GROUPS WITH SPECIAL PROPERTIES

by K. M. RANGASWAMY,* *University of Madras, Madurai*

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This paper contains two parts. In the first part, we consider groups whose subgroups of a particular type have some special properties. With any given group G, we can associate various classes of subgroups of G. Among the associated classes there exists a natural partial ordering relation by set inclusion. In the first part, we investigate the nature of a group G in relation to the coincidence or otherwise of specific pairs of these classes. The second part of the paper is, in a sense, dual to the first. We say that H is a super-group of G if G is a subgroup of H. Given a group G, we consider various classes of super-groups of G and investigate how far the coincidence relations among the classes of super-groups influence the properties of G. This point of view includes some of the well-known classes of groups.

I. INTRODUCTION

The interesting problem of classifying groups according to certain properties and then analysing their structure has been the subject of consideration in a series of papers (Kertesz 1951; Kertesz and Szele 1952; Fuchs, Kertesz and Szele 1953). With any given group G, we can naturally associate various classes of subgroups like the class $\mathcal{S}$ of all subgroups, the class $\mathcal{D}$ of divisible subgroups, the class $\mathcal{A}$ of absolute direct summands, the class $\mathcal{D}'$ of direct summands, the class $\mathcal{P}$ of pure subgroups, the class $\mathcal{N}$ of neat subgroups, the class π of π subgroups, the class $\mathcal{A}'$ of absorbing subgroups, the class $\mathcal{F}$ of full subgroups, etc. Among the associated classes there exists a natural partial ordering relation by set inclusion. Thus $\mathcal{D} \subset \mathcal{A} \subset \mathcal{D}' \subset \mathcal{P} \subset \mathcal{N} \subset \pi \subset \mathcal{S}$; $\mathcal{A}' \subset \mathcal{F}$; $\mathcal{A}' \subset \mathcal{P} \subset \mathcal{N} \subset \pi \subset \mathcal{S}$. This order is not necessarily strict, and it is interesting to study the nature of the group G in relation to the coincidence or otherwise of specific pairs of these classes. This is the motivation of our investigation in III of this paper. For example, it turns out (theorem 3.3) that in a group G, $\mathcal{N} = \mathcal{P}$ if and only if either (i) G_t is divisible, or (ii) G is torsion and for each prime p, either G_p is divisible or $G_p = \Sigma c(p^n) + \Sigma c(p^{n+1})$ for some non-negative integer n, the summation being arbitrary. Similarly it is interesting to note that groups G for which $\mathcal{N} = \mathcal{A}$ are precisely those groups in which every endomorphism (automorphism) of

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* N.I.S.I. Research Fellow. *Present address:* Department of Mathematics, Panjab University, Chandigarh.

each subgroup can be extended to an endomorphism (automorphism) of the whole group G.

In IV we consider situations which are, in a sense, dual to those considered in III. We consider groups which behave in a special way whenever they are subgroups of a particular type of their containing groups. For convenience of expression, we say that a group H is a *super-group* of G if H contains G as a subgroup. Given the group we may consider the various classes of super-groups, like, for example, the class $\mathfrak{N}^*$ of all super-groups in which G is neat, the class $\mathfrak{P}^*$ of all super-groups in which G is pure, etc. Similarly the classes $\mathcal{A}^*$, $(\mathcal{D}')^*$, $(\mathcal{A}')^*$, $\mathcal{F}^*$, π^*, $\mathcal{S}^*$ are defined. The properties of the group G are, to a large extent, influenced by the coincidence relations among the various classes of super-groups of G and these are investigated in IV. A result that would perhaps be of interest in itself is that (4.8) the class of groups for which $\mathfrak{N}^* = \mathfrak{P}^*$ is precisely the class of groups G in which nG is a summand for every integer n. In this connection we consider two questions raised by Kertesz and Szele (1952). Let G be a reduced group wherein, for each integer n, nG is a summand, or equivalently where $\mathfrak{N}^* = \mathfrak{P}^*$. Then,

(i) Is every endomorphic image of G a direct summand?

(ii) Is the group G determined uniquely (up to isomorphism) by the full system of invariants of G_t and G/G_t?

We prove that every endomorphic image of such a group is always pure. We also show that (ii) has a negative answer.

II. Preliminaries

All groups considered in this paper are Abelian.

(2.1) A subgroup S of a group G is said to be *dense* in G if S meets every non-zero subgroup A of G, that is $S \cap A \neq 0$.

(2.2) A subgroup N is *neat* in G if, for each prime p, $N \cap pG = pN$. An interesting known result is that if a subgroup H is maximally disjoint with another subgroup K, then H is neat. In fact, given K, H is maximally disjoint with K if and only if $H \cap K = 0$, H is neat and $\{H, K\}$ is dense in G. If N is neat and is further dense in G, then $N = G$. A neat subgroup minimally containing a subgroup S is called its *Neat Hull.*

(2.3) We call the subgroup $G^1 = \bigcap_{n=1}^{n=\infty} nG$ as the *Radical* of G. The subgroup $\phi(G) = \bigcap p\,G$, where p runs over the set of all primes, is called the *ϕ-subgroup* or the *Frattini subgroup*. It is known (Dlab 1960) that if $G = \Sigma G_\lambda$ and $H = \Pi G_\lambda$, then $\phi(G) = \Sigma \phi(G_\lambda)$ and $\phi(H) = \Pi \phi(G_\lambda)$.

(2.4) G is said to be *elementary* if every element of G is of square free order. If, further, G is a p-group, then G is called *p-elementary*. If S is a neat subgroup of G and is p-elementary, then S is a direct summand of G.

(2.5) The maximal (largest) elementary subgroup of G is called its *Socle*. A subgroup F of a group G is said to be *Full* in G if Socle of $F =$ Socle of G. A subgroup S is *absorbing* if $nx \in S \Rightarrow x \in S$. Clearly an absorbing subgroup is pure. For properties of full and absorbing subgroup we refer to Rangaswamy (1964). A neat subgroup is full if and only if it is absorbing. A non-neat subgroup is full if and only if it has a unique neat hull.

(2.6) A subgroup S is called a *π-subgroup* if it is the intersection of neat subgroups of G. For characterization and properties of π-subgroups *see* Rangaswamy (1965).

(2.7) Given A and B, a group G is an extension of A by B if A is isomorphic to a subgroup A' of G and G/A' is isomorphic to B. The set of all extensions of A by B under a suitable equivalence relation becomes a group denoted by Ext (B, A). The subset of Ext (B, A) consisting of all those extensions in which A is neat is denoted by Next (B, A). Similarly Pext (B, A) denotes the set of all extensions in which A is pure. It is known (Fuchs 1960*a*) that Next (B, A) is the Frattini subgroup and Pext (B, A) is the Radical of Ext (B, A).

(2.8) A group G is called *co-torsion* if G is reduced and Ext $(X, G) = 0$ for every torsion-free group X. For important properties of co-torsion groups and further homological results we refer to Harrison (1954). A co-torsion group is *adjusted* if it has no torsion-free summand. A co-torsion group G is adjusted if and only if G/G_t is divisible. It can be easily verified that a subgroup S of a co-torsion group C is itself co-torsion if and only if C/S is co-torsion. A torsion-free co-torsion group G always contains a subgroup isomorphic to the group of p-adic integers for some prime p and hence $\phi\ (G) = 0$.

(2.9) If G is a group, G_t denotes the maximal torsion subgroup of G while G_p denotes the p-component of G_t. It is a known simple result that if S is pure in G, then $\{G_t, S\}$ is again pure in G.

For other standard concepts, results and terminology we refer to Fuchs (1960*a*).

III. Groups with Special Classes of Subgroups

We first note that the class $\mathcal{S}$ will coincide with any one of the classes $\mathcal{A}$, $\mathcal{D}'$, $\mathcal{P}$, $\mathcal{N}$ and π if and only if G is elementary. Similarly the class $\mathcal{F}$ will be a subclass of any one of the classes π, $\mathcal{N}$, $\mathcal{P}$, $\mathcal{D}'$, $\mathcal{A}$ and $\mathcal{A}'$ if and only if G is elementary.

We now consider the groups G for which $\mathcal{N}$ and $\mathcal{P}$ coincide. Clearly every torsion-free group has this property.

The following lemma is of some interest:

Lemma 3.1. Let G be a reduced group for which $\mathcal{N} = \mathcal{P}$. Then $G^1 = 0$.

Proof. We note that if N is a neat hull of G^1, then N is pure so that $N^1 = G^1$

and since G^1 is dense in N, N becomes divisible. But G is reduced so that N and hence G^1 reduces to the identity.

Let G be a mixed group for which $\mathfrak{N} = \mathfrak{P}$. Let H be a subgroup maximally disjoint form G_t. Then H is neat and so is pure in G. But then $\{G_t, H\} = G_t+H$ is pure in G. Also $\{G_t, H\}$ is dense in G and hence it follows that $G = G_t+H$. Thus G splits and G_t becomes an absolute direct summand. This means (*see* Fuchs 1960*a*) that either G_t is divisible or G/G_t is torsion. Thus either $G = D+R$, D torsion-divisible and R torsion-free *or* G is a torsion group in which $\mathfrak{N} = \mathfrak{P}$. So we are led to investigate all the torsion groups for which $\mathfrak{N} = \mathfrak{P}$ which reduces itself to the consideration of p-groups with the stated property.

THEOREM 3.2. Every neat subgroup of a p-group G is pure if and only if either G is divisible or is of the form $G = \Sigma c(p^n)+\Sigma c(p^{n+1})$ where n is some non-negative integer and where the number of components in each summation is arbitrary.

PROOF. Let G be a p-group in which every neat subgroup is pure. If G is divisible we have nothing to prove. Let G be non-divisible. Consider the set of all elements of height zero in G. Since G is not divisible, this set is not empty. Let x be an element of this set and of least order, say, p^n. Then there cannot be any element y of order $> p^{n+1}$. If there exists an element y of order p^{n+2}, then the element $z = py+x$ is of zero height and of order p^{n+1}. This means that the subgroup $\{z\}$ is neat and hence, by hypothesis, pure in G. But $\{z\}$ contains an element $p^n z = p^{n+1}y$ which is divisible by p^{n+1} in G but not in $\{z\}$, contradicting the purity of $\{z\}$. Thus every element of G is of order $\leqq p^{n+1}$ so that G is bounded and hence is a direct sum of cyclic groups. Further G cannot have a cyclic summand of order $< p^n$. For, otherwise, G will have an element (namely the generator of the cyclic summand) with zero height and order $< p^n$ and this is not possible since x is an element of least order and height zero and $o(x) = p^n$. Thus G is of the form, $G = \Sigma c(p^n)+\Sigma c(p^{n+1})$, where the summation is arbitrary, the Σ denoting the sum of a number of identical copies.

Conversely, if G is a divisible p-group, then it is immediate that $\mathfrak{N} = \mathfrak{P}$. On the other hand, let $G = \Sigma c(p^n)+\Sigma c(p^{n+1})$. We must show that every neat subgroup is pure in G. A direct calculation will show that we can restrict ourselves to proving that all cyclic neat subgroups of G are pure in G, or, what is the same thing, every element of height zero generates a pure subgroup of G.

Let a be an element of height zero. Let $a = l_1a_1+ \dots +l_ra_r$ where the a_i's are elements of a fixed basis B of G and the l_i's are integers. Since the height of a is zero, there exists at least one element, say, l_ka_k, the height of which is zero and this means that $(l_k, p) = 1$. Hence, we can, without loss in generality, replace $l_k a_k$ by a_k and also suppose that $k = 1$, and we assume that this has been done. We now show that $\{a\}$ is pure.

Let $0 \neq p^s x = ta$, where $x \in G$ and where clearly $s \leqq n$. Expanding both sides with respect to the basis elements of the fixed basis B of G, we have

$$\sum_{i=1}^{n} p^s m_i x_i = ta = ta_1 + tl_2 a_2 + \ldots + tl_r a_r,$$

where x_i's are basis elements of G which make up x and where m_i's are integers. Transferring all terms to one side and clubbing the like terms, we see that each term vanishes due to the independence of the basis elements. We now distinguish two cases:

CASE 1. There exists an x_i such that $x_i = a_1$. Then $p^s m_i x_i - ta_1 = 0$, i.e. $(p^s m_i - t)a_1 = 0$. Since $o(a_1) = p^n$ or p^{n+1}, $(p^s m_i - t)$ should be a multiple of p^n. Further, $s \leqq n$ and so we can write $t = p^s . h$, where h is a suitable integer. But then, $ta = p^s ha = p^s . a'$, where $a' = ha \in \{a\}$.

CASE 2. No x_i equals a_1. Then $ta_1 = 0$, so that $t = p^n . t' = p^s . t''$. But then $ta = p^s t'' a = p^s a''$, $a'' \in \{a\}$. Thus in any case it follows that $\{a\}$ is pure in G and this completes the proof.

Theorem 3.2 and the remarks in the paragraph preceding the theorem lead us to the following:

THEOREM 3.3. Every neat subgroup of a group G is pure if and only if either (i) G is a direct sum of a torsion-divisible group and a torsion-free group, or (ii) G is torsion and for each prime p, G_p is either divisible or $G_p = \Sigma c(p^n) + \Sigma c(p^{n+1})$ where n is some non-negative integer and the summation is arbitrary.

The following theorem describes groups G for which $\mathfrak{N} = \mathfrak{D}'$:

THEOREM 3.4. Every neat subgroup of G is a direct summand if and only if either (i) G is torsion and for each prime p, G_p is either divisible or is of the form $G_p = \Sigma c(p^n) + \Sigma c(p^{n+1})$, or (ii) $G = D + F$ where D is torsion-divisible and $F = \Sigma Q + \sum_m Q'$ where Q is the additive group of rational numbers, Q' is a subgroup of Q and m is an integer.

PROOF. In a torsion-free group $\mathfrak{N} = \mathfrak{P}$ and theorem $2a$ of Fuchs, Kertesz and Szele (1953) asserts that in a torsion-free group F, $\mathfrak{P} = \mathfrak{D}'$ if and only if $F = \Sigma Q + \sum_m Q'$ where Q' is a subgroup of the additive group Q of rational numbers and m is an integer. This along with theorem 3.3 proves the necessity part.

Conversely, let G be any one of the stated groups. Then we show that $\mathfrak{N} = \mathfrak{D}'$. In particular, we need only to show that if $G = \Sigma c(p^n) + \Sigma c(p^{n+1})$, then every neat subgroup is a direct summand. By theorem 3.2, every neat subgroup of G is pure. But G, and therefore every subgroup of G, is bounded and hence it follows that every neat subgroup of G is a direct summand.

REMARK 3.5. If G is a group for which $\mathfrak{N} = \mathfrak{D}'$, then every neat subgroup N of G has an interesting property, namely it is not only a direct summand but every subgroup maximally disjoint with N is also a direct summand. But it is to be noted that such a summand need not be an absolute summand. For example let $G = c(8)+c(4) = \{a\}+\{b\}$. It follows from theorem 3.4 that every neat subgroup of G is a summand. The subgroup $S = \{s\}$ where $s = 2a+b$ is maximally disjoint with $B = \{b\}$ and hence is a summand of G. But S is not a complementary summand since the element $a \notin \{B, S\}$ and so $G \neq \{B, S\}$.

The following theorem characterizes groups in which $\mathfrak{N} = \mathscr{A}$:

THEOREM 3.6. Every neat subgroup of a group G is an absolute direct summand if and only if G is any one of the following groups: (i) G is divisible; (ii) $G = D+R$, where D is torsion-divisible and R is a subgroup of rational numbers; (iii) G is a non-divisible torsion group and for each prime p, G_p is either divisible or $G_p = \Sigma c(p^n)$ where n is a non-negative integer.

PROOF. Now $\mathfrak{N} = \mathscr{A}$ implies $\mathfrak{N} = \mathfrak{D}'$ and hence by theorem 3.4 either (i) G is torsion where the non-divisible p-components are of the form $G_p = \Sigma c(p^n)+\Sigma c(p^{n+1})$, or (ii) $G = D+F$, where D is torsion-divisible and $F = \Sigma Q+\sum_m Q'$, m is finite.

CASE 1. G is torsion. Consider those G_p which are not divisible. Let $S = \{a\}$ be a cyclic summand of order p^n. By assumption S is an absolute summand and since $a \notin pS$, by theorem 22.2 of Fuchs (1960*a*), $p^n[(G/S)_p] = 0$, i.e. $G_p = \Sigma c(p^n)$.

CASE 2. $G = D+F$, D torsion divisible and $F = \Sigma Q+\Sigma Q'$, $Q' \subset Q$. If the reduced component of F vanishes, then G is divisible. If it does not vanish then Q' is a summand of G and hence is absolute. This implies that G/Q' is torsion. In other words, $G = D+Q'$.

The proof of the converse part is straightforward.

REMARK 3.7. Groups, in which $\mathfrak{N} = \mathscr{A}$, seem to possess interesting properties. Misina (1962) has recently shown that these are precisely the groups in which every endomorphism (automorphism) of each subgroup can be extended to an endomorphism (automorphism) of the whole group G. Further arguments similar to those used in proving § 3.6 show that groups for which $\mathfrak{N} = \mathscr{A}$ are precisely the groups for which $\mathscr{P} = \mathscr{A}$.

The following corollary describes groups for which $\mathfrak{D}' = \mathscr{A}$, the proof being obtained by employing similar arguments.

COROLLARY 3.8.* Every direct summand of a group G is absolute if and only if G has one of the following properties:

(i) G is divisible;

(ii) $G = D + R$, where D is torsion-divisible and R is an indecomposable torsion-free group;

(iii) G is torsion and for each prime p, relevant to G, G_p is either divisible or $G_p = \Sigma c(p^n)$ for some integer n.

In the following we consider situations when the class π coincides with any one of the classes $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{D}'$, $\mathcal{A}$.

THEOREM 3.9. Let G be a group.

1. Every π-subgroup is neat in G if and only if $G = c(p^r)$, $(0 \leqq r \leqq \infty)$ or G is elementary or G is torsion-free.
2. Every π-subgroup is pure if and only if $G = c(p^r)$, $0 \leqq r \leqq \infty$ or G is elementary or G is torsion-free.
3. Every π-subgroup is a direct summand if and only if $G = c(p^r)$, $(0 \leqq r \leqq \infty)$ or G is elementary or $G = \Sigma Q + \sum_m Q'$, Q' being a subgroup of the group Q of rational numbers and m is finite.
4. Every π-subgroup is an absolute direct summand if and only if $G = c(p^r)$, $0 \leqq r \leqq \infty$ or G is elementary or G is a subgroup of rational numbers or G is torsion-free divisible.

PROOF. 1. Let every π-subgroup be neat, i.e. intersection of neat subgroups is again neat. Case 1. Let every subgroup be neat. Then G becomes elementary. Case 2. Not every subgroup is neat. In this case every non-neat subgroup S has a proper unique neat hull, namely the intersection of all neat subgroups containing S. Then it follows from Rangaswamy (1964) that S is full. Thus all non-neat subgroups are full in G. Since their neat hulls will also be full, it follows without difficulty that all subgroups of G are full in G. A direct calculation then shows that either G is torsion-free or $G = c(p^r)$, $0 \leqq r \leqq \infty$ for any prime p.

The converse part is easily verified.

Assertions 2, 3, 4 can be verified in a similar manner.

We now consider situations where each one of the classes π, $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{D}'$ is contained in $\mathcal{F}$ or $\mathcal{A}'$. The results are stated in a series of theorems; the proofs are omitted.

THEOREM 3.10. Let G be a group. Then the following are equivalent: (1) $\pi = \mathcal{A}'$; (2) $\mathfrak{N} = \mathcal{A}'$; (3) $\mathfrak{P} = \mathcal{A}'$; (4) $\mathfrak{D}' \subset \mathcal{A}'$; (5) G is torsion-free or G

* Note that the remarks given as answer to Ex. 11a, Ch. IV of Fuchs (1960a), are not complete. Corollary 3.8 supplies the complete answer.

is an indecomposable torsion group, i.e. $G = c(p^r)$, $0 \leq r \leq \infty$, where p is any prime.

If S is a π-subgroup and is also full in G, then (*see* Rangaswamy 1964) S is neat and hence is absorbing. Thus $\pi \subset \mathcal{F}$ implies that $\pi = \mathcal{A}'$. Similarly $\mathcal{N} \subset \mathcal{F} \Rightarrow \mathcal{N} = \mathcal{A}'$, etc., and thus we are led to the following:

THEOREM 3.11. The following properties of a group G are equivalent: (i) $\pi \subset \mathcal{F}$; (ii) $\mathcal{N} \subset \mathcal{F}$; (iii) $\mathcal{P} \subset \mathcal{F}$; (iv) $\mathcal{D}' \subset \mathcal{F}$; (v) G is torsion-free or an indecomposable torsion group.

THEOREM 3.12. (i) Every absorbing subgroup of a group G is a direct summand if and only if $G = T + F$, where T is an arbitrary torsion group and $F = \Sigma Q + \underset{m}{\Sigma} Q'$, where m is an integer, Q' is a subgroup of the additive group Q of rational numbers.

(ii) Every absorbing subgroup is an absolute direct summand if and only if G is divisible or G is torsion or $G = T + R$, where T is torsion-divisible and R is a torsion-free group of rank one.

The assertion (i) follows from theorem 2*a* of Fuchs, Kertesz and Szele (1953) together with the fact that every absorbing subgroup of G is a pure subgroup containing G_t. Assertion (ii) is immediate from (i) and the characterizing property of absolute direct summands (Fuchs 1960*a*).

IV. Groups with Special Classes of Super-groups

This section is devoted to the study of groups which are, in a sense, dual to those discussed in III. For convenience of expression we say that a group H is a *super-group* of G if H contains G as a subgroup. As before we consider the various classes of super-groups of a group G like the class $\mathcal{N}^*$ of groups in which G is neat, the class $\mathcal{P}^*$ of groups in which G is pure, etc. Similarly the classes π^*, $(\mathcal{D}')^*$, $\mathcal{A}^*$, $(\mathcal{A}')^*$ are defined. The properties of G are, to a large extent, influenced by the coincidence relations among the various classes of super-groups of G and this is investigated in the present section. A result that would perhaps be of interest in itself is that the class of groups for which $\mathcal{N}^* = \mathcal{P}^*$ is precisely the class of groups G for which nG is a direct summand of G for every integer n—the latter class having been investigated by Kertesz and Szele (1952). In this connection we consider two questions raised by them. Let G be a reduced group wherein nG is a summand of G for every integer n or, equivalently, where $\mathcal{N}^* = \mathcal{P}^*$. Then,

(i) Is every endomorphic image of G a direct summand?

(ii) Is the group G determined uniquely (up to isomorphism) by the full systems of invariants of G_t and G/G_t?

We show that every endomorphic image of such a group G is pure in G. We also show that (ii) has a negative answer.

Theorem 4.1. Let G be a group. Then the following are equivalent:

(i) $\mathscr{S}^*$ coincides with one of the classes π^*, $\mathfrak{N}^*$, $\mathscr{P}^*$, $(\mathfrak{D}')^*$, $\mathscr{A}^*$.

(ii) π^* coincides with one of the classes $\mathfrak{N}^*$, $\mathscr{P}^*$, $(\mathfrak{D}')^*$, $\mathscr{A}^*$.

(iii) $\mathscr{A}^*$ coincides with one of the classes $\mathfrak{N}^*$, $\mathscr{P}^*$, $(\mathfrak{D}')^*$.

(iv) G is divisible.

That (i) $\Longleftarrow\!\Longrightarrow$ (iv) is well known while equivalence of (ii) and (iv) follows from the known fact that every group G is a π- subgroup of a suitable divisible group E (for example, if D is a divisible hull of G, set $E = D + Q/Z$). That assertions (iii) $\Longleftarrow\!\Longrightarrow$ (iv) can be easily verified.

Theorem 4.2. Let G be a group. Then the following properties are equivalent:

(i) The class $\mathscr{F}^*$ is contained in one of the classes π^*, $\mathfrak{N}^*$, $\mathscr{P}^*$, $(\mathfrak{D}')^*$, $(\mathscr{A}')^*$.

(ii) $(\mathscr{A}')^* \subset \mathscr{A}^*$.

(iii) G is divisible.

The proof is straightforward and is omitted.

We note that for a number of cases the possibility of coincidence of some of the concerned classes of super-groups does not arise at all. For example, there exists no group for which $\mathfrak{N}^* = (\mathscr{A}')^*$, $\mathscr{P}^* = (\mathscr{A}')^*$, $(\mathfrak{D}')^* \subset (\mathscr{A}')^*$, $\pi^* \subset \mathscr{F}^*$, $\mathfrak{N}^* \subset \mathscr{F}^*$, etc. Groups for which $\mathscr{P}^* = (\mathfrak{D}')^*$ are precisely the algebraic compact groups of Kaplansky (1954), while groups for which $(\mathscr{A}')^* \subset (\mathfrak{D}')^*$ are the B-groups defined by Fuchs (1960*b*), or, equivalently, groups which are direct sums of a divisible group and a co-torsion group.

We now consider groups for which $\mathfrak{N}^* = (\mathfrak{D}')^*$.

Theorem 4.3. G is a direct summand of every group in which it is neat if and only if G is a direct sum of a divisible group and a direct product of p-elementary groups†.

Proof. Let G be a direct summand of all those groups in which it is neat. Since each summand of G also has the stated property, we assume, without loss in generality, that G is reduced. Our hypothesis says that, for any group X, all neat extensions of G by X are splitting or, equivalently, the Frattini subgroup of Ext (X, G) vanishes for every X.

Since $\mathfrak{N}^* = (\mathfrak{D}')^*$ implies $(\mathscr{A}') \subset (\mathfrak{D}')^*$, it follows that the (reduced) group G is co-torsion. Then

$$G \cong \text{Ext}\,(Q/Z, G)$$

† I find that this assertion has been proved, in a different way, in a pre-print entitled 'High Extensions of Abelian Groups' which Prof. E. A. Walker had kindly sent me some time back.

and so

$$\phi(G) \cong \phi\,[\text{Ext}\,(Q/Z, G)] = 0.$$

In particular $\phi(G_t) = 0$ which means that G_t is elementary. Further, G cannot have a subgroup isomorphic to the group P of p-adic integers, since P has a non-zero ϕ-subgroup, namely $p\,.\,P$. Thus G is adjusted co-torsion and hence

$$G \cong \text{Ext}\,(Q/Z, G_t) = \text{Ext}\left(\sum_p c(p^\infty), \sum_p \sum_{n_p} c(p)\right)$$

where n_p is a cardinal depending on p,

$$\cong \Pi\,\text{Ext}\left[c(p^\infty), \left(\sum_{n_p} c(p) + \sum_q{}' \sum_{n_q} c(q)\right)\right]$$

where in Σ' the summation is over-all relevant primes other than p,

$$\cong \Pi\,\text{Ext}\left[c(p^\infty), \sum_{n_p} c\,(p)\right] \cong \Pi\,\Sigma c(p).$$

For, since $\Sigma c(p)$ is co-torsion,

$$\Sigma c(p) \cong \text{Ext}\,[Q/Z, \Sigma c(p)] \cong \text{Ext}\,[c(p^\infty), \Sigma c(p)].$$

Thus G is a direct product of p-elementary groups.

Conversely, let $G = D + \Pi\Sigma c(p)$, where D is divisible. Then for any group L,

$$\begin{aligned}\phi\,[\text{Ext}\,(L, G)] &\cong \phi\,[\text{Ext}\,(L, \Pi\Sigma c(p))] \cong \phi\,[\Pi\,\text{Ext}\,(L, \Sigma c(p))] \\ &= \Pi\phi\,[\text{Ext}\,(L, \Sigma c(p))] \cong 0,\end{aligned}$$

since $\phi[\text{Ext}\,(L, \Sigma c(p))] = 0$, due to the simple fact that if E is p-elementary and is neat in H, then E is a direct summand of H. Thus $\phi[\text{Ext}\,(X, G)] = 0$ for every X and so $\mathfrak{N}^* = (\mathfrak{D}')^*$.

REMARK 4.4. It becomes clear from the proof of the above theorem that a reduced group G possesses the stated property if and only if G is adjusted co-torsion and G_t is elementary.

THEOREM 4.5. *Let G be a reduced group for which $\mathfrak{N}^* = (\mathfrak{D}')^*$. Then every endomorphic image of G is a direct summand.*

PROOF. Let G be a reduced group with $\mathfrak{N}^* = (\mathfrak{D}')^*$. By 4.4, G is adjusted co-torsion, G_t elementary and $\phi(G) = 0$. Let I be an endomorphic image of G, say, $I = \alpha(G)$. Then it is easy to see that I is co-torsion and I_t elementary. Further $\phi(I) = 0$ and this will imply, by arguments similar to those employed in § 4.3, that I is adjusted co-torsion. It then follows from § 4.4 that I is a summand of every group in which it is neat. Now I_t is a summand of the elementary group and hence pure in G. Further, since I is adjusted, I/I_t is divisible and so is pure in G/I_t. Thus it follows that I is pure in G and hence is a direct summand of G.

We now consider groups for which $\mathfrak{N}^* = \mathfrak{P}^*$.

Theorem 4.6. G is pure in every group in which it is neat if and only if $G = D+R$, where D is divisible, R is reduced with R_t elementary and R/R_t divisible.

Proof. Let G be a group which is pure in every group in which it is neat. This means that for any group X, all the neat extensions of G by X are also pure, i.e. for any X, the ϕ-subgroup of $E = \text{Ext}(X, G)$ coincides with its radical E^1. In particular, take $X = c(p^r)$, $r > 1$. Then $\text{Ext}(X, G) \cong G/p^rG = G^*$ say. Now $(G^*)^1 = 0$, since G^* is bounded while $\phi(G^*) = pG^*$. Hence by assumption $pG^* = (G^*)^1 = 0$. This means that $pG = p^rG$, $r > 1$, i.e. pG is p-divisible. So if G is a p-group for which $\mathfrak{N}^* = \mathfrak{P}^*$, then G should be a direct sum of a divisible group and a p-elementary group. Thus a torsion group will have the stated property if and only if it is a direct sum of a divisible group and an elementary group.

Let G be a mixed group which is pure whenever it is neat in a supergroup. We assume, without loss in generality, that G is reduced. A direct calculation shows that G_t also possesses the same property and hence G_t becomes an elementary group. Let $G^* = \text{Ext}(Q/Z, G)$. By hypothesis, $\phi(G^*) = (G^*)^1 \ldots$ (1). This implies that G^* is adjusted. For, if $G^* = A+R$, A adjusted, R torsion-free co-torsion, then, since $R^1 = 0$, relation (1) gives that $\phi(A)+\phi(R) = A^1$ and this implies that $\phi(R) = 0$, a contradiction (by Remark 2.8). Thus G^* is adjusted and G^*/G_t^* is divisible. Now it is known (Harrison 1959) that G is pure in G^* and $G_t = (G^*)_t$. In particular, G/G_t is pure in the divisible group G^*/G_t and hence is itself divisible. Thus a reduced group G with the stated property is such that G_t is elementary and G/G_t is divisible.

Conversely, if $G = D+R$, where D is divisible and R_t is elementary with R/R_t divisible, we can readily check up that $\mathfrak{N}^* = \mathfrak{P}^*$.

From theorem 4.6, we get an interesting

Corollary 4.7. A torsion G is a direct summand of every torsion group H in which it is neat if and only if G is a direct sum of a (torsion) divisible group and an elementary group.

Proof. If G is a summand of every torsion group in which it is neat, then G will be pure in every group in which it is neat and hence, by theorem 4.6, G is a direct sum of a (torsion) divisible group and an elementary group.

The converse implication follows on noting that if G is neat in a torsion group H, then, for each p, G_p is (p-elementary+divisible) and is neat in H_p and hence is a direct summand of H_p.

The following theorem gives an interesting characterization of groups for which $\mathfrak{N}^* = \mathfrak{P}^*$:

Theorem 4.8. For a group G, $\mathfrak{N}^* = \mathfrak{P}^*$ if and only if for each integer n, nG is a summand of G.

PROOF. Let G be a group which is pure whenever it is neat. Then, by theorem 4.6, G_t is divisible and elementary and G/G_t is divisible. We assume without loss in generality that G is reduced. Now G_p is p-elementary and is pure in G and hence is a summand, $G = G_p+C$. Now G/G_t is divisible and hence G/G_p is p-divisible, i.e. C is p-divisible, which means that $C \subset pG$. But already $pG \cap G_p = 0$, G_p being p-elementary and neat. Hence it follows that $C = pG$. In other words pG is a summand of G. Thus for each p, pG is a summand. (Note that the decomposition is unique since both the components are fully invariant). By an easy calculation we then see that for each integer n, nG is a summand of G.

Conversely, let for each prime p, pG be a summand of G. This implies, in particular, that pG is p-divisible. Let G be a neat subgroup of H. We show that G is pure in H. Let $nx = a \in G$, where $x \in H$ and $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$. Since G is neat, a is divisible in G by each one of the primes $p_1, \dots p_k$. i.e. $a \in p_i G$ for $i = 1, \dots, k$ and since $p_i G$ is p_i-divisible, a is divisible by each of the factors $p_1^{r_1}, \dots, p_k^{r_k}$ in G and hence is divisible by their L.C.M., namely n itself, in G. Thus G is pure in H.

Now Kertesz and Szele (1952) proved that a reduced group G with G_t elementary and G/G_t divisible will be an interdirect sum of the components $G_p = \Sigma c(p)$. The following theorem collects all information about the groups under study:

THEOREM 4.9. Let G be a group. Then the following properties are equivalent:

(1) G possesses the property that $\mathfrak{N}^* = \mathfrak{P}^*$.

(2) G_t is a direct sum of a (torsion) divisible group and an elementary group and G/G_t is divisible.

(3a) For each integer n, nG is a summand of G.

(3b) For each prime p, pG is a summand of G.

(4) $G = D+R$, where (i) D is divisible, (ii) R is reduced with $R_t = \Sigma R_p$ elementary, and (iii) R is an interdirect sum of the components R_p with R/R_t divisible.

(5) $G = D+R$, D divisible and R is a pure subgroup of a direct product of p-elementary groups.

PROOF. Equivalence of (1), (2), (3a), (3b) has already been proved while the remark in the para preceding the above theorem shows that (4) $<= =>$ (2).

We now assume (4). We take without loss of generality that G is reduced. G is an interdirect sum of G_p with G_t elementary and G/G_t divisible. Let $H = \Pi G_p$. Now $G_t = H_t$ and G/G_t, being divisible, is pure in H/G_t. Hence G is pure in H, thus proving (4).

On the other hand, assume (5). Let G be pure in $H = \Pi E_p$ where E_p's are p-elementary. Then $H_t = \Sigma E_p$ is elementary. Since G is pure in H, $\{H_t, G\}$ is also pure in H. Then $\{H_t, G\}/H_t \cong G/G \cap H_t = G/G_t$ is pure in the divisible group H/H_t and hence is divisible. This proves (2) and hence (4).

THEOREM 4.10. Let G be a reduced group for which $\mathfrak{N}^* = \mathfrak{P}^*$. Then every endomorphic image of G is pure in G.

PROOF. Now G is pure in every group in which it is neat and so, by theorem 4.8, pG is a summand of G for every prime p and, in fact, we see that $G = pG + G_p$. Let α be an endomorphism of G and $S = \alpha G$. Since, for each p, pG and G_p are fully invariant, $S = \alpha G = \alpha(pG) + \alpha(G_p) = pS + S_p$. Thus, for each p, pS is a summand of S and hence, by theorem 4.9, S_t is elementary and S/S_t is divisible. Since G_t is elementary, S_t is pure in G. Also S/S_t, being divisible, is pure in G/S_t. This implies that S is pure in G. Thus every endomorphic image of G is pure and also possesses the stated property.

THEOREM 4.11. Let G be a group for which $\mathfrak{N}^* = \mathfrak{P}^*$. Then the following are true:

(1) Next $(T, G) = 0$ for any torsion group T.

(2) $\mathfrak{N}^* = (\mathfrak{D}')^*$ (that is G is a direct summand of every group H in which it is neat) if and only if G is a B-Group.

PROOF. (1) Now, for G, $\mathfrak{N}^* = \mathfrak{P}^*$ and hence, by theorem 4.9, G_t is (divisible +elementary) and G/G_t is divisible. The neat exact sequence $0 \to G_t \to G \to G/G_t \to 0$ gives, for any torsion group T, the exact sequence, Next $(T, G_t) = 0 \to$ Next $(T, G) \to$ Next $(T, G/G_t) = 0$, where the first term is zero since G_t is (divisible+elementary) (*see* Corollary 4.7) while the last term is zero since G/G_t is divisible. The exactness of the sequence implies that Next $(T, G) = 0$.

(2) Let G have the stated property and further be a B-group. Let X be an arbitrary group. The neat exact sequence $0 \to X_t \to X \to X/X_t \to 0$ gives the exact sequence Next $(X/X_t, G) \to$ Next $(X, G) \to$ Next (X_t, G). Now, by (1), Next $(X_t, G) = 0$. Further since G is a B-group, $0 =$ Ext $(X/X_t, G) =$ Next $(X/X_t, G)$. Then by the exactness of the last sequence we get Next $(X, G) = 0$ for any group X. Thus $\mathfrak{N}^* = (\mathfrak{D}')^*$. The converse is obvious.

Let G be a reduced group for which $\mathfrak{N}^* = \mathfrak{P}^*$. Then G_t is elementary and G/G_t is divisible. Let P be the set of primes relevant to G. Since G/G_t is torsion-free divisible, it is a vector space over the field of rational numbers. Also, for each $p_i \in P$, G_{p_i} is a vector space over the field of integers mod p_i. Now G determines uniquely a sequence of cardinals $(r, r_{p_1} \ldots r_{p_n} \ldots)$ $p_i \in P$, where r is the dimension of the vector space G/G_t while r_{p_i} is that of G_{p_i}. Kertesz and Szele (1952) ask whether the above sequence of cardinals determine uniquely (up to isomorphism) the group G. The following example answers this in the negative:

EXAMPLE 4.12. Let G be a direct product of p-elementary groups G_p such that G/G_t is of rank r, r being some infinite cardinal. It is always possible to find one such G. Now, by Remark 4.4, this group G is adjusted co-torsion. Hence $G/G_t = \underset{r}{\Sigma} Q$. Let H^* be a direct summand of G/G_t such that $G/G_t = H^* + \underset{n}{\Sigma} Q$, where n is finite. Clearly, the rank of H^* is r. Let H be the pre-image of H^* in G. Then $H_t = G_t$ and so is elementary. Further, $H/H_t \cong G/G_t$ since they are of the same rank r. Thus G and H define the same sequence of cardinals $(r, r_{p_1}, \ldots r_{p_n} \ldots)$. Now a subgroup S of a co-torsion group C is co-torsion if and only if C/S is co-torsion. Here G is co-torsion. But G/H is divisible and hence H is not co-torsion. Thus G and H are not isomorphic even though they define the same sequence of cardinals.

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