SECOND ORDER WAVES IN NON-LINEAR MAGNETO-ELASTICITY

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The differential equation governing the strength of an acceleration wave in magneto-elasticity is found, which shows that the strength does not remain constant at the front and that the wave has a distinct possibility of terminating into a shock.

1. INTRODUCTION

Anisotropic wave propagation has been studied much during recent years (Bazer and Fleishman 1960; Lighthill 1960; Ludwig 1961; Duff 1963). These studies, however, have been limited to linearized problems. In a number of papers recently communicated by us (Nariboli 1966; Nariboli and Ranga Rao; Nariboli and Juneja *communicated*) we developed a technique based on the theory of singular surfaces (Thomas 1957) and on the ray theory (Courant and Hilbert 1962). This enables us to discuss anisotropic wave propagation in a straightforward manner. After the present paper was completed we came across the work by E. Varley (1965) in which essentially the same ideas have been exploited. We present here an application of the method for discussing the acceleration wave fronts in non-linear elasticity in the presence of a magnetic field.

In section 2 we present basic ideas of the ray theory relevant to our problem. In the next section we obtain the wave speeds and the modes of propagation. We also give there certain additional results for later use. In the fourth section we derive the ordinary differential equation governing the strength of the wave. We give its complete integral and establish the relationship of our results with the results already known.

2. THE THEORY OF SINGULAR SURFACES AND THE RAY THEORY

Let x_i be an orthogonal Cartesian system of reference and t be the time. Consider a surface $\Sigma(t)$ described by x_i or by a Gaussian system of surface

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coordinates $u^\alpha(\alpha = 1, 2)$ as represented by

$$f(x_1, x_2, x_3, t) = 0, \quad x_i = x_i(u^\alpha, t). \quad \ldots \quad (2.1)$$

Latin suffixes after a comma will denote partial differentiation with respect to x_i and Greek suffixes after a comma will denote covariant differentiation with respect to u^α. Let $(g_{\alpha\beta})$ and $(b_{\alpha\beta})$ be the first and the second fundamental forms of $\Sigma(t)$ and G, the velocity of $\Sigma(t)$ normal to itself. Let (x_i, t) and $(x_i+\Delta x_i, t+\Delta t)$ be two consecutive points in space-time. Then we can write the increment in a function $F(x_i, t)$ quite generally

$$\Delta F = \frac{\partial F}{\partial t}\Delta t + F_{,j}\Delta x_j. \quad \ldots \quad (2.2)$$

If the points lie on a normal trajectory and on the successive positions of $\Sigma(t)$, then we take $\Delta x_i = \delta x_i = Gn_i\Delta t$ and get

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + F_{,j}Gn_j, \quad \ldots \quad (2.3)$$

where n_i is a unit normal to $\Sigma(t)$.

If the points are again on successive positions of $\Sigma(t)$ but in some other direction (identified with ray-direction, later defined) the velocity along which is V_i, we then have $\Delta x_i = dx_i = V_i\Delta t$ and so we get

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + F_{,j}V_j. \quad \ldots \quad (2.4)$$

Since the normal component of V_i is Gn_i, we have

$$\frac{dF}{dt} = \frac{\delta F}{\delta t} + F_{,j}(V_j - Gn_j) \quad \ldots \quad (2.5)$$

$$= \frac{\delta F}{\delta t} + V^\alpha F_{,\alpha},$$

where

$$V_\alpha = V_i x_{i,\alpha}, \quad V^\alpha = g^{\alpha\beta}V_\beta.$$

This important relation connects the time derivative along the rays with that along the normal trajectories. By Hadamard's lemma, a jump in the tangential derivative of a quantity is equal to the tangential derivative of the jump. Hence the above relation continues to hold when F happens to be a jump in any quantity.

Since $f(x_i, t) = 0$ continues to remain a wave front, its δ time derivative must vanish. So we have

$$\frac{\partial f}{\partial t} + f_{,i}Gn_i = 0. \quad \ldots \quad (2.6)$$

Assuming that the partial derivative of f with respect to time does not identically vanish, which must be true for a propagating surface $\Sigma(t)$, we can write $f(x_i, t) = 0$ as $W(x_i) - t = 0$. Let now

$$p_i = f_{,i} = W_{,i}, \quad p_i = pn_i. \quad \ldots \quad (2.7)$$

From (2.6) and (2.7) we now obtain

$$H \equiv Gp-1 = 0. \qquad (2.8)$$

Such a first order differential equation is solved by Charpit's method (more commonly called the ray method) (Courant and Hilbert 1962) by solving the equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad (2.9)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}. \qquad (2.10)$$

The system (2.9) and (2.10) constitutes a set of ordinary differential equations for x_i, p_i regarded as *independent.* Their solution subject to the initial conditions will yield the solution of the eqn. (2.8). We take the variable t in (2.9) and (2.10), which may be any parameter, as time. The curves described by (2.9) are called rays. Assuming that G depends on n_i only and that it is independent of x_i, we see that p_i (and hence n_i also) is constant along the rays. We further obtain now the ray velocity as

$$V_i = \frac{dx_i}{dt}$$

$$= Gn_i + (\delta_{ij} - n_i n_j)\frac{\partial G}{\partial n_j}, \qquad (2.11a)$$

$$V_\alpha = x_{i,\alpha}\frac{\partial G}{\partial n_i}. \qquad (2.11b)$$

We remark that since n_i is orthogonal to $(x_{i,\alpha})$ the δ-time derivative and differentiation with respect to u^α commute. Using this we get

$$\frac{\delta x_i}{\delta t} = Gn_i, \qquad (2.12a)$$

$$\frac{\delta g_{\alpha\beta}}{\delta t} = -2Gb_{\alpha\beta}, \qquad (2.12b)$$

$$\frac{\delta g^{\alpha\beta}}{\delta t} = 2Gb^{\alpha\beta}. \qquad (2.12c)$$

Finally, we note two more formulae, one of Weingarten's and the other given by Thomas (1957).

$$b_\alpha{}^\beta = -g^{\beta\gamma} n_{i,\alpha} x_{i,\gamma}, \qquad (2.13a)$$

$$\frac{\delta n_i}{\delta t} = -g^{\alpha\beta} G_{,\alpha} x_{i,\beta}. \qquad (2.13b)$$

Using these we get

$$\frac{\delta b_\alpha{}^\beta}{\delta t} = g^{\beta\gamma} G_{,\alpha\gamma} + Gb_\alpha{}^\gamma b_\gamma{}^\beta. \qquad (2.14)$$

We also have, from (2.11b),

$$G,\alpha = -\frac{\partial G}{\partial n_j} b_\alpha{}^\beta x_{j,\,\beta} \qquad (2.15)$$

$$= -V_\beta b_\alpha{}^\beta.$$

Using (2.5) and the above results we obtain

$$\frac{db_\alpha{}^\beta}{dt} = (G b_\alpha{}^\gamma b_\gamma{}^\beta - g^{\beta\gamma} V_{\delta,\,\gamma} b_\alpha{}^\delta) + (V^\gamma b_{\alpha,\,\gamma}^{\ \beta} - g^{\beta\gamma} V_\delta b_{\alpha,\,\gamma}^{\ \delta}). \qquad (2.16)$$

The Mainardi-Coddazi formulae given as

$$b_{\alpha\beta,\,\gamma} = b_{\alpha\gamma,\,\beta} \qquad (2.17)$$

enable us to prove that the latter group in (2.16) vanishes. After a straightforward evaluation we get

$$\frac{d(\log \bar{b})}{dt} = G b_\alpha{}^\alpha - V^\alpha{}_{,\,\alpha}. \qquad (2.18)$$

Here b is the determinant of $(b_\alpha{}^\beta)$ and gives the Gaussian curvature of $\Sigma(t)$.

3. Modes and Wave-speeds

We consider the acceleration waves in a most general non-linear elastic medium in the presence of a magnetic field. Let ρ be the density, u_i the displacement vector, e_{ij} and t_{ij} the strain and the stress tensors, H_i the magnetic field vector and μ_0 the magnetic permeability. We take the conductivity to be infinite and, without going into thermodynamical considerations, assume that the following system forms a complete system:

$$\frac{\partial \rho}{\partial t} + \rho_{,\,i} v_i + \rho v_{i,\,i} = 0, \qquad (3.1)$$

$$\frac{\partial H_i}{\partial t} + v_j H_{i,\,j} - H_j v_{i,\,j} + H_i v_{j,\,j} = 0, \qquad (3.2)$$

$$\rho \frac{dv_i}{dt} = t_{ij,\,j} + \mu_0 (H_j H_{i,\,j} - H_j H_{j,\,i}), \qquad (3.3)$$

where v_i is the velocity vector given by $v_i = (\partial u_i/\partial t) + v_j u_{i,\,j}$.

The first is the continuity equation, the second is the equation of magnetic induction and the last one is the equation of motion. We assume the medium to be isotropic with the constitutive law

$$t_{ij} = a\delta_{ij} + b e_{ij} + c e_{ik} e_{kj}, \qquad (3.4a)$$

with

$$2e_{ij} = u_{i,\,j} + u_{j,\,i} - u_{k,\,i} u_{k,\,j}. \qquad (3.4b)$$

We assume that a, b, c are arbitrary functions of the invariants (I), (II), (III) of e_{ij} only.

We now take it that across a moving singular surface $\Sigma(t)$, ρ, $u_{i,j}$ (H_i) are continuous and $(u_{i,jk})$, $(\rho_{,t})$, $(H_{i,j})$ discontinuous. We use square brackets to denote discontinuities. Also we always take it that the jump is the difference between a quantity that is just behind and one that is just ahead. Let us denote

$$[\rho_{,i}]n_i = \zeta,\ [u_{i,jk}]n_j n_k = \eta_i,\ [H_{i,j}]n_j = \xi_i. \quad \ldots \quad \ldots \quad (3.5)$$

We assume that the medium ahead is at rest and unstrained with constant values of $\rho = \rho_0$ and $H_i = H_{0i} = l_i H_0$, which defines l_i. We note that the presence of the magnetic field contributes to the stress and hence the above assumptions are consistent and analogous to those in hydrodynamics. Using the compatibility conditions we now get

$$\zeta + \rho_0 \eta_N = 0, \quad \ldots \quad (3.6)$$

$$\xi_i = H_0(l_N \eta_i - l_i \eta_N), \quad \ldots \quad (3.7)$$

$$\rho_0 G^2 \eta_i = \alpha_1 \eta_N n_i + \tfrac{1}{2}\beta_0(\eta_i + \eta_N n_i) + \mu_0 H_0^2 \{ l_N(l_N \eta_i - l_i \eta_N) - l_j n_i (l_N \eta_j - l_j \eta_N), \quad (3.8)$$

where (3.7) is used in (3.8) for (ξ_i) and

$$\alpha_1 = \left(\frac{\partial a}{\partial I}\right)_0, \quad \beta_0 = (b)_0,$$

and the suffix N denotes the normal component.

Using (3.8) we get

$$G^2 \eta_i = a_{ik} \eta_k \quad \ldots \quad (3.9a)$$

with

$$a_{ik} = (a_0^2 - c_0^2) n_i n_k + c_0^2 \delta_{ik} + b_0^2 \left\{ n_i n_k + l_N^2 \delta_{ik} - l_N(l_0 n_k + l_k n_i) \right\}, \quad \ldots \quad (3.9b)$$

where

$$\rho_0 a_0^2 = \alpha_1 + \beta_0, \quad \rho_0 c_0^2 = \tfrac{1}{2}\beta_0, \quad \mu_0 H_0^2 = \rho_0 b_0^2. \quad \ldots \quad (3.9c)$$

From (3.9a) we note that η_i is an eigen-vector of the symmetric matrix a_{ik} with eigen-value G^2. If L_i is a normalized eigen-vector of a_{ik}, we must have

$$\eta_i = L_i \psi. \quad \ldots \quad (3.10)$$

Since η_i is a jump, ψ may be called the strength of the discontinuity.

For each root G^2 of the cubic

$$|G^2 \delta_{ik} - a_{ik}| = 0 \quad \ldots \quad (3.11)$$

which must hold since $\eta_i \neq 0$, for $\Sigma(t)$ to be singular, we obtain one mode of propagation with corresponding eigen-vector L_i. To obtain the modes we choose l_i as (0, 0, 1) without any loss of generality. Let θ be the angle that n_i makes with the x_3-axis. Further let us denote

$$\Delta = P^2 + Q^2, \quad P = (a_0^2 - c_0^2)(1 - 2n_3^2) + b_0^2,$$

$$Q = 2n_3\sqrt{1 - n_3^2}\,(a_0^2 - c_0^2), \quad \cot \chi = \frac{\sqrt{\Delta} - P}{Q}. \quad \ldots \quad (3.12)$$

We can now write the three modes as

$$\overset{(1)}{M} : 2G^2 = a_0^2+b_0^2+c_0^2+\sqrt{\Delta},\ L_i = (\sin\chi\cos\phi,\ \sin\chi\sin\phi,\ \cos\chi), \qquad (3.13a)$$

$$\overset{(2)}{M} : 2G^2 = a_0^2+b_0^2+c_0^2-\sqrt{\Delta},\ L_i = (\cos\chi\cos\phi,\ \cos\chi\sin\phi,\ -\sin\chi), \qquad (3.13b)$$

$$\overset{(2)}{M} : G^2 = c_0^2+b_0^2n_3^2,\ L_i = (-\sin\phi,\ \cos\phi,\ 0). \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.13c)$$

Here ϕ is the angular coordinate in the cylindrical system $(\sqrt{x_1{}^2+x_2{}^2}, \theta, \phi)$. We also take t_i and s_i as unit tangents to θ and ϕ curves on $\Sigma(t)$, which are identified with u^1, u^2 respectively. Clearly $\overset{(3)}{L_i}$ is s_i. The mode remains a shear mode giving rotation about x_3-axis. As $b_0 \to 0$, the first two speeds reduce to dilatational and shear-wave speeds and the eigen-vectors to n_i and t_i respectively. However, the modes have no such interpretation for $b_0 \neq 0$. Each of the first two modes is accompanied by both dilatation and rotation. However, the modes are orthogonal and hence they can be studied independently of each other (Courant and Hilbert 1962).

Whatever the mode, some common properties continue to hold. We note them here for later reference. Since L_i is a unit vector we can write from (3.9a)

$$G^2 = a_{ik}L_iL_k. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.14)$$

Differentiating this with respect to u^β and assuming that b^β_α is not identically zero, we get

$$GV_\beta = (a_0^2-c_0^2)L_NL_\beta+b_0^2\{l_Nl_\beta+L_NL_\beta-l_iL_i(l_NL_\beta+L_Nl_\beta)\}. \quad \ldots (3.15)$$

Differentiate this with respect to u^α and then multiply by $g^{\alpha\beta}$. We get

$$V^\alpha G_{,\alpha}+GV^\alpha{}_{,\alpha} = A+B-C, \quad \ldots \quad \ldots \quad \ldots (3.16a)$$

where

$$A = (a_0^2-c_0^2)(L^\alpha L_{i,\alpha}n_i+g^{\alpha\beta}L_NL_{i,\alpha}x_{i,\beta})+b_0^2\{L_\alpha L_{i,\alpha}n_i+g^{\alpha\beta}L_NL_{i,\alpha}x_{i,\beta}$$
$$-l_iL_{i,\alpha}(l_NL^\alpha+L_Nl^\alpha)-l_iL_i(l_NL_{j,\alpha}x_{j,\beta}g^{\alpha\beta}+l^\alpha L_{j,\alpha}n_j)\}, \quad \ldots (3.16b)$$

$$B = b_\alpha{}^\alpha\{(a_0^2-c_0^2)L_N^2+b_0^2(l_N^2+L_N^2-2l_iL_il_NL_N)\} = (G^2-c_0^2)b_\alpha{}^\alpha, \quad \ldots (3.16c)$$

$$C = b^{\alpha\beta}\{(a_0^2c_0^2)L_\alpha L_\beta+b_0^2(l_\alpha l_\beta+L_\alpha L_\beta)-b_0^2l_iL_i(L_\beta l_\alpha+l_\beta L_\alpha)\}. \quad \ldots (3.16d)$$

Using (2.18) we can write (3.16a) as

$$G\frac{d\log\bar{b}}{dt} = G^2b_\alpha^\alpha-(A+B-C-V^\alpha G_{,\alpha}). \quad \ldots \quad \ldots (3.17)$$

4. Growth of the Wave

In order to obtain the growth equation we need third order compatibility conditions. These can be obtained on lines as those used by others (Thomas 1957) and are just noted.

Let

$$[Z] = 0,\ [Z,_i] = 0,\ [Z,_{ij}]\, n_i n_j = k_2,\ [Z,_{ijk}]\, n_i n_j n_k = k_3. \quad \ldots \quad (4.1)$$

Then we have

$$[Z,_{ijk}] = k_3 n_i n_j n_k + g^{\alpha\beta} k_{2,\alpha}(x_{i,\beta} n_j n_k + x_{j,\beta} n_k n_i + x_{k,\beta} n_i n_j)$$
$$- k_2 b^{\alpha\beta}(n_i x_{j,\alpha} x_{k,\beta} + n_j x_{i,\alpha} x_{i,\beta} + n_k x_{i,\alpha} x_{j,\beta}), \quad \ldots \quad (4.2)$$

$$\left[\frac{\partial^3 Z}{\partial x_i \partial x_j \partial t}\right] = \left(-Gk_3 + \frac{\delta k_2}{\delta t}\right) n_i n_j - G g^{\alpha\beta} k_{2,\alpha}(n_i x_{j,\beta} + n_j x_{i,\beta})$$
$$+ k_2 \left(\frac{\delta}{\delta t}(n_i n_j) + b^{\alpha\beta} x_{i,\alpha} x_{j,\beta}\right), \quad \ldots \quad (4.3)$$

$$\left[\frac{\partial^3 Z}{\partial x_i \partial t^2}\right] = \left(G^2 k_3 - 2G\frac{\delta k_2}{\delta t} - k_2 \frac{\delta G}{\delta t}\right) n_i + \left(G^2 g^{\alpha\beta} k_{2,\alpha} x_{i,\beta} - 2Gk_2 \frac{\delta n_i}{\delta t}\right). \quad \ldots \quad (4.4)$$

Differentiating now (3.2) and (3.3) with respect to x_p and multiplying by n_p, we get

$$\rho_0 \left\{G^2 \eta_i^* - 2G\frac{\delta \eta_i}{\delta t} - \eta_i \frac{\delta G}{\delta t} + G^2 \eta_N \eta_i\right\}$$
$$= (\alpha_1 + \tfrac{1}{2}\beta_0)\eta_N^* n_i + \tfrac{1}{2}\beta_0 \eta_i^* + \mu_0(H_{0N}\xi_i^* - H_{0j}\xi_j^* n_i) + f_i^L + f_i^Q, \quad \ldots \quad \ldots \quad (4.5a)$$

$$\xi_i^* - H_{0N}\eta_i^* + H_{0i}\eta_N^* = g_i^L + g_i^Q, \quad \ldots \quad \ldots \quad \ldots \quad (4.6a)$$

where

$$f_i^L = \rho_0 \{ (a_0^2 - c_0^2) g^{\alpha\beta}(\eta_{p,\alpha} x_{p,\beta} n_i + \eta_{p,\alpha} n_p x_{i,\beta}) - (a_0^2 - c_0^2) b^{\alpha\beta} \eta_\beta x_{i,\alpha}$$
$$- c_0^2 b_\alpha^\alpha \eta_i \} + \mu_0 (H_0^\alpha \xi_{i,\alpha} - g^{\alpha\beta} H_{0j} \xi_{j,\alpha} x_{i,\beta}), \quad \ldots \quad \ldots \quad (4.5b)$$

$$f_i^Q = \rho_0 \left(-a_0^2 \eta^2 n_i + \frac{a_0^2}{\alpha_1 + \beta_0} \left\{ \alpha_{11} \eta_N^2 n_i + \tfrac{1}{2}\alpha_2(\eta_N^2 - \eta^2) n_i + \beta \eta_N(\eta_i + \eta_N n_i) \right.\right.$$
$$\left.\left. + \tfrac{1}{2}\gamma_0 \left(2\eta_N \eta_i + \overline{\eta^2 + \eta_N^2} n_i\right) \right\} \right) - \mu_0 \xi^2 n_i, \quad \ldots \quad (4.5c)$$

$$g_i^L = H_0^\alpha \eta_{i,\alpha} - g^{\alpha\beta} H_{0i} \eta_{j,\alpha} x_{j,\beta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6b)$$

$$g_i^Q = 2\eta_N (H_{0j}\eta_i - H_{0i}\eta_N), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6c)$$

$$\alpha_{11} = \left.\frac{\partial^2 a}{\partial I^2}\right)_0,\ \alpha_2 = \left.\frac{\partial a}{\partial II}\right)_0,\ \beta_1 = \left.\frac{\partial b}{\partial I}\right)_0,\ \gamma_0 = C)_0,$$

$$[u_{i,jkl}] n_j n_k n_l = \eta_i^*,\ [H_{i,jk}] n_j n_k = \xi_i^*. \quad \ldots \quad \ldots \quad (4.7)$$

If we omit the magnetic field terms, this agrees with our earlier results (Juneja and Nariboli, *in press*). Here the superscripts L, Q are used to denote linear and quadratic terms. Using now (4.6) in (4.5) we obtain

$$\rho_0(G^2 \eta_i^* - a_{ik}\eta_k^*) = 2\rho_0 G \frac{\delta \eta_i}{\delta t} + h_i^L + h_i^Q, \quad \ldots \quad \ldots \quad (4.8a)$$

where

$$h_i^L = f^L + \mu_0 H_{0N} g_i^L - \mu_0 H_{0j} n_i g_j^L + \eta_i \frac{\delta G}{\delta t}, \quad \ldots \quad \ldots \quad (4.8b)$$

$$h_i^Q = f_i^Q - 2G^2 \eta_N \eta_i + \mu_0 H_{0N} g_i^Q - \mu_0 H_{0j} n_i g_j^Q. \quad \ldots \quad \ldots \quad (4.8c)$$

Now, since L_i is an eigen-vector of the symmetric matrix a_{ik} with principal values G^2, multiplication of (4.8a) by L_i reduces the left members to zero. So the product of the right-hand members with L_i equated to zero is the growth equation. By somewhat lengthy but straightforward manipulations we obtain

$$h_i^L L_i = \rho_0 \psi (A + B - C - G^2 b_\alpha^\alpha) + 2GV^\alpha \psi_{\ \alpha}. \quad \ldots \quad \ldots \quad (4.9)$$

So the linear terms reduce to

$$\rho_0 G \left(2 \frac{d\psi}{dt} + \psi (V^\alpha_{\ ,\alpha} - G b_\alpha^\alpha) \right)$$

$$= \rho_0 G \left(2 \frac{d\psi}{dt} - \psi \frac{d \log \bar{b}}{dt} \right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.10)$$

where (2.18) has been used.

Thus the growth equation becomes

$$2 \frac{d\psi}{dt} - \frac{d \log \bar{b}}{dt} \psi + D\psi^2 = 0, \quad \ldots \quad \ldots \quad \ldots \quad (4.11a)$$

where

$$h_i^Q L_i = \rho_0 G D \psi^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.11b)$$

We note that all functions of n_i are to be treated as constant in this time-differentiation along the ray. The eqn. (4.11a) can be integrated to give

$$\frac{\bar{b}^{\frac{1}{2}}}{\psi} - \frac{\bar{b}_0^{\frac{1}{2}}}{\psi_0} = \tfrac{1}{2} D \int_0^t \bar{b}^{\frac{1}{2}}\, dt, \quad \ldots \quad \ldots \quad \ldots \quad (4.12)$$

with

$$\psi = \psi_0,\ \bar{b} = \bar{b}_0 \text{ for } t = 0.$$

The power of this result is self-evident. To bring out the features more clearly, we take the initial manifold to be a surface of revolution given by

$$x_i^0 = (g(\theta) \cos \phi,\ g(\theta) \sin \phi,\ f(\theta)), \quad \ldots \quad \ldots \quad \ldots \quad (4.13a)$$

$$\dot{f} = -\dot{g} \tan \theta, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.13b)$$

where dots denote differentiation with respect to θ.

The latter condition ensures that the normal to $\Sigma(t)$ makes an angle θ with the x_3-axis. We also have now

$$V_i = Gn_i + \dot{G} t_i, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.14)$$

$$x_i = x_i^0 + V_i t. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.15)$$

We thus obtain the curvatures K_1 and K_2 of θ and ϕ curves as

$$K_1 = \frac{1}{\dot{g} \sec \theta + (G+\ddot{G})t}, \quad K_2 = \frac{1}{g \operatorname{cosec} \theta + (G+\dot{G} \cot \theta)t}. \quad \text{.. (4.16)}$$

As is known (Lighthill 1960; Ludwig 1961), the curvatures of the reciprocal wave speed locus $\mathcal{R}$ play an important role. This is the surface of revolution obtained by revolving the curve (G^{-1}, θ). The curvatures of this surface are

$$K_1^R = (G+\ddot{G})\frac{G^3}{V^3}, \quad K_2^R = (G+\dot{G} \cot \theta)\frac{G}{V}. \quad \text{.. .. (4.17)}$$

The result (4.12) now reduces to

$$\frac{\sqrt{K_1 K_2}}{\Psi} - \frac{\sqrt{K_{01}K_{02}}}{\Psi_0} = \frac{D}{\{(G+\ddot{G})(\dot{G} \cot \theta + G)\}^{\frac{1}{2}}}$$

$$\times \log\left\{\frac{\sqrt{(G+\ddot{G})t+\dot{g} \sec \theta}+\sqrt{(\dot{G} \cot \theta + G)t + g \operatorname{cosec} \theta}}{\sqrt{\dot{g} \sec \theta}+\sqrt{g \operatorname{cosec} \theta}}\right\}. \quad (4.18)$$

Setting $D = 0$ gives the linearized problem. From (4.16)–(4.17) we then obtain: at points of $\Sigma(t)$ corresponding to those of $\mathcal{R}$ where the curvatures of the latter are non-zero, the strength varies as t^{-1}; at points of $\Sigma(t)$ corresponding to those of $\mathcal{R}$ where one curvature vanishes, the asymptotic strength is as $t^{-\frac{1}{2}}$, i.e. cylindrical. Finally, at points of $\Sigma(t)$ corresponding to those where both the curvatures of $\mathcal{R}$ vanish, the propagation is planar, i.e. the strength is constant. These asymptotic features correspond to the results obtained earlier (Lighthill 1960; Ludwig 1961; Duff 1963).

However, for the non-linear problem such an asymptotic description may not always be true. There may exist a finite time when the strength becomes infinite.

We thus obtain a distinct possibility of the indefinite growth of the wave-strength for each mode. Depending on the velocities, the front $\Sigma(t)$ can also develop caustics when one of the curvatures becomes infinite. For a vanishing magnetic field the result clearly reduces to our earlier result (Juneja and Nariboli *in press*).

Conclusion

The differential equation governing the strength of an acceleration wave is an ordinary differential equation along the rays. Even though the initial strength may be the same, subsequently the strength on the front varies. There exists a distinct possibility of the wave terminating into a shock depending on the elasticities and the magnetic field.

References

Bazer, J., and Fleishman, O. (1960). Propagation of weak hydromagnetic discontinuities. Research Report No. M.H. 10, N.Y.V.

Courant, R., and Hilbert, D. (1962). Methods of Mathematical Physics, Vol. II. Interscience Publishers, New York, 577–630.

Duff, G. F. D. (1963). On wave fronts and boundary waves. M.R.C. Technical Report, **434**. Madison, Wisconsin.

Juneja, B. L., and Nariboli, G. A. (*in press*). Growth of acceleration waves in an unstrained non-linear isotropic elastic medium. *Int. J. monlinear Mech.*

Lighthill, M. J. (1960). Studies on magneto-hydrodynamic waves and other anisotropic wave motions. *Phil. Trans. R. Soc.*, A **252**, 397–430.

Ludwig, D. (1961). The singularities of the Riemann function. A.E.C. Research and Development Report (1961). *N.Y.O.*-9351.

Nariboli, G. A. (1966). Wave propagation in completely anisotropic media. *J. math. Analysis Applic.*, **16**, 108–122.

Nariboli. G. A., and Juneja, B. L. (*communicated*). Wave propagation in an initially stressed hypo-elastic media.

Nariboli, G. A., and Ranga Rao, M. P. (*communicated*). Anisotropic wave propagation in magneto-gas-dynamics.

Thomas, T. Y. (1957). Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics. *J. Math. Mech.*, **6**, 311–322.

Varley, E. (1965). Acceleration fronts in visco-elastic materials. *Archs. ratívn. Mech. Analysis*, **19**, 215–225.