GREEN'S THEOREM IN A NON-COMPACT AFFINE MANIFOLD

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In the present note it is shown how Green's theorem, which states that for any arbitrary vector field λ^i in a compact orientable manifold M, $\int_M \lambda^i_{,i}\, dv = 0$, can be proved in an orientable affine manifold which is not necessarily compact but satisfies the property (P), i.e. every open cover of M admits a finite subcollection whose closures cover M. To prove the theorem, affine connection has been suitably restricted.

Green's theorem, which states that for any arbitrary vector field λ^i in a compact orientable manifold M, $\int_M \lambda^i_{,i}\, dv = 0$, has been used extensively in establishing various results in such manifolds (Bochner 1937; Yano and Bochner 1953; Goldberg 1956). In the present note Green's theorem is shown to hold good in an orientable affine manifold which is not necessarily compact but satisfies the property (P), i.e. every open cover of M admits a finite subcollection whose closures cover M.

An affine manifold of dimension n is characterized by the existence of n contravariant parallel vector fields $h^i_{(\alpha)}$ $(\alpha = 1, 2, \ldots, n)$ which are analytic functions of x^i coordinates ; as $h^i_{(\alpha)}$ satisfy the equation

$$\frac{\partial h^i_{(\alpha)}}{\partial x^k} + h^j_{(\alpha)} \Lambda^i_{jk} = 0,$$

it follows that the connection coefficients Λ^i_{jk} are also analytic functions of x^i (Thomas 1934).

Theorem: If M be an orientable manifold covered by rectangular coordinate neighbourhoods and satisfying the following two conditions:

I. It has the property (P).

II. To every neighbourhood A_P of point P there is associated a positive integer l_i (not necessarily distinct) and a scalar $f(l_i, x)(1 \leqslant l_i \leqslant n,\ i = 1, 2, \ldots, n)$ such that for any arbitrary vector field $\lambda^i(x)$ and affine connection Λ^i_{jk}, one has $A^i_{ik}\lambda^k = \frac{\partial}{\partial x^{l_i}}[f(l_i, x)\lambda^{l_i}]$, then

$$\int_M \lambda^i_{,i}\, dv = 0.$$

Proof: Let $\mathcal{A} = \{A_P : P \in M\}$ denote the given cover of M. Since A_P is a 'rectangle' neighbourhood of P we can obtain two open neighbourhoods U_P and V_P of P such that $\bar{U}_P \subset V_P \subset \bar{V}_P \subset A_P$, then $\mathcal{U} = \{U_P : P \in M\}$ is an open cover of M and as M satisfies the property (P) we can choose a subcollection $\{U_1, U_2 \ldots U_N\}$ such that $\{\bar{U}_a, a = 1, 2 \ldots, N\}$ cover M. Moreover, $\bar{U}_a \subset V_a$ for every $a = 1, 2, \ldots, N$; therefore we can cover M by a finite number of neighbourhoods $V_1, V_2, \ldots, V_N$ whose closures are contained in the rectangles $A_1, A_2, \ldots, A_N$ respectively.

Since M is a differentiable manifold we can find a non-negative function ϕ_a defined over a neighbourhood $W_a \subset V_a \subset \bar{W}_a \subset A_a$ such that $\phi_a > 0$ in W_a and is zero outside $\bar{W}_a$. We denote the carrier of ϕ_a by $\bar{\mathfrak{W}}_a$; clearly the collection $\{\bar{\mathfrak{W}}_a\}$ covers M. We further assume that the family $\Phi = \{\phi_a : a = 1 \ldots N\}$ forms a partition of unity, i.e. $\sum_{a=1}^{N} \phi_a(P) = 1$ for every $P \in M$ (Goldberg 1962; Auslander and Mackenzie 1963).

We now put

$$\lambda^i_{(a)} = \phi_a \lambda^i;$$

that

$$\sum_{a=1}^{N} \lambda^i_{(a)} = \lambda^i \sum_{a=1}^{N} \phi_a = \lambda^i.$$

Consequently

$$\lambda^i_{,i} = \sum_{a=1}^{N} \lambda^i_{(a),i}$$

and

$$\int \lambda^i_{,i}\, dv = \int \sum \lambda^i_{(a),i}\, dv = \sum_{a=1}^{N} \int \lambda^i_{(a),i}\, dv.$$

Taking the integral over the whole manifold, we have

$$\int_M \lambda^i_{,i}\, dv = \sum_{a=1}^{N} \int_M \lambda^i_{(a),i}\, dv.$$

Consider an A_q $(Q = 1, 2, \ldots, N)$; since A_q is a 'rectangle' $a^i_q < x^i < b^i_q$, A_q is a bounded set contained in a coordinate neighbourhood, A_q itself being a coordinate neighbourhood.

Now,

$$\int_{A_q} \lambda^i_{(q),\,i}\, dv = \int_{A_q} \left(\frac{\partial \lambda^i_{(q)}}{\partial x^i} + A^i_{ik}\lambda^k_{(q)}\right) dx^1 dx^2 \ldots dx^n$$

$$= \int_{A_q} \left[\frac{\partial \lambda^i_{(q)}}{\partial x^i} + \frac{\partial}{\partial x^{li}}\left\{f(li,\, x)\, \lambda^{li}_{(q)}\right\}\right] dx^1 dx^2 \ldots dx^n$$

$$= \int_{A_q} \frac{\partial \lambda^i_{(q)}}{\partial x^i}\, dx^1 dx^2 \ldots dx^n + \int_{A_q} \frac{\partial}{\partial x^{li}}\left\{ f(li,\, x)\, \lambda^{li}_{(q)}\right\} dx^1 dx^2 \ldots dx^n = 0,$$

(since $\lambda^i_{(q)}$ vanishes on the boundary of A_q). Again, λ_q^i vanishes over the open set $M \sim A_q$ due to the choice of ϕ_q; therefore integral $\lambda^i_{(q),\,i}$ is zero over $M \sim A_q$.

Hence

$$\int_M \lambda^i_{(q),\,i}\, dv = \int_{A_q} \lambda^i_{(q),\,i}\, dv + \int_{M \sim A_q} \lambda^i_{q,\,i}\, dv = 0+0 = 0.$$

This is true for any $q = 1, 2, \ldots, N$; substituting in I, we get the result

$$\int_M \lambda^i_{,\,i}\, dv = 0$$

which proves the theorem.

It may be remarked that Green's theorem for paracompact manifolds has also been established (Nirmala Prakash, *in press*), but since every such manifold can be metrized, the author felt interested in defining a manifold which need only have an affine structure and still Green's theorem would be satisfied on it.

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