THERMAL INSTABILITY PROBLEM OF A FLUID SPHERE AND THE EFFECT OF RADIATIVE TRANSFER

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We have considered in this paper the thermal instability of an incompressible fluid sphere heated within when the fluid absorbs and emits thermal radiation. Two asymptotic cases of the radiative transfer equation, namely, when the fluid is optically thin and optically thick, have been examined. The variational principle and the validity of the principle of exchange of stabilities have been established assuming the temperature gradient to be constant. The analysis shows that radiative transfer stabilizes the fluid.

1. INTRODUCTION

In two recent publications Murgai and Khosla (1962 and 1963) (hereafter referred to as I and II) had examined the effect of radiative transfer on the gravitational stability of a fluid enclosed between two parallel surfaces. In the first the fluid was assumed to be ionized and the problem studied in hydromagnetic approximation in the presence of a uniform vertical magnetic field; in the latter it was unionized but rotating. Two asymptotic cases, viz. when the fluid is optically thin and when it is optically thick, were examined and the necessary and sufficient conditions for the validity of the principle of exchange of stabilities derived. The values of the critical Rayleigh number obtained for the different temperature levels (within the two approximations mentioned above) showed that the radiation tends to stabilize the fluid motion. These investigations were essentially an extension of the earlier work of Goody (1956) who had studied the problem for a simple hot fluid enclosed between two free surfaces. It is of interest to examine the radiative transfer effects in problems of stability relating to other more realistic geometries because of their importance in astrophysical and terrestrial contexts. We have done this here for an incompressible fluid sphere which is reacting and generating heat within it. This problem in its simplest version has been the subject of investigations by several authors (Wasiutynski 1946; Jeffrey and Bland 1951; Chandrasekhar 1961). The formulations of the last two differ in the choice of the dependent variable. We follow here the method due to Chandrasekhar who had used radial velocity for the dependent variable. This method lends itself to a correct use of available boundary conditions. The investigations embodied in the present paper have been carried out, as in our previous publications, for two approximations of the radiative transfer equation, one appropriate to the transparent and the other to an opaque medium.

It was first established (Goody 1956) that this mode of energy transfer results in a variable temperature gradient in the equilibrium state. This introduces a difficulty in the definition of the Rayleigh number in such problems, the characteristic values of which determine the critical state for the onset of instability. However, one can still reduce this case to an eigenvalue problem by introducing $\bar{\beta}$ which represents a suitable average of the temperature gradient. Although this helps us define the Rayleigh number, the variable nature of $(\beta/\bar{\beta})$, which inevitably accompanies it, introduces additional difficulties in these problems. For example, the questions relating to the stability in a plane geometry with rigid boundaries cannot be based on a variational principle and the principle of exchange of stabilities can only be proved when the two boundary surfaces are free. In the present case even when the bounding surface is free, it has not been possible to establish the variational principle for a variable $(\beta/\bar{\beta})$. The reason for this is that the problem is not completely defined as far as the boundary conditions are concerned. To establish the principle for variable $(\beta/\bar{\beta})$ we require the value of the third derivative of radial velocity on the surface of the sphere. This is not available in the problem. This difficulty, though present in Chandrasekhar's (1961) case for constant β, has been overcome by him by artificially reducing the order of the differential equations of the problem by defining an auxiliary function F which is proportional to the disturbance in temperature. However, for variable $(\beta/\bar{\beta})$ even this does not help. This is so because the system of differential equations is no longer self-adjoint. The calculation of Rayleigh number for marginal stability based on variational method for a variable β does not arise. We have assumed the radial temperature gradient to be constant and solved the problem. The Rayleigh number based on this is different from the one defined by Chandrasekhar (1961) and may be taken to be that based on a suitable average of the actual temperature gradient. As is evident from I and II, the variable nature of β influences the final expression for Rayleigh number through a factor in the denominator whose value is always less than unity within the limitation of transparent approximation and always equal to one for opaque case. It means that the variable β, which owes its origin to the radiative transfer process only, increases the final value of Rayleigh number over the one based on a constant β as hitherto being done. Thus in a sense the values of Rayleigh number presently obtained would represent a lower limit to the values one would have obtained for variable β as done in I and II for example. Hence the assumption of a constant β helps us to achieve a greater mathematical simplification without seriously affecting the final results. An additional advantage of this assumption is that it makes possible the discussion of the principle of exchange of stabilities, the validity of which rules out the presence of overstability in the present case and so requires the discussion of marginal state for convection, the only mode of instability which the system is capable of taking. The above-mentioned difficulties in regard to variable β appear for a transparent medium only and require the constant β assumption. Whereas in the opaque case the radiative transfer behaves like molecular conduction and its effect can be taken into account by modifying the thermal diffusivity and the formal analysis of Chandrasekhar (1961) can easily be carried out without any assumption in regard to the temperature gradient. The general results of the present paper are in conformity with the previous conclusions, viz. the radiative transfer has a stabilizing influence on the fluid motion.

In Sec. 2 the basic equations of the problem are obtained. Section 3 is devoted to the derivations of equations for marginal stability and the evaluation of the critical Rayleigh for onset of convection when the bounding surface is free.

2. Basic Equations

In order to investigate the effect of radiative transfer on the thermal instability of a fluid sphere of radius R under the effect of its own gravitation with uniform distribution of heat sources, one has to deal with the hydrodynamical equations of continuity, momentum, energy and the integro-differential equation of radiative transfer. These in vector notation are:

$$\frac{\partial \rho}{\partial t}+\nabla\,.\left(\rho\vec{u}\right)=0 \qquad (1)$$

$$\rho\left[\frac{\partial \vec{u}}{\partial t}+\left(\vec{u}\,.\,\nabla\right)\vec{u}\right]=-\nabla P+\rho\nabla V+\rho\nu\nabla^2\vec{u} \qquad (2)$$

$$\frac{\partial T}{\partial t}+\left(\vec{u}\,.\,\nabla\right)T=K\nabla^2 T+\epsilon+\Phi/C_p \qquad (3)$$

$$\frac{dI}{ds}=k(B^*-I(s)) \qquad (4)$$

$$\Phi=-\int\frac{dI}{ds}\,d\omega=-4\pi kB^*+k\int I(s)\,d\omega \qquad (5)$$

$$\rho=\rho_0(1-\alpha T) \qquad (6)$$

$$\nabla V=-\tfrac{4}{3}\pi\bar{\rho}G\vec{r}. \qquad (7)$$

In the foregoing equations ρ is the density of the fluid, $\vec{u}$ the velocity vector, P the pressure, ν the kinematic viscosity, V the gravitational potential,

T the temperature, K the thermal diffusivity, C_p the specific heat per unit volume, Φ the radiative heating per unit volume, I the intensity of radiation at any point, k the absorption coefficient (inverse of mean free path of radiation), B^* the Planck function, s and ω are elements of length and solid angle respectively, α the coefficient of volume expansion and G denotes the constant of gravitation. As is usually done in problems of this kind we split up T, P and Φ into $(T_0+\theta)$, (P_0+p) and $(\Phi_0+\phi)$ respectively, where the quantities with subscript 0 refer to the equilibrium or the static state and θ, p and ϕ are perturbations. $\vec{u}$ in this context is the disturbance velocity. The perturbations θ, p, ϕ and $\vec{u}$ are assumed to be small and hence satisfy a linearized version of the equations (1) to (7). The equations for the static state are the pressure and temperature distributions given by

$$\frac{dP_0}{dr} = -\frac{4\pi}{3}\bar{\rho}G\left[1-\alpha\int 2\beta r\,dr\right]r\rho_0 \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$K\left(\frac{d^2T_0}{dr^2}+\frac{2}{r}\frac{dT_0}{dr}\right)+\epsilon+\Phi_0/C_p = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

where β is the radial temperature gradient and is to be obtained from the solution of (9). A complete solution of (9) for arbitrary value of k seems difficult. However, we solve this equation for the two asymptotic cases mentioned earlier. In general, depending upon the value of the mean free path of radiation, the two expressions for Φ be written as (Secs. I and II)

$$\Phi = \begin{bmatrix} -4\pi kB^* & kR \ll 1 \\ \frac{4\pi}{3k}\nabla^2 B^* & kR \gg 1 \end{bmatrix} \qquad \begin{matrix} \ldots \quad \ldots \quad \ldots \quad (10a) \\ \ldots \quad \ldots \quad \ldots \quad (10b) \end{matrix}$$

where R is the radius of the sphere and the temperature of outer surface has been assumed to be zero and so does not contribute to Φ in (10a). Thus Φ_0 may be written as

$$\Phi_0 = \begin{bmatrix} -4\pi k\frac{\sigma}{\pi}T_0^4 & kR \ll 1 \\ \frac{4\sigma}{3k}\nabla^2 T_0^4 & kR \gg 1 \end{bmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

utilizing (11), equation (9) can be solved as

$$T_0 = \begin{bmatrix} \frac{\epsilon}{\lambda^2}\left\{\frac{I_{\frac{1}{2}}(\lambda r)}{\sqrt{r}I_{\frac{1}{2}}(\lambda)}-1\right\} & kR \ll 1 \\ \beta_1(1-r^2) & kR \gg 1 \end{bmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

where

$$\lambda^2 = 3k^2R^2\chi\,;\ \chi = \frac{4\pi S}{3KkC_p}\,;\ S = \frac{4\sigma}{\pi}T_0^3$$

and

$$\beta_1 = \epsilon R^2/6K(1+\chi),$$

σ being Stefan's constant.

In evaluating the temperature distribution, the vanishing of T_0 at the surface and its non-singularity at $r = 0$ determine the two constants of integration. Also we have assumed T_0^3 to be constant. The later assumption is not bad for incompressible fluids and has been discussed in some detail by Goody (1956). Now in conformity with the small disturbance theory, the linearized version of the equations of the problem in the Boussinesq approximation may be written as

$$\text{div}\,\vec{u} = 0 \qquad \ldots \quad (13a)$$

$$\frac{\partial \vec{u}}{\partial t} = -\nabla(p/\rho_0)+\nu\nabla^2\vec{u}+\gamma\theta\vec{r} \qquad \ldots \quad (13b)$$

$$\frac{\partial \theta}{\partial t} = K\nabla^2\theta+2\beta\vec{u}\,.\,\vec{r}+\phi/C_p \qquad \ldots \quad (13c)$$

where

$$\gamma = \frac{4\pi}{3}\bar{\rho}G\chi.$$

The boundary conditions to be satisfied by the solutions of equations (13) are not affected by radiation and remain the same as those for Chandrasekhar's (1961) corresponding case. They are for the free bounding surfaces.

$$\left.\begin{array}{l}\theta = u_r = 0 \\ d^2(ru_r)/dr^2 = 0\end{array}\right\} \qquad \ldots \quad (14)$$

at the surface of the sphere and θ and u_r should behave like r^l as $r \to 0$, u_r being the radial component of velocity.

3. Marginal Stability

Before we actually derive the equations for marginal stability, it seems worth while to reduce the general equations of the problem into the standard form by analysing the disturbances into normal modes and so we write

$$\left.\begin{array}{r}\vec{u}\,.\,\vec{r} = W(r)y_l^m(\Theta,\psi)e^{p^*t} \\ \theta = \theta(r)y_l^m(\Theta,\psi)e^{p^*t} \\ y_l^m = P_l^m(\cos\Theta)e^{\pm im\psi}\end{array}\right\} \qquad \ldots \quad (15)$$

where

P_l^m are Associated Legendre Polynomials. Now eliminating p from equations (13) by taking curl of (13b), using (13a) and equation (15) we have

$$D_l\left(D_l-\frac{p^*R^2}{\nu}\right)W = \frac{\gamma}{\nu}l(l+1)R^4\theta. \qquad \ldots \quad (16)$$

Also utilizing the expression for ϕ in terms of θ (Sec. I) and with the help of (15), the energy equation (13*c*) in the two approximations reduces to

$$(D_l-\lambda^2-\sigma_1)\theta = \frac{-2\beta}{K}R^2W\ (kR \ll 1) \quad \ldots \quad (17a)$$

$$(D_l-\sigma_2)\theta = \frac{-2\beta}{K(1+\chi)}W\ (kR \gg 1) \quad \ldots \quad (17b)$$

where

$$\sigma_1 = \frac{p^*R^2}{K} \text{ and } \sigma_2 = \frac{p^*R^2}{K(1+\chi)}.$$

The boundary conditions (14) then become

$$\left.\begin{aligned} \theta = W = 0 \\ \frac{d^2W}{dr^2} = 0 \end{aligned}\right\} \text{at } r=1 \quad \ldots \quad (18)$$

and θ and W should behave like r^l as $r \to 0$.

From now onwards we shall regard β to be constant and replace it by $\bar{\beta}$ in (17*a*). Now from equations (16) and (17) it can easily be proved that the principle of exchange of stabilities is valid which rules out the presence of over-stability (Appendix). The marginal state for convection is characterized by $\frac{\partial}{\partial t} = 0$. Thus putting $p^* = 0$ in equations (16) and (17) and eliminating θ from these equations, we obtain the equations of marginal stability as

$$(D_l-\lambda^2)F = -l(l+1)C_lW \quad (kR \ll 1) \quad \ldots \quad (19a)$$

$$D_lF = -l(l+1)\frac{C_lW}{1+\chi} \quad (kR \gg 1) \quad \ldots \quad (19b)$$

$$F = D_l^2W \quad \ldots \quad (19c)$$

where

$$C_l = 2\gamma\frac{\bar{\beta}R^6}{\nu K}$$

is the Rayleigh number whose characteristic values determine the marginal state. For opaque case $\bar{\beta}$ is to be replaced by β_1. Equations (18) and (19) are in the appropriate form for the discussion of variational principle which can now easily be established. Multiplying equations (19*a*, *b*) by r^2F and using (19*c*), we have after integrating by parts

$$l(l+1)C_l = \frac{\int_0^1\left[r^2\left(\frac{dF}{dr}\right)^2+l(l+1)F^2\right]dr+\lambda^2\int_0^1 r^2F^2\,dr}{\int_0^1 r^2(D_lW)^2\,dr} \quad (kR \ll 1)$$

and

$$l(l+1)C_l = \left\{ \frac{\int_0^1 \left[r^2\left(\frac{dF}{dr}\right)^2 + l(l+1)F^2 \right] dr}{\int_0^1 r^2(D_l W)^2\, dr} \right\} (1+\chi)(kR \gg 1).$$

Having put C_l as the ratio of two positive definite integrals it can easily be shown that corresponding to any arbitrary variation δW in W, compatible with the boundary conditions, the variation δC_l in C_l is identically zero provided W satisfies the differential equations of the problem. This provides us with a basis for evaluating the Rayleigh number based on the variational principle proved above. Therefore, if one chooses a function F satisfying the boundary conditions but not necessarily the differential equation and containing a few arbitrary parameters, one can find an approximate value of $C_{l\ \text{min.}}$ by minimizing with respect to them the expression of C_l. The accuracy of this minimum value can be increased by increasing the number of these parameters in the trial function. However, it is found that only one or two such parameters are sufficient for a fairly accurate value of the Rayleigh number. The numerical results presented in the present paper are for the first approximation only. Since radiation does not affect the boundary conditions of the problem, we can still use the same trial function for F given for the case of no radiative transfer. So we put

$$F = \frac{1}{\sqrt{r}} \sum_j A_j J_{l+1/2}(\alpha_{l,j}\, r) \qquad \ldots \quad \ldots \quad \ldots \quad (20)$$

where $J_{l+\frac{1}{2}}$ denotes the Bessel functions of order $l+\frac{1}{2}$ and $\alpha_{l,j}$ are its jth zero. A_j's are the various variational parameters. The value of the Rayleigh number can now be computed which for the first approximation comes out to be

$$l(l+1)C_l = \left. \begin{array}{ll} \dfrac{(\alpha_{l,1}^2+\lambda^2)\alpha_{l,1}^6}{\alpha_{l,1}^2 + \dfrac{4(2l+3)}{2l+1}} & (kR \ll 1) \\[2em] \dfrac{\alpha_{l,1}^8(1+\chi)}{\alpha_{l,1}^2 + \dfrac{4(2l+3)}{2l+1}} & (kR \gg 1) \end{array} \right\} \qquad \ldots \quad \ldots \quad (21)$$

In the limit of χ or $\lambda \to 0$, equations (21) reduce to Chandrasekhar's (1961) result. It may be seen that corresponding to $l = 1$ are the modes which can be easily excited and so instability first arises for this value of l. Tables I and II give the values of C_l and C_l/C_l^0 respectively for different λ's for the transparent case, where C_l^0 is the value of the Rayleigh number obtained by Chandrasekhar (1961). Fig. 1 is the plot of log C_l vs. l for various values of λ's shown on curves. The above-mentioned calculations were performed for the transparent case only. In the opaque case, as mentioned in the introduction, the values of the Rayleigh number can easily be obtained from those given by Chandrasekhar (1961) by multiplying them with $(1+\chi)$. The stabilizing influence of radiative transfer is quite apparent from the tables as well as the figures where the curves for $\lambda = 10^{-1}$ and 1 nearly coincide with that for $\lambda = 0$ which represents no radiative transfer.

It may be pointed out that this variational principle can easily be established even for the cases of rigid bounding surface and spherical shell.

4. Discussion

1. As will be apparent from Table I, the radiative transfer has got stabilizing influence on the fluid. This is borne out by the fact that radiative transfer will tend to damp out any disturbance of the fluid which may result due to heat transfer.

2. From Figs. 1–3 it will be clear that for small values of λ the radiative transfer has no effect on the convective motion of the fluid. Now small values of λ can result due to two reasons, viz. either the temperature level of the medium is low which means negligible radiation effects or the absorption coefficient of the medium is very small which means that the mean free path of the photons is so large that they do not collide or interact with the matter. So the results for small λ's are physically justifiable.

3. Both the above points are very well supported from Table II which brings out the ratio C_l/C_l^0 where C_l^0 is the value of Rayleigh number in the absence of radiative transfer. This ratio increases with λ but decreases with l, the order of harmonic. This shows that the effect of radiative transfer reduces for the disturbances of higher harmonic modes. This result is true only for the transparent case. The variable nature of β will make a contribution to this ratio for higher order harmonics. It is very difficult to comment whether the results based on $\beta(r)$ will support this conclusion or not.

It may not be out of place to remark about the possible effects of variable β on the validity of the principle of exchange of stabilities. It is expected that even if β is a variable quantity it is hardly going to produce overstability. This remark is based on the results of an earlier investigation by Khosla (1963) which indicate that the type of modes of wave propagation in the medium determines the presence of overstability in such systems. Since there seems to be no reason* for variable β introducing additional modes of wave propagation, the presence of overstability may be ruled out. The truth of this remark can only be proved if the required boundary condition is prescribed.

The present investigation clearly brings out the different mathematical

* Because of incompressibility, the longitudinal waves are not affected by the energetics of the system.

TABLE I

The values of C_l for different λ and l

l / λ	10^{-1}	1	10	10^2	10^3	10^4
1	$3{\cdot}0936\times10^3$	$3{\cdot}2451\times10^3$	$1{\cdot}8394\times10^4$	$1{\cdot}5333\times10^6$	$1{\cdot}5302\times10^8$	$1{\cdot}5302\times10^{10}$
2	$5{\cdot}2264\times10^3$	$5{\cdot}3820\times10^3$	$2{\cdot}0941\times10^4$	$1{\cdot}5768\times10^6$	$1{\cdot}5716\times10^8$	$1{\cdot}5716\times10^{10}$
3	$8{\cdot}7802\times10^3$	$8{\cdot}9580\times10^3$	$2{\cdot}6738\times10^4$	$1{\cdot}8048\times10^6$	$1{\cdot}7961\times10^8$	$1{\cdot}7960\times10^{10}$
4	$1{\cdot}3993\times10^4$	$1{\cdot}4199\times10^4$	$3{\cdot}4864\times10^4$	$2{\cdot}1013\times10^6$	$2{\cdot}0875\times10^8$	$2{\cdot}0873\times10^{10}$
5	$2{\cdot}1214\times10^4$	$2{\cdot}1454\times10^4$	$4{\cdot}5420\times10^4$	$2{\cdot}4420\times10^6$	$2{\cdot}4210\times10^8$	$2{\cdot}4208\times10^{10}$

TABLE II

The ratio C_l / C_l^0 for different values of λ and l

l / λ	10^{-1}	1	10	10^2	10^3	10^4
1	1·0005	1·0495	5·9488	$4{\cdot}9588\times10^2$	$4{\cdot}9488\times10^4$	$4{\cdot}9488\times10^6$
2	1·0003	1·0301	4·0079	$3{\cdot}0782\times10^2$	$3{\cdot}0079\times10^4$	$3{\cdot}0079\times10^6$
3	1·0002	1·0205	3·0459	$2{\cdot}0560\times10^2$	$2{\cdot}0460\times10^4$	$2{\cdot}0459\times10^6$
4	1·0001	1·0149	2·4919	$1{\cdot}5019\times10^2$	$1{\cdot}4920\times10^4$	$1{\cdot}4919\times10^6$
5	1·00009	1·0114	2·1412	$1{\cdot}1512\times10^2$	$1{\cdot}1413\times10^4$	$1{\cdot}1412\times10^6$

difficulties confronted in the solution of this problem and stresses the need for an additional boundary condition.

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Appendix

The Principle of Exchange of Stabilities

The principle is proved by assuming β to be constant. Denoting the left-hand side of equation (16) by F and eliminating θ from equations (16) and (17) we get

$$[D_l-\lambda^2-\sigma_1]F = -l(l+1)C_l W. \qquad (22)$$

The boundary conditions in terms of F are

$$W = F = 0 \quad \text{at } r = 1$$

$$\frac{d^2W}{dr^2} = 0 \quad \text{on the free surface.} \qquad (23)$$

Multiplying both sides of equation (22) by r^2F and integrating over the range of r, we get after making use of the above boundary conditions

$$\int_0^1 \left[r^2\left(\frac{dF}{dr}\right)^2+l(l+1)F^2+\lambda^2r^2F^2+\sigma_1{}^2F^2\right] dr$$

$$= l(l+1)C_l\left[-2\left\{r\left(\frac{dW}{dr}\right)^2\right\}_0^1+\int_0^1 r^2(D_lW)^2\,dr\right.$$

$$\left.+\overset{*}{p}\overset{*}{\sigma}\int_0^1\left\{r^2\left(\frac{dW}{dr}\right)^2+l(l+1)W^2\right\}dr\right]. \qquad (24)$$

The real and imaginary parts of (24) must vanish separately and the vanishing of imaginary parts gives

$$I_m(\sigma)\left[\int_0^1 r^2F^2dr+\overset{*}{p}l(l+1)C_l\int_0^1\left\{r^2\left(\frac{dW}{dr}\right)^2+l(l+1)W^2\right\}dr\right] = 0. \qquad (25)$$

In equation (25) the expression within the bracket is not zero and we get

$$I_m(\sigma) = 0.$$

Hence σ is real and the onset of stability must be via a marginal state. The principle is therefore valid.