1/\log n), \text{ uniformly in } E \text{ (by hypothesis)},$$
$$= o(1), \text{ uniformly in } E, \quad .. \quad .. \quad .. \quad (4.9)$$

where $N_n(t) = O(n)$ for all t such that $0 < t \leqslant \delta$ (Siddiqi, 1948).

In order to show that $L_2 = o(1)$, uniformly in E, we require a suitable estimation for the kernel $N_n(t)$ in the interval $(1/n, \delta)$. Since

$$N_n(t) = \frac{1}{2\pi\log n\,.\,\sin(1/2)t}\left\{\sin(n+3/2)t\sum_{k=0}^{n}\frac{\cos(k+1)t}{k+1}-\cos(n+3/2)t\right.$$
$$\left.\times\sum_{k=0}^{n}\frac{\sin(k+1)t}{k+1}\right\},$$

by making use of Lemmas I and II, we get

$$N_n(t) = O\left\{\frac{1}{t \log n}(1+\log^+ (1/t))\right\}$$

$$= O\left\{\frac{1}{t \log n}\log (1/t)\right\},$$

since $\log^+ (1/t) = \log (1/t)$ for every $t \epsilon (1/n, \delta)$.
Hence

$$L_2 = \int_{1/n}^{\delta} \phi(t) N_n(t)\, dt = O\left(\int_{1/n}^{\delta} |\phi(t)||N_n(t)|\, dt\right)$$

$$= O\left((1/\log n)\int_{1/n}^{\delta} |\phi(t)|(1/t) \log (1/t) dt\right)$$

$$= O\left((1/\log n)\Big[\Phi(t)(1/t) \log (1/t)\Big]_{1/n}^{\delta} + (1/\log n)\int_{1/n}^{\delta} \Phi(t)\frac{(1+\log \overline{1/t})}{t^2}\, dt\right)$$

$$= \Bigg\{ O(1/\log n) + o(1/\log n) + o\left((1/\log n)\int_{1/n}^{\delta} \frac{dt}{t \log (1/t)}\right)$$

$$+ o\left((1/\log n)\int_{1/n}^{\delta} (1/t)\, dt\right)\Bigg\}, \text{ uniformly in } E,$$

$$= \left\{ o(1) + o\left(\left[\frac{\log \log (1/t)}{\log n}\right]_{1/n}^{\delta}\right) + o\left(\left[\frac{\log (1/t)}{\log n}\right]_{1/n}^{\delta}\right)\right\}, \text{ uniformly in } E,$$

$$= o(1), \text{ uniformly in } E. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.10)$$

The theorem follows from (4.8), (4.9) and (4.10).

Theorem 2. If

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o(t/\log (1/t)) \quad \ldots \quad \ldots \quad \ldots \quad (4.11)$$

uniformly in a set E, as $t \to +0$, then the series (1.2) is summable by Harmonic means uniformly in E to the sum $\bar{f}(x)$, provided that limit (1.3) exists uniformly in E.

Proof of Theorem 2. It is well known that

$$\bar{S}_n(x) = (1/\pi)\int_0^{\pi} \psi(t)\frac{\cos (1/2)t - \cos (n+1/2)t}{\sin (1/2)t}\, dt.$$

Therefore

$$\bar{S}_n(x) - \bar{f}(x) = -(1/\pi)\int_0^{\pi} \psi(t)\frac{\cos (n+1/2)t}{\sin (1/2)t}\, dt.$$

Now

$$(1/\log n)\sum_{\nu=0}^{n}\frac{1}{\nu+1}\{\bar{S}_{n-\nu}(x)-\bar{f}(x)\}$$

$$=-\int_0^{\pi}\psi(t)\frac{1}{\pi\log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1}\frac{\cos(n-\nu+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$=\int_0^{\pi}\psi(t)M_n(t)\,dt, \text{ where}$$

$$M_n(t)=-\frac{1}{\pi\log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1}\frac{\cos(n-\nu+\frac{1}{2})t}{\sin\frac{1}{2}t}.$$

In order to prove Theorem 2, we have to show that

$$\int_0^{\pi}\psi(t)M_n(t)\,dt=o(1), \text{ uniformly in } E.$$

We set

$$\int_0^{\pi}\psi(t)M_n(t)\,dt=\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}+\int_{\delta}^{\pi}\right)\psi(t)M_n(t)\,dt$$

$$=H_1+H_2+H_3, \text{ say.} \qquad \ldots \quad \ldots \quad \ldots \quad (4.12)$$

Since limit (1.3) exists uniformly in E,

$$\frac{1}{\pi}\int_0^{\frac{1}{n}}\psi(t)\cot\tfrac{1}{2}t\,dt=o(1), \text{ uniformly in } E.$$

Also

$$\frac{1}{\pi\log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1}\frac{\cos(\frac{1}{2})t-\cos(n-\nu+\frac{1}{2})t}{\sin(\frac{1}{2})t}=\frac{2}{\pi\log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1}\sum_{k=1}^{n-\nu}\sin kt$$

$$=O\left(\frac{1}{\log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1}\sum_{k=1}^{n-\nu}|\sin kt|\right)$$

$$=O\left(\frac{n}{\log n}\sum_{\nu=0}^{n}\frac{1}{\nu+1}\right)$$

$$=O(n),$$

for all t such that $0<t\leqslant\pi$.

Hence

$$H_1 = \int_0^{1/n} \psi(t) M_n(t)\, dt$$

$$= -\int_0^{1/n} \psi(t) \frac{1}{\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \frac{\cos (n-\nu+\frac{1}{2})t}{\sin (1/2)t}\, dt$$

$$= \int_0^{1/n} \psi(t) \frac{1}{\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \frac{\cos (1/2)t - \cos (n-\nu+\frac{1}{2})t}{\sin (1/2)t}\, dt$$

$$- \frac{1}{\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \int_0^{1/n} \psi(t) \cot (1/2)t\, dt$$

$$= O\left(n \int_0^{1/n} |\psi(t)|\, dt\right) + o(1), \text{ uniformly in } E,$$

$$= O(n\Psi(1/n)) + o(1), \text{ uniformly in } E,$$

$$= o(1/\log n) + o(1), \text{ uniformly in } E,$$

$$= o(1), \text{ uniformly in } E. \qquad \ldots \quad (4.13)$$

Now

$$H_3 = -\frac{1}{2\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \int_\delta^\pi f(x+t) \frac{\cos (n-\nu+\frac{1}{2})t}{\sin(1/2)t}\, dt$$

$$+ \frac{1}{2\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \int_\delta^\pi f(x-t) \frac{\cos (n-\nu+\frac{1}{2})t}{\sin (1/2)t}\, dt$$

$$= -\frac{1}{2\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \int_{-\pi}^{-\delta} f(x+t) \frac{\cos (n-\nu+\frac{1}{2})t}{\sin (1/2)t}\, dt$$

$+o(1)$, uniformly in E (Hardy and Rogosinski, 1944)

$= o(1)$, uniformly in E (Hardy and Rogosinski, 1944). .. (4.14)

In order to show that $H_2 = o(1)$, uniformly in E, we require a suitable estimation for the kernel $M_n(t)$ in the interval $(1/n, \delta)$. Since

$$M_n(t) = -\frac{1}{\pi \log n} \sum_{\nu=0}^{n} \frac{1}{\nu+1} \frac{\cos (n-\nu+\frac{1}{2})t}{\sin (1/2)t}$$

$$= -\frac{2}{\pi t \log n} \left\{ \cos (n+\tfrac{3}{2})t \sum_{\nu=0}^{n} \frac{\cos (\nu+1)t}{\nu+1} + \sin (n+\tfrac{3}{2})t \sum_{\nu=0}^{n} \frac{\sin (\nu+1)t}{\nu+1} \right\}.$$

By making use of Lemmas I and II, we get

$$M_n(t) = O\left\{ \frac{1}{t \log n} (1+\log^+ (1/t)) \right\}$$

$$= O\left(\frac{\log (1/t)}{t \log n} \right)$$

since $\log^+ (1/t) = \log (1/t)$ for every $t \in (1/n, \delta)$.

Hence

$$H_2 = O\left(\frac{1}{\log n}\int_{1/n}^{\delta} |\psi(t)| \frac{\log(1/t)}{t}\,dt\right)$$

$$= o(1), \text{ uniformly in } E, \text{ as in the proof of Theorem 1.} \tag{4.15}$$

Thus from (4.12), (4.13), (4.14) and (4.15), we have

$$(1/\log n)\sum_{\nu=0}^{n} \frac{1}{\nu+1}\{\bar{S}_{n-\nu}(x)-\bar{f}(x)\} = o(1), \text{ uniformly in } E.$$

This proves the theorem.

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