ANALYSIS OF A FINITE HOLLOW CYLINDER SUBJECTED TO AXISYMMETRIC END LOADS

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A multiple Fourier solution has been given for the analysis of a finite hollow cylinder subjected to axisymmetric normal loads on the ends. The curved surfaces are assumed to be free of external loads and the solution is presented through Love function. Numerical work has been done with an Elliot 803 digital computer and the results are compared with those for a semi-infinite hollow cylinder.

NOMENCLATURE

A_n, B_n, C_n, D_n; P_m, Q_m, R_m, S_m = constants, *see* eqn. (13)

a, b = outer and inner radii of the hollow cylinder

l = length of the hollow cylinder

m, n = integer variables

$(k_n a)$ = roots of the transcendental equation (14)

c = outer radius of loaded area

$J_0(k_n r)$, $J_1(k_n r)$ = Bessel functions of I kind and of orders zero and one respectively

$Y_0(k_n r)$, $Y_1(k_n r)$ = Bessel functions of II kind and of orders zero and one respectively

(r, z) = radial and axial coordinates

$T_n = -J_1(k_n a)/Y_1(k_n a) = -J_1(k_n b)/Y_1(k_n b)$

$H_0(k_n r) = J_0(k_n r) + T_n Y_0(k_n r)$

$H_1(k_n r) = J_1(k_n r) + T_n Y_1(k_n r)$

$I_0(\alpha_m r)$, $I_1(\alpha_m r)$ = modified Bessel functions of I kind and of orders zero and one respectively

$K_0(\alpha_m r)$, $K_1(\alpha_m r)$ = modified Bessel functions of II kind and of orders zero and one respectively

$\sigma_z, \sigma_\theta, \sigma_r, \tau_{rz}$ = axial, hoop, radial and shearing stress components

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

$\alpha_m = m\pi/l$

q = total load, $P/\pi(a^2-b^2)$

q' = total load, $P/\pi(c^2-b^2)$

E = Young's modulus

μ = Poisson's ratio

ϕ = Love function

$\epsilon = r/a$

$\beta = z/a$

Introduction

Filon (1902) had analysed the problem of finite solid cylinders subjected to various types of boundary forces of practical significance. The solution given by him is essentially one of a single Fourier type where all of the boundary conditions of the problem are not satisfied. Pickett (1944) demonstrated the applicability of multiple Fourier method to the analysis of finite solid cylinders subjected to axisymmetric loads that satisfied all the boundary conditions. Numerical results were presented by him and compared with Filon's results. Douglas and Trahair (1960) analysed the problem of a hollow cylinder subjected to axisymmetric self-equilibrating normal end loads. They used a single Fourier series and arrived, as a first approximation, at a solution which did not satisfy some of the boundary conditions. But these boundary conditions which were not satisfied were allowed for later by proper superposition in a second approximation. But this involved the violation of those boundary conditions which had been observed already. Hodgkins (1962) presented a numerical solution, based on the method of finite differences for the end deformation problem of cylinders.

In this paper, a multiple Fourier solution has been presented for a finite hollow cylinder subjected to axisymmetric normal end loads. The loading considered is of self-equilibrating type as shown in Fig. 1(*b*). By adding a uniform axial compression—Fig. 1(*c*)—to the axial stress, the stress distribution in a hollow cylinder subjected to normal end loads as shown in Fig. 1(*a*) can be obtained. Theoretical analysis is presented for the problem in Fig. 1(*b*). Finally the numerical results are compared with those for a semi-infinite hollow cylinder subjected to the same end load (Yogananda 1965; Sundara Raja Iyengar and Yogananda 1966). The comparison throws light on the validity of St. Venant's principle for such problems.

Analysis of the problem:

The boundary conditions of the problem in Fig. 1(*b*) are—

when $z = 0, \quad \tau_{rz} = 0$ .. .. (1)

,, $z = 0, \quad \sigma_z = f_1^*(r)$ .. .. (2)

,, $r = a, \quad \tau_{rz} = 0$ .. .. (3)

,, $r = a, \quad \sigma_r = 0$ .. .. (4)

,, $r = b, \quad \tau_{rz} = 0$ .. .. (5)

,, $r = b, \quad \sigma_r = 0$ .. .. (6)

,, $z = l, \quad \tau_{rz} = 0$ .. .. (7)

,, $z = l, \quad \sigma_z = 0$ .. .. (8)

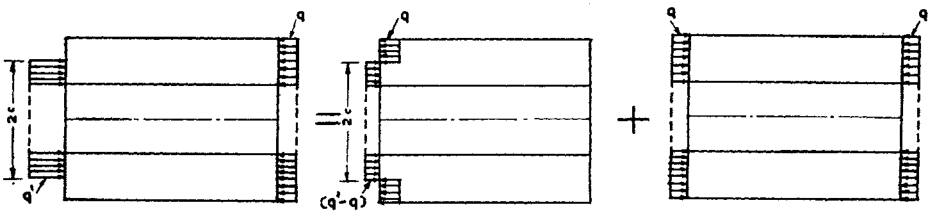

FIG. 1(*a*). Compressive normal end loads on the finite hollow cylinder.

FIG. 1(*b*). Self-equilibrating normal end load on the finite hollow cylinder.

FIG. 1(*c*). Uniform compression on the finite hollow cylinder.

Note: $q = \dfrac{P}{\pi(a^2 - b^2)}$; $q' = \dfrac{P}{\pi(c^2 - b^2)}$

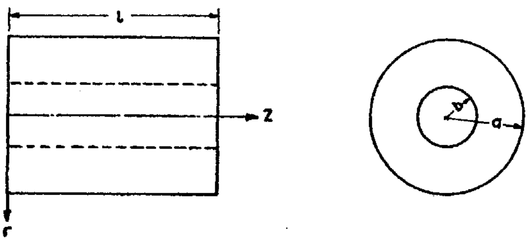

FIG. 1(*d*). Coordinate axes for the finite hollow cylinder.

In eqn. (2), $f_1^*(r)$ is a known function of r.

The Love function for an axisymmetric problem must satisfy

$$\nabla^2 \nabla^2 \phi = 0. \quad .. \quad .. \quad .. \quad .. \quad (9)$$

The stresses are given by

$$\left.\begin{aligned} \sigma_z &= \frac{\partial}{\delta z}\left[(2-\mu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi \\ \sigma_\theta &= \frac{\partial}{\partial z}\left[\mu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right]\phi \\ \sigma_r &= \frac{\partial}{\partial z}\left[\mu\nabla^2 - \frac{\partial^2}{\partial r^2}\right]\phi \\ \tau_{rz} &= \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi \end{aligned}\right\} \quad \ldots \quad (10)$$

The self-equilibrating normal load on the end ($z = 0$) is expanded in Fourier-Bessel series thus:

$$f_1^*(r) = \sum_{n=1}^{\infty} a_n[J_0(k_n r) + T_n Y_0(k_n r)], \quad \ldots \quad (11)$$

where

$$a_n = \frac{2\int_b^a r f_1^*(r)[J_0(k_n r) + T_n Y_0(k_n r)]\,dr}{\left[a^2 H_0^2(k_n a) - b^2 H_0^2(k_n b)\right]},$$

which finally reduces to

$$a_n = \frac{2(c/a)[J_1(k_n c) + T_n Y_1(k_n c)]\left[\dfrac{1-b^2/a^2}{(c^2/a^2)-(b^2/a^2)}\right]q}{(k_n a)\left[(b/a)^2 H_0^2(k_n b) - H_0^2(k_n a)\right]}. \quad \ldots \quad (12)$$

The Love function for the problem is chosen in the following form (Yogananda 1965):

$$\phi = \sum_{m=1}^{\infty} \frac{1}{\alpha_m^3}[P_m I_0(\alpha_m r) + Q_m \alpha_m r I_1(\alpha_m r) + R_m K_0(\alpha_m r) + S_m \alpha_m r K_1(\alpha_m r)] \sin \alpha_m z$$

$$+ \sum_{n=1}^{\infty} \frac{1}{k_n^3}[A_n \sinh k_n z + B_n \cosh k_n z + C_n k_n z \sinh k_n z + D_n k_n z \cosh k_n z]$$

$$\times [J_0(k_n r) + T_n Y_0(k_n r)], \quad \ldots \quad (13)$$

where $(k_n a)$ is a root of the transcendental equation

$$\left.\begin{aligned} J_1(k_n a) + T_n Y_1(k_n a) &= 0 \\ J_1(k_n b) + T_n Y_1(k_n b) &= 0 \end{aligned}\right\}. \quad \ldots \quad (14)$$

Using eqns. (13) and (10), we can derive the expressions for the stresses as

$$\sigma_z = \sum_{m=1}^{\infty} \{[P_m+2(2-\mu)Q_m]I_0(\alpha_m r)+Q_m\alpha_m r I_1(\alpha_m r)+[R_m-2(2-\mu)S_m]K_0(\alpha_m r)$$
$$+S_m\alpha_m r K_1(\alpha_m r)\} \cos \alpha_m z+ \sum_{n=1}^{\infty} \{[1-2\mu)D_n-A_n] \cosh k_n z$$
$$+[(1-2\mu)C_n-B_n] \sinh k_n z-C_n k_n z \cosh k_n z-D_n k_n z \sinh k_n z\}$$
$$\times\{J_0(k_n r)+T_n Y_0(k_n r)\},$$

$$\sigma_\theta = \sum_{m=1}^{\infty} \left[-P_m \frac{I_1(\alpha_m r)}{\alpha_m r}-(1-\mu)Q_m I_0(\alpha_m r)+R_m \frac{K_1(\alpha_m r)}{\alpha_m r}+(1-\mu)S_m K_0(\alpha_m r)\right]$$
$$\times \cos \alpha_m z+\sum_{n=1}^{\infty} \{2\mu[C_n \sinh k_n z+D_n \cosh k_n z] [J_0(k_n r)+T_n Y_0(k_n r)]$$
$$+[(A_n+D_n) \cosh k_n z+(B_n+C_n) \sinh k_n z+C_n k_n z \cosh k_n z+D_n k_n z \sinh k_n z]$$
$$\times[J_1(k_n r)+T_n Y_1(k_n r)]/(k_n r)\},$$

$$\sigma_r = \sum_{m=1}^{\infty} \left\{[(2\mu-1)Q_m-P_m]I_0(\alpha_m r)+\left[\frac{P_m}{\alpha_m r}-Q_m\alpha_m r\right]I_1(\alpha_m r)+[(1-2\mu)S_m-R_m]\right.$$
$$\left.\times K_0(\alpha_m r)-\left[\frac{R_m}{\alpha_m r}+S_m\alpha_m r\right]K_1(\alpha_m r)\right\} \cos \alpha_m z+\sum_{n=1}^{\infty} \{[A_n+(1+2\mu)D_n]$$
$$\times \cosh k_n z+[B_n+(1+2\mu)C_n] \sinh k_n z+C_n k_n z \cosh k_n z+D_n k_n z \sinh k_n z\}$$
$$\times[J_0(k_n r)+T_n Y_0(k_n r)]-\sum_{n=1}^{\infty} [(A_n+D_n) \cosh k_n z+(B_n+C_n) \sinh k_n z$$
$$+C_n k_n z \cosh k_n z+D_n k_n z \sinh k_n z][J_1(k_n r)+T_n Y_1(k_n r)]/(k_n r),$$

$$\tau_{rz} = \sum_{m=1}^{\infty} \{[P_m+2(1-\mu)Q_m]I_1(\alpha_m r)+Q_m\alpha_m r I_0(\alpha_m r)-[R_m-2(1-\mu)S_m]K_1(\alpha_m r)$$
$$-S_m\alpha_m r K_0(\alpha_m r)\} \sin \alpha_m z+\sum_{n=1}^{\infty} [(A_n+2\mu D_n) \sinh k_n z+(B_n+2\mu C_n) \cosh k_n z$$
$$+C_n k_n z \sinh k_n z+D_n k_n z \cosh k_n z][J_1(k_n r)+T_n Y_1(k_n r)]. \quad \ldots \quad \ldots \quad (15)$$

Equating the shear stress on the ends ($z = 0$ and l) and the outer and inner curved rims to zero, we get

$$B_n+2\mu C_n = 0, \quad \ldots \quad (16)$$

$$C_n = -\left[\left(\frac{A_n}{k_n l}\right)+\left(\frac{2\mu}{k_n l}+\cosh k_n l\right) D_n\right], \quad \ldots \quad (17)$$

$$I_1(\alpha_m a)P_m+[2(1-\mu)I_1(\alpha_m a)+\alpha_m aI_0(\alpha_m a)]Q_m-K_1(\alpha_m a)R_m$$
$$+[(1-\mu)2\,.\,K_1(\alpha_m a)-\alpha_m aK_0(\alpha_m a)]S_m=0, \quad \ldots \quad \ldots \quad (18)$$

$$I_1(\alpha_m b)P_m+[2(1-\mu)I_1(\alpha_m b)+\alpha_m bI_0(\alpha_m b)]Q_m-K_1(\alpha_m b)R_m$$
$$+[2(1-\mu)K_1(\alpha_m b)-\alpha_m bK_0(\alpha_m b)]S_m=0, \quad \ldots \quad \ldots \quad (19)$$

together with the transcendental eqn. (14).

The vanishing of the radial stresses on outer and inner curved rims yield two equations which upon simplification using eqns. (16) to (19) lead to

$$\left[a_{32}^m+a_{31}^m f_1(m)+a_{33}^m f_3(m)\right]Q_m+\left[a_{34}^m+a_{31}^m f_2(m)+a_{33}^m f_4(m)\right]S_m$$
$$=\sum_{n=1}^{\infty}[W_1(m,n)A_n+X_1(m,n)D_n] \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

and

$$\left[a_{42}^m+a_{41}^m f_1(m)+a_{43}^m f_3(m)\right]Q_m+\left[a_{44}^m+a_{41}^m f_2(m)+a_{43}^m f_4(m)\right]S_m$$
$$=\sum_{n=1}^{\infty}[W_2(m,n)A_n+X_2(m,n)D_n], \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where

$$a_{31}^m=\frac{I_1(\alpha_m a)}{\alpha_m a}-I_0(\alpha_m a)$$

$$a_{32}^m=(2\mu-1)I_0(\alpha_m a)-\alpha_m aI_1(\alpha_m a)$$

$$a_{33}^m=-\frac{K_1(\alpha_m a)}{\alpha_m a}-K_0(\alpha_m a)$$

$$a_{34}^m=(1-2\mu)K_0(\alpha_m a)-\alpha_m aK_1(\alpha_m a)$$

and

$$W_1(m,n)=-4\frac{\alpha_m^2}{l^2}\frac{H_0(k_n a)[1-\cos m\pi\cosh k_n c]}{[\alpha_m^2+k_n^2]^2}$$

$$X_1(m,n)=-4\frac{\alpha_m^2}{l^2}\frac{H_0(k_n a)}{(\alpha_m^2+k_n^2)^2}\left[\frac{k_n c}{\sinh k_n c}(\cosh k_n c-\cos m\pi)-2\mu(\cos m\pi\cosh k_n c-1)\right]$$

$$a_{41}^m=\frac{I_1(\alpha_m b)}{\alpha_m b}-I_0(\alpha_m b)$$

$$a_{42}^m=(2\mu-1)I_0(\alpha_m b)-\alpha_m bI_1(\alpha_m b)$$

$$a_{43}^m=-\frac{K_1(\alpha_m b)}{\alpha_m b}-K_0(\alpha_m b)$$

$$a_{44}^m=(1-2\mu)K_0(\alpha_m b)-\alpha_m bK_1(\alpha_m b)$$

$$W_2(m,n)=-4\frac{\alpha_m^2}{l^2}\frac{H_0(k_n b)}{[\alpha_m^2+k_n^2]^2}[1-\cos m\pi\cosh k_n c]$$

and

$$X_2(m, n) = -4\frac{\alpha_m^2}{l^2}\frac{H_0(k_n b)}{[\alpha_m^2+k_n^2]^2}\left[\frac{k_n c}{\sinh k_n c}(\cosh k_n c-\cos m\pi)-2\mu(\cos m\pi \cosh k_n c-1)\right].$$

Equating the axial stress on the $z = 0$ end to the prescribed value and equating the axial stress to zero on the $z = l$ end, we get two equations which upon simplification using Fourier-Bessel inverse transforms yield

$$a_n+[A_n-(1-2\mu)D_n] = \sum_{m=1}^{\infty}[g_1(m, n)Q_m+g_2(m, n)S_m] \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

and

$$\left[\frac{\sinh k_n l}{k_n l}\right]A_n-\left[\frac{(k_n l)^2-2\mu \sinh^2 k_n l}{k_n l \sinh k_n l}\right]D_n = \sum_{m=1}^{\infty}(-1)^m[g_1(m, n)Q_m+g_2(m, n)S_m], \quad \ldots \quad (23)$$

where

$$g_1(m, n) = \frac{2\{[f_1(m)+2(2-\mu)]M_1(\alpha_m, k_n)+\alpha_m M_2(\alpha_m, k_n)+f_3(m)M_3(\alpha_m, k_n)\}}{[a^2H_0^2(k_n a)-b^2H_0^2(k_n b)]}$$

$$g_2(m, n) = \frac{2\{[f_4(m)-2(2-\mu)]M_3(\alpha_m, k_n)+\alpha_m M_4(\alpha_m, k_n)+f_2(m)M_1(\alpha_m, k_n)\}}{[a^2H_0^2(k_n a)-b^2H_0^2(k_n b)]}$$

in which

$$M_1(\alpha_m, k_n) = \frac{[\alpha_m a I_1(\alpha_m a)H_0(k_n a)-\alpha_m b I_1(\alpha_m b)H_0(k_n b)]}{(\alpha_m^2+k_n^2)}$$

$$M_2(\alpha_m, k_n) = \frac{[\alpha_m a^2 I_0(\alpha_m a)H_0(k_n a)-\alpha_m b^2 I_0(\alpha_m b)H_0(k_n b)]}{(\alpha_m^2+k_n^2)} + \frac{2[-\alpha_m^2 a I_1(\alpha_m a)H_0(k_n a)+\alpha_m^2 b I_1(\alpha_m b)H_0(k_n b)]}{(\alpha_m^2+k_n^2)^2}$$

$$M_3(\alpha_m, k_n) = \frac{[-\alpha_m a K_1(\alpha_m a)H_0(k_n a)+\alpha_m b K_1(\alpha_m b)H_0(k_n b)]}{(\alpha_m^2+k_n^2)}$$

and

$$M_4(\alpha_m, k_n) = \frac{[-\alpha_m a^2 H_0(k_n a)K_0(\alpha_m a)+\alpha_m b^2 K_0(\alpha_m b)H_0(k_n b)]}{(\alpha_m^2+k_n^2)} + \frac{2[-\alpha_m^2 a K_1(\alpha_m a)H_0(k_n a)+\alpha_m^2 b K_1(\alpha_m b)H_0(k_n b)]}{(\alpha_m^2+k_n^2)^2}. \quad \ldots \quad \ldots \quad (24)$$

Hence, eqns. (20), (21), (22) and (23) yield the required set of infinite equations for an infinite number of unknowns.

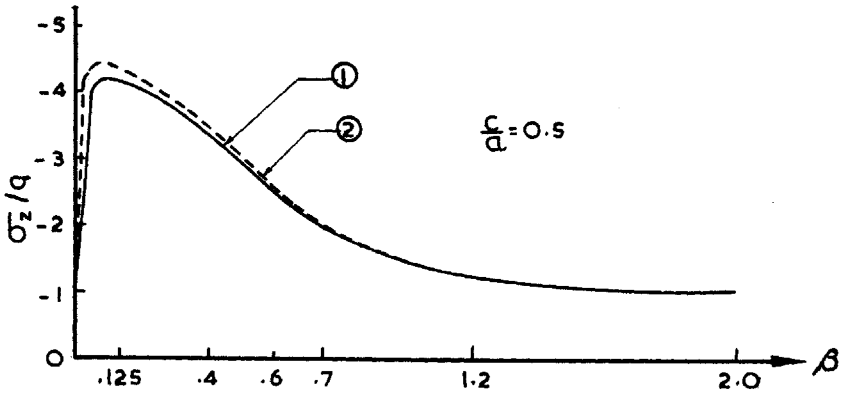

FIG. 2. Non-dimensional axial stress on inner rim, $\epsilon = 0{\cdot}125$.

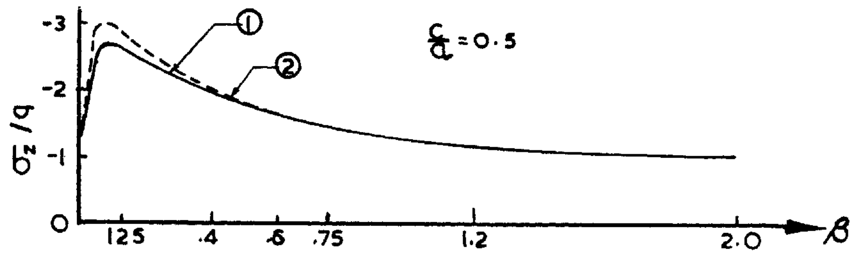

FIG. 3. Non-dimensional axial stress on layer, $\epsilon = 0{\cdot}41667$.

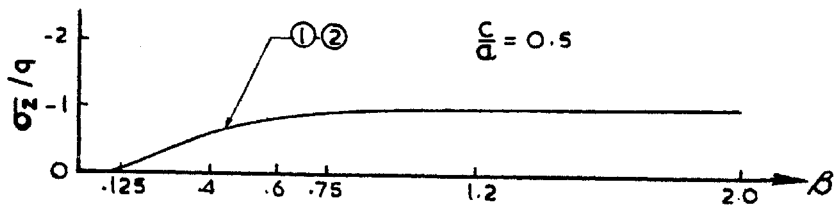

FIG. 4. Non-dimensional axial stress on layer, $\epsilon = 0{\cdot}70833$.

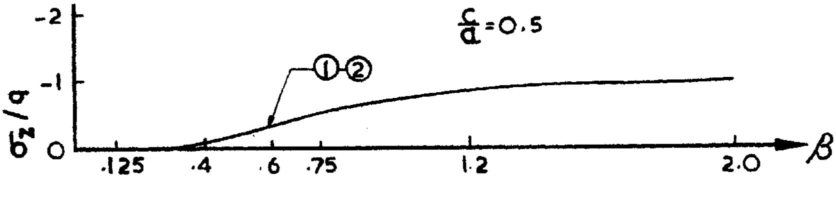

FIG. 5. Non-dimensional axial stress on outer layer, $\epsilon = 1{\cdot}0$.

Note: (1) Finite hollow cylinder solution; (2) Semi-infinite hollow cylinder solution.

Non-dimensional axial stress diagrams

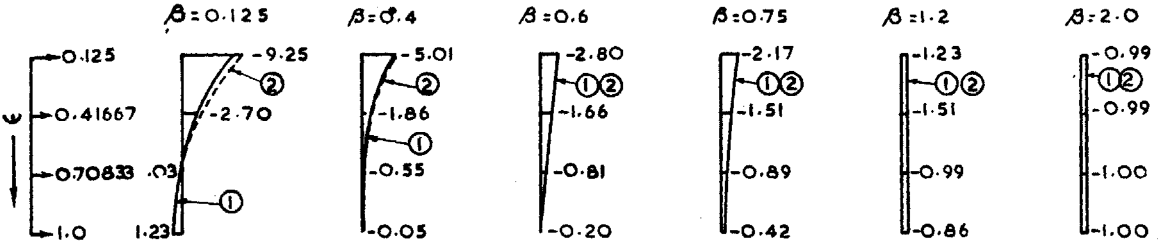

FIG. 6. $c/a = 0{\cdot}34375$.

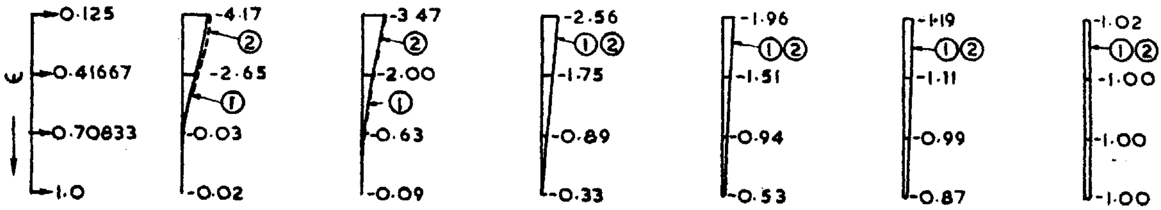

FIG. 7. $c/a = 0{\cdot}5$.

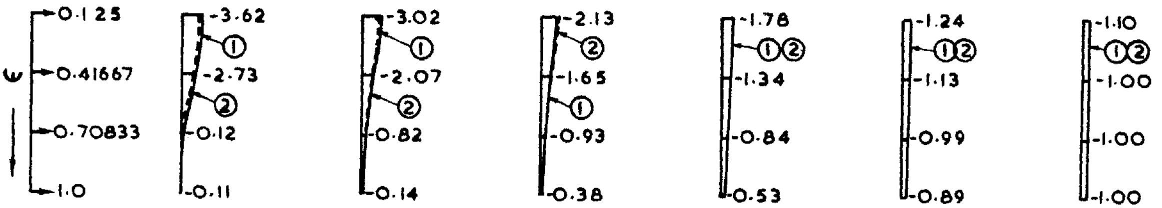

FIG. 8. $c/a = 0{\cdot}5625$.

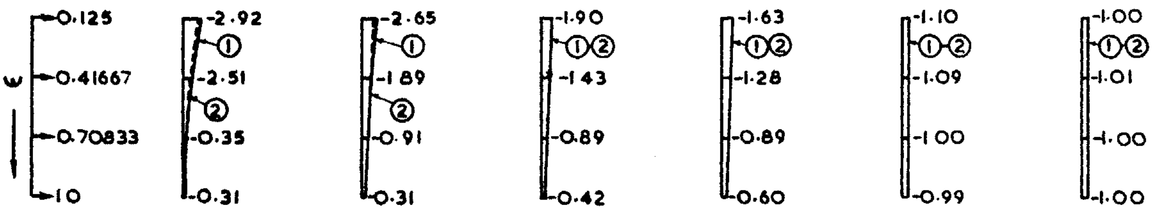

FIG. 9. $c/a = 0{\cdot}625$.

Note: (1) Finite hollow cylinder solution; (2) Semi-infinite hollow cylinder solution. Numerical values are stresses as per (1).

Numerical work and discussion:

Detailed numerical work has been done with an Elliot 803 digital computer using 16 terms in each of the series. In order to check the convergence of the series, results of 12 terms' solution have been compared with those of 16 terms. It was observed that the deviation was of the order of $6\frac{1}{4}$ per cent which was assumed quite small. The properties of the finite hollow cylinder considered were: $a/b = 8$, $l/a = 4$ and $\mu = 0{\cdot}15$. Various load-bearing areas are considered—$c/a = 0{\cdot}34375$, $0{\cdot}5$, $0{\cdot}5625$ and $0{\cdot}625$.

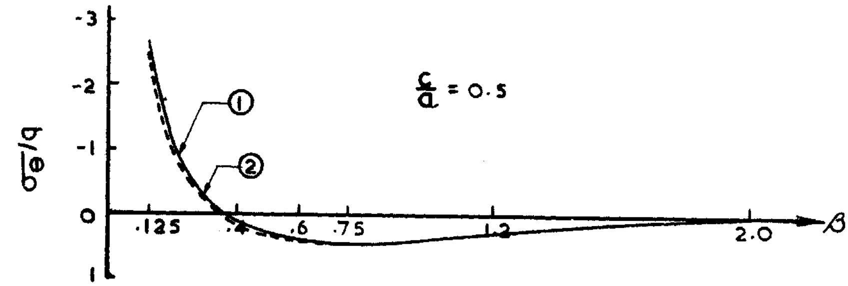

FIG. 10. Non-dimensional hoop stress on inner rim, $\epsilon = 0{\cdot}125$.

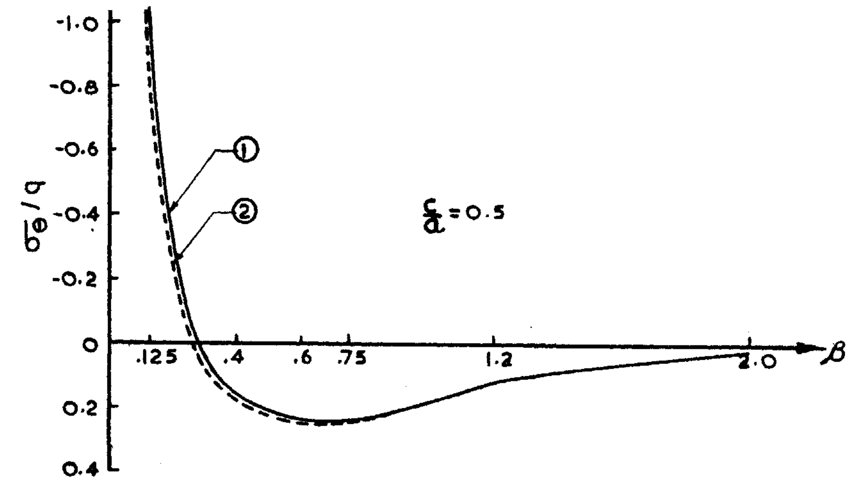

FIG. 11. Non-dimensional hoop stress on layer, $\epsilon = 0{\cdot}41667$.

Final stresses are calculated for the problem shown in Fig. 1(*a*)—compressive normal end loads—by adding a uniform axial compression to the stress distribution obtained for the problem shown in Fig. 1(*b*).

It is known that long cylinders with end loads can be treated as two separate semi-infinite hollow cylinders. It would be of interest to compare the present finite hollow cylinder solution with the solution for a semi-infinite hollow cylinder with the same end load. The numerical results for a semi-infinite hollow cylinder with the same elastic properties and subjected to the same normal load on the end are available in the literature on the subject (Douglas and Trahair 1960; Yogananda 1965). A comparison of the results

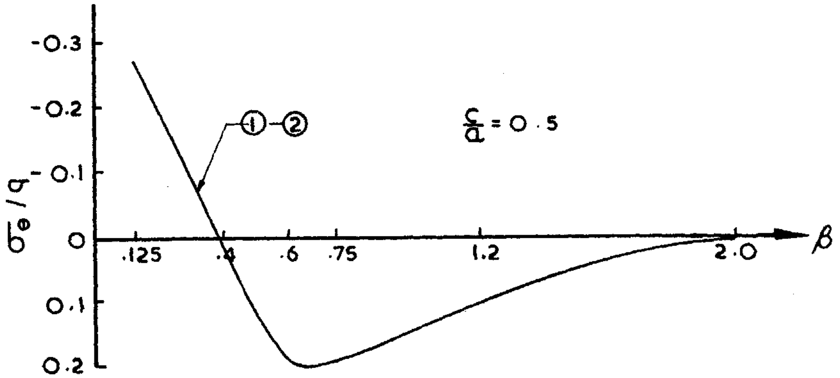

FIG. 12. Non-dimensional hoop stress on layer, $\epsilon = 0{\cdot}70833$.

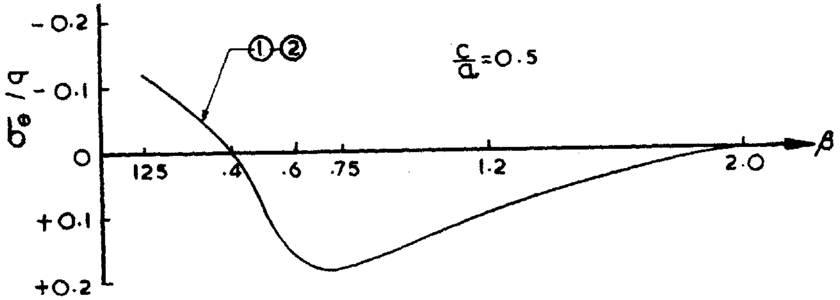

FIG. 13. Non-dimensional hoop stress on outer rim, $\epsilon = 1{\cdot}0$.

Note: (1) Finite hollow cylinder solution; (2) Semi-infinite hollow cylinder solution.

obtained in the present analysis with those for a semi-infinite hollow cylinder (Figs. 2 to 17) leads us to the following conclusions:

(1) St. Vennat's principle is valid and the zone extends to an axial distance of about one outer diameter of the hollow cylinder.

(2) Hollow cylinders longer than four times the outer radius can be analysed as two separate semi-infinite hollow cylinders.

(3) The maximum deviation of the results of finite hollow cylinder solution from the semi-infinite hollow cylinder solution is about $8\frac{1}{2}$ per cent.

(4) The agreement of the two results is better on the outer curved rim than on the inner curved rim.

Non-dimensional hoop stress diagrams

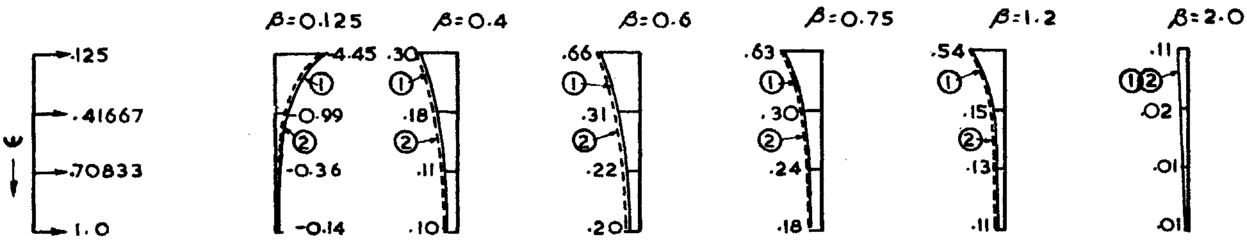

FIG. 14. $c/a = 0{\cdot}34375$.

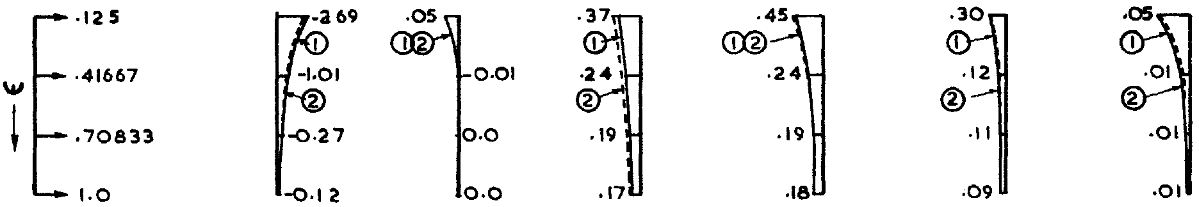

FIG. 15. $c/a = 0{\cdot}5$.

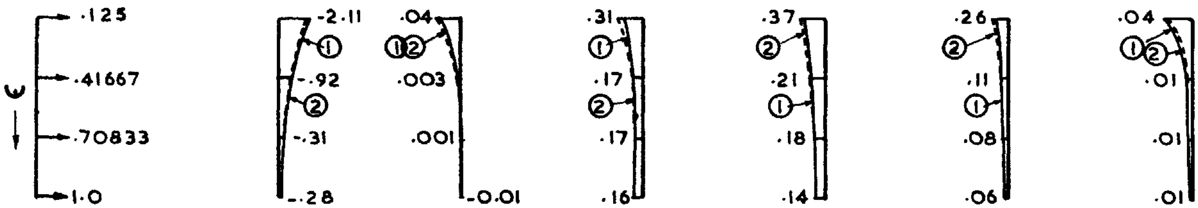

FIG. 16. $c/a = 0{\cdot}5625$.

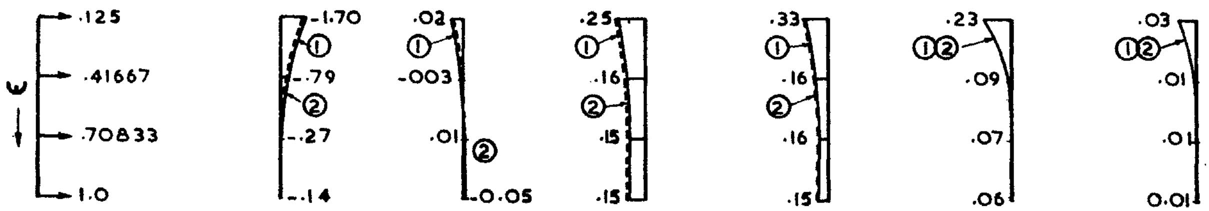

FIG. 17. $c/a = 0{\cdot}625$.

Note: (1) Finite hollow cylinder solution; (2) Semi-infinite hollow cylinder solution. Numerical values are stresses as per (1)

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