THEORETICAL CONSIDERATION OF TEMPERATURE DISTRIBUTION IN AN ELLIPTIC TUBE

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Expressions are obtained for the temperature distributions in presence of viscous fluid (i) when flow is steady, (ii) when flow is unsteady and $\frac{\partial Q}{\partial t}$, the rate of heat produced per unit volume, is an oscillatory function of time, (iii) in the case when flow is unsteady $\frac{\partial Q}{\partial t} = 0$ and temperature distribution is an exponential function of time and (iv) for steady flow with temperature distribution varying as $T = Az+g(x, y)$, A being the constant temperature gradient.

INTRODUCTION

The temperature distributions in a circular tube have been discussed by several authors, notably by Graetz (1883, 1885), Nusselt (1910), Eagle and Ferguson (1930), Goldstein (1931) and Lighthill (1953). Many of these papers have been cited by Goldstein (1931, § 266). Eagle and Ferguson have obtained the expression for the temperature distribution when a circular brass tube is heated by an alternating current and viscous incompressible fluid is passing through it. They have also dealt with the experimental arrangements needed to obtain the expression for the temperature distribution as well as the experimental proof to evaluate k_0, the coefficient of heat transfer. Lighthill (1953) has considered theoretically the free convection in tubes. In his paper, methods have been formulated for predicting the flow and heat transfer due to free convection in heated vertical tubes, closed at the bottom and opening into a reservoir of cool fluid at the top. The theory is extended to apply to a tube closed at both ends, a part heated and a part cooled.

On the basis of the above papers for circular tube and Sexl (1930), I consider the temperature distributions in an elliptic tube whose axis is along the z-axis. The paper is divided into five sections. In § 1, a general energy equation is given for the viscous incompressible fluid. In § 2, a solution for temperature distribution is given in the presence of dissipation term. In §§ 3, 4 and 5, the expressions for the temperature distributions have been obtained when viscous incompressible fluid is flowing through the tube. However, the dissipation due to friction has been neglected in the last three sections on the assumption that the velocity of flow is very small. In all the four cases discussed in this paper, it has been assumed that μ and k_0, the coefficients of viscosity and thermal conductivity, are constant with respect to temperature. The four cases discussed are:

Case I. When the flow is steady and the dissipation due to friction has been taken into consideration.

Case II. When the flow is unsteady, $\frac{\partial Q}{\partial t} \neq 0$, the pipe is heated by an alternating current and the dissipation due to friction is neglected as a small quantity.

Case III. When the flow is unsteady, T is an exponentially decreasing function of time, dissipation due to friction is neglected in comparison to other terms of the energy equation.

Case IV. When the flow is steady, dissipation due to friction is neglected but T is a function of x, y and z such that $T = Az+g(x, y)$.

§ 1. The energy equation is

$$\rho C_v \frac{DT}{Dt} = \frac{\partial Q}{\partial t}+k_0\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\phi, \qquad \text{(1.1)}$$

where ϕ is the dissipation function, defined for the present purpose by

$$\phi = \mu\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right], \qquad \text{(1.2)}$$

following Pai (1956, p. 27).

The dissipation function is a form of the strain energy which should be independent of the coordinate system used. For general energy equation ϕ includes the mechanical equivalent of heat as given in Goldstein's book (1938) in eqn. (1.1),

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}. \qquad \text{(1.3)}$$

The symbols are:

Q = quantity of heat added,
k_0 = coefficient of thermal conductivity of the liquid,
C_v = specific heat at constant volume,
ρ = density of the liquid,
T = temperature at any point in the pipe.

When $w = f(x, y)$, we know (Ramsey: Hydrodynamics, 1954 edition, p. 385) that the expression of the velocity at any point (x, y) is

$$w = \frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right). \qquad \text{(1.4)}$$

where $P = -\dfrac{\partial p}{\partial z}$, is the constant pressure gradient along the axis of the pipe, μ the coefficient of viscosity and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ is the equation of the cross-section of the tube.

§ 2. *Case I.* The motion is steady. $\dfrac{\partial^2 T}{\partial z^2}$ is very small in comparison to other terms of the energy equation. Thus from the above equations, we have

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k_0}\left[\frac{P^2}{\mu^2}\,\frac{a^4b^4}{(a^2+b^2)^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\right], \qquad \text{(2.1)}$$

away from the inlet.
To solve (2.1), we put

$$T = u_1+B-D\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}\right), \qquad \text{(2.2)}$$

where

$$D = \frac{\mu}{k_0}\,\frac{P^2}{12\mu^2}\cdot\frac{a^4b^4}{(a^2+b^2)^2}, \qquad \text{(2.3)}$$

and u_1 satisfies

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2} = 0. \qquad \text{(2.4)}$$

Put

$$u_1 = A(x^4-6x^2y^2+y^4)+B(x^2-y^2)+C_1, \qquad \text{(2.5)}$$

where A, B and C_1 are certain constants.
Therefore

$$T = A(x^4-6x^2y^2+y^4)+B(x^2-y^2)+C-D\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}\right). \qquad \text{(2.6)}$$

The boundary condition is

$$T = T_0 \text{ on } \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

i.e.

$$T = T_0 \text{ on } y^2 = b^2\left(1-\frac{x^2}{a^2}\right).$$

Putting this in eqn. (2.6), we get

$$T_0 = x^4\left[A\left(1+\frac{6b^2}{a^2}+\frac{b^4}{a^4}\right)-D\left(\frac{2}{a^4}\right)\right]+x^2\left[-A\left(6b^2+\frac{2b^4}{a^2}\right)+B\left(1+\frac{b^2}{a^2}\right)+\frac{2D}{a^2}\right]$$
$$+(Ab^4-Bb^2-D+C). \qquad \text{(2.7)}$$

Equating to zero the coefficients of x^4 and x^2, we get

$$A = \frac{2D}{a^4+6a^2b^2+b^4},$$

$$B = -\frac{2D(a^2-b^2)}{a^4+6a^2b^2+b^4}.$$

The constant term reduces to

$$a^2b^2A-D+C.$$

Therefore

$$C = T_0+\frac{D(a^2+b^2)^2}{(a^4+6a^2b^2+b^4)}. \qquad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

Thus the expression for the temperature distribution T is known from eqn. (2.7).

§ 3. *Case II.* The temperature distribution is oscillatory. The dissipation due to friction is neglected, since w is considered to be a small quantity. Thus the approximate form of the energy equation (Pai 1956, p. 43) is

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_v}\frac{\partial Q}{\partial t}+k'\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right], \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where

$$k' = \frac{k_0}{\rho c_v} = \text{constant}.$$

If the rate of heat addition be periodic with period $\frac{2\pi}{\sigma}$, we assume that

$$\frac{1}{\rho c_v}\frac{\partial Q}{\partial t} = \alpha e^{i\sigma t}$$

and

$$T = f(x, y)e^{i\sigma t}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where α is real and in eqns. (3.12) and (3.13) the real parts are to be taken.

Substituting in eqn. (3.1) and putting $F = f+\frac{i\alpha}{\sigma}$, $h^2 = \frac{i\sigma}{k'}$, we see that F satisfies the equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}-h^2F = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

Let the boundary of the tube be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, and we substitute $x+iy = c\cosh(\xi+i\eta)$, $c^2 = a^2-b^2$.

In these coordinates, eqn. (3.3) transforms into

$$\frac{\partial^2 F}{\partial \xi^2}+\frac{\partial^2 F}{\partial \eta^2}-2k^2(\cosh 2\xi-\cos 2\eta)F = 0, \qquad \ldots \quad \ldots \quad (3.4)$$

where

$$4k^2 = h^2c^2 = \frac{ic^2\sigma}{k'}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

Let the solution of eqn. (3.4) be $F = \phi(\xi)\psi(\eta)$; then we see that $\phi(\xi)$ satisfies the modified Mathieu equation

$$\frac{d^2\phi}{d\xi^2} - (a+2k^2 \cosh 2\xi)\phi = 0, \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

and $\psi(\eta)$ satisfies the Mathieu equation

$$\frac{d^2\psi}{d\eta^2} + (a+2k^2 \cos 2\eta)\psi = 0. \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

But T is symmetrical about the axes of the ellipse and is periodic with period π; hence ψ is the periodic Mathieu function (McLachlan 1947, p. 321)

$$Ce_{2n}(\eta, -k^2)$$

of order $2n$. The function ϕ is then the modified Mathieu function $Ce_{2n}(\xi, -k^2)$.

Also we have (McLachlan 1947, pp. 21, 27)

$$\left.\begin{aligned} Ce_{2n}(\eta, -k^2) &= (-1)^n \sum_{r=1}^{\infty} (-1)^r A_{2r}^{(2n)} \cos 2r\eta \\ Ce_{2n}(\xi, -k^2) &= (-1)^n \sum_{r=1}^{\infty} (-1)^r A_{2r}^{(2n)} \cosh 2r\xi \end{aligned}\right\}, \quad \ldots \quad \ldots \quad (3.8)$$

where the coefficients $A_{2r}^{(2n)}$ are function of k^2.

The appropriate general expression for f is then

$$f = -\frac{i\alpha}{\sigma} + \sum_{n=0}^{\infty} C_{2n} \,.\, Ce_{2n}(\eta, -k^2)Ce_{2n}(\xi, -k^2). \quad \ldots \quad (3.9)$$

If $\xi = \xi_0$ be the boundary of the ellipse, we have the boundary condition:

$$\left.\begin{aligned} T &\to f_1 \cos \sigma t \\ f &\to f_1 \end{aligned}\right\} \quad \begin{aligned} &\text{at } \xi = \xi_0 \\ &\text{for all } \eta \end{aligned}$$

and $\alpha = 0$ at $t = 0$. Thus

$$f_1 + \frac{i\alpha}{\sigma} = \sum_{n=0}^{\infty} C_{2n} \cdot Ce_{2n}(\xi_0, -k^2)Ce_{2n}(\eta, -k^2). \quad \ldots \quad \ldots \quad (3.10)$$

Multiplying both sides of eqn. (3.10) by $Ce_{2n}(\eta, -k^2)$ and integrating with respect to η from 0 to 2π and using the orthogonality relations (McLachlan 1947, p. 321), we have

$$C_{2n} = \frac{2\pi(-1)^n A_0^{(2n)}}{Ce_{2n}(\xi_0, -k^2)I_{2n}} \left(\frac{i\alpha}{\sigma} + f_1\right), \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

where

$$I_{2n} = \int_0^{2\pi} Ce_{2n}^2(\eta, -k^2)\, d\eta. \qquad \ldots \quad (3.12)$$

Hence

$$T = \frac{\alpha}{\sigma} \sin \sigma t + R\left[e^{i\sigma t} \sum_{n=0}^{\infty} C_{2n} \cdot Ce_{2n}(\eta, -q) Ce_{2n}(\xi, -q), \right. \qquad \ldots \quad (3.13)$$

where R denotes the real part and q has been put for k^2.

Discussion: (i) The case when $|q|$ is small may be discussed, following McLachlan (1947, pp. 15, 46), as

$$Ce_0(\eta, -q) \simeq 1+\tfrac{1}{2}q \cos 2\eta$$

$$Ce_2(\eta, -q) \simeq \cos 2\eta + q(\tfrac{1}{12} \cos 4\eta - \tfrac{1}{4})$$

$$A_0^{(2n)} \simeq O(q^n)\,;\; A_0^{(2)} \simeq \frac{q}{4}+O(q^3)$$

$$A_0^{(0)} \simeq 1\,;\; A_2^{(0)} \simeq 1$$

and

$$C_0 = \left(1-\frac{q}{2} \cosh 2\xi_0\right)\left(\frac{i\alpha}{\sigma}+f_1\right)$$

$$C_2 = -\left(\frac{q}{2 \cosh 2\xi_0}\right)\left(\frac{i\alpha}{\sigma}+f_1\right).$$

Hence from eqn. (3.13), the expression of T is approximately

$$T = \frac{c^2}{8k'} \sin (\sigma t+E)(\cosh 2\xi_0 - \cos 2\eta - \cosh 2\xi + \frac{\cosh 2\xi \cos 2\eta}{\cosh 2\xi_0}$$

$$+\left(f_1+\frac{c^2\alpha}{8k'}\right) \cos \sigma t, \qquad \ldots \quad \ldots \quad (3.14)$$

where

$$\alpha = \sin E, \quad \sigma f_1 = \cos E.$$

Using polar coordinates and putting

$$a = c \cosh \xi_0,\; b = c \sinh \xi_0,$$

we have

$$T = \frac{\sin (\sigma t+E)}{2k'} \cdot \frac{a^2b^2}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\left(f_1+\frac{c^2\alpha}{8k'}\right) \cos \sigma t, \qquad \ldots \quad (3.15)$$

which satisfies the boundary conditions.

(ii) When $|q|$ is large, we have the asymptotic formula (McLachlan 1947, pp. 165, 230):

$$Ce_0(\xi, -q) \simeq \left(\frac{2}{i \sinh \xi}\right)^{\frac{1}{2}} K_0 \cosh\left[2k \cosh \xi - \tanh^{-1} \tan\left(\frac{\pi}{2}-\frac{i\xi}{4}\right)\right], \quad (3.16)$$

where
$$K_0 = \frac{Ce_0^{(0)} Ce_0(\frac{1}{2}\pi)}{A_0^{(0)}(2\pi k)^{\frac{1}{2}}}$$

and $A_0^{(2n)}$ is very small when $n \geqslant 1$ and $A_0^{(0)} = 1$.

From eqn. (3.5)

$$4k^2 = \frac{ic^2\sigma}{k'}.$$

Hence $2k = Ac(1+i)$ where $A = \left(\frac{\sigma}{2k'}\right)^{\frac{1}{2}}$. But when x is large, $\cosh x = \frac{1}{2}e^x$ approximately and hence

$$\cosh\left[2k \cosh \xi - \tanh^{-1} \tan\left(\frac{\pi}{4} - \frac{i\xi}{2}\right)\right]$$
$$\simeq \tfrac{1}{2} \exp\left\{Ac(1+i) \cosh \xi - \tanh^{-1} \tan\left(\frac{\pi}{4} - \frac{i\xi}{2}\right)\right\}$$
$$= \tfrac{1}{2}\left(\tanh \frac{\xi}{2}\right)^{\frac{1}{2}} \exp\{Ac(1+i) \cosh \xi\}. \quad \ldots \quad \ldots \quad \ldots (3.17)$$

From eqns. (3.9) to (3.12), we have

$$f = -\frac{i\alpha}{\sigma} + \left(\frac{i\alpha}{\sigma} + f_1\right) \frac{Ce_0(\xi, -q)}{Ce_0(\xi_0, -q)} Ce_0(\eta, -q). \quad \ldots \quad \ldots (3.18)$$

Substituting from (3.17) and the above equations in (3.18), we have

$$T = \frac{\alpha}{\sigma} \sin \sigma t + \frac{\cosh \xi_0/2}{\cosh \xi/2} \exp\{-Ac(\cosh \xi_0 - \cosh \xi)\}$$
$$+ R\left[\left(\frac{i\alpha}{\sigma} + f_1\right) \exp i\{\sigma t - Ac(\cosh \xi_0 - \cosh \xi)\} Ce_0(\eta, -q)\right], \quad \ldots (3.19)$$

where R stands for the real part.

§ 4. *Case III.* Let the temperature variation be exponentially decreasing and $\frac{\partial Q}{\partial t} = 0$. Also let the heat generated due to friction be negligibly small. Thus the energy equation to be solved is

$$\frac{\partial T}{\partial t} = k'\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots (4.1)$$

where
$$k' = \frac{k_0}{\rho c_v} = \text{constant}.$$

Taking
$$T_1 = T - T_0 = f(x, y) e^{-\lambda^2 k' t},$$

eqn. (4.1) reduces to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \lambda^2 f = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (4.2)$$

Transforming to elliptic coordinates, we get

$$\frac{\partial^2 f_1}{\partial \xi^2}+\frac{\partial^2 f_1}{\partial \eta^2}+2k^2(\cosh 2\xi-\cos 2\eta)f_1 = 0, \qquad (4.3)$$

where $k^2 = q = \frac{1}{4}\lambda^2 h^2$ and f_1 is a function of (ξ, η).
Let

$$f_1 = \phi(\xi)\psi(\eta). \qquad (4.4)$$

Therefore

$$\frac{d^2\phi}{d\xi^2}-(a-2q \cosh 2\xi)\phi = 0 \qquad (4.5)$$

and

$$\frac{d^2\psi}{d\eta^2}+(a-2q \cos 2\eta)\psi = 0. \qquad (4.6)$$

The general solution is

$$T = f_1(\xi, \eta) \exp(-\lambda^2 k' t) = \sum_{n=0}^{\infty} C_{2n} \,.\, Ce_{2n}(\xi, q)Ce_{2n}(\eta, q) \exp(-\lambda^2 k' t). \qquad (4.7)$$

The boundary conditions are

(i) $T = 0$, $t < 0$ throughout the pipe,

(ii) $T = T_0$, $t \geqslant 0$ at the surface of the pipe,

(iii) $T = T_0$, $t = \infty$, $0 \leqslant \xi \leqslant \xi_0$.

At the surface of the pipe $\xi = \xi_0$ and when $t \geqslant 0$, $T_1 = T - T_0 = 0$, so that from eqn. (4.7),

$$Ce_{2n}(\xi_0, q) = 0$$

and q has those values $q_{2n,m}$ which make $Ce_{2n}(\xi_0, q)$ zero.

Thus λ depends on m and n.

Now at $t = 0$, $T = 0$, $T_1 = -T_0$.

Hence

$$-T_0 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{2n} \,.\, Ce_{2n}(\xi, q_{2n,m})Ce_{2n}(\eta, q_{2n,m}),$$

where

$$C_{2n} = \frac{-T_0\int_0^{\xi_0} Ce_{2n}(\xi, -q_{2n,m})\left[2A_0^{(2n)} \cosh 2\xi - A_0^{(2n)}\right] d\xi}{\int_0^{\xi_0} Ce_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \Theta_{2n}]\, d\xi}$$

and C_{2n} depends on m as well as on n.

Following McLachlan (1947, p. 329) the expression for T may be deduced.

§ 5. *Case IV.* If the motion be steady and the dissipation term is neglected in comparison with other terms, then eqn. (1.1) near the inlet gives

$$w\frac{\partial T}{\partial z} = k'\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right), \quad \ldots \quad \ldots \quad \ldots \quad (5.1)$$

which from eqn. (1.4) reduces to

$$\frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\frac{\partial T}{\partial z} = k'\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \quad \ldots \quad (5.2)$$

But $\dfrac{\partial^2 T}{\partial z^2}$ is very small in comparison to other terms (in fully developed flow) in the r.h.s. of eqn. (5.2); so we neglect this term and assume

$$T = Az+g(x, y). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

Thus eqn. (5.2) is reduced to

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2} = B\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right), \quad \ldots \quad \ldots \quad \ldots \quad (5.4)$$

where

$$B = \frac{AP}{2\mu k'}\cdot\frac{a^2b^2}{a^2+b^2} = \text{constant}. \quad \ldots \quad \ldots \quad \ldots \quad (5.5)$$

Thus

$$g = f_1(x+iy)+f_2(x-iy)+B\left[\frac{x^2+y^2}{12}-\frac{x^4}{12a^2}-\frac{y^4}{12b^2}\right], \quad \ldots \quad (5.6)$$

where f_1 and f_2 are arbitrary functions of x and y. Knowing g, we can get an expression for the temperature distribution. The exact solution of eqn. (5.4) is determined by transforming $g(x, y)$ to $g_1(\xi, \eta)$.

Following Goldstein (1938, § 267) (for the circular tube), let the wall of the pipe be kept at a uniform temperature gradient.

Let $g = 0$ at the inner surface of the pipe.

Thus

$$g = B'\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

where B' is a constant to be determined from the boundary condition. Hence in this case eqn. (5.4) can be easily solved by taking

$$g = C_1(x^4-6x^2y^2+y^4)+C_2(x^2-y^2)+C_3+B'\left[\frac{x^2+y^2}{4}-\frac{1}{12}\left(\frac{x^4}{a^2}+\frac{y^4}{b^2}\right)\right], \quad (5.7)$$

as has been done in § 2; here C_1, C_2, C_3 and B' are constants.

Now $g = 0$ on the inner surface of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ which can be written as

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

Thus substituting for y^2 from eqn. (5.8) in eqn. (5.7), we get

$$O = x^4\left[C_1\left(1+\frac{6b^2}{a^2}+\frac{b^4}{a^4}\right)-\frac{B'}{12}\left(\frac{1}{a^2}+\frac{b^2}{a^4}\right)\right]$$
$$+x^2\left[C_1\left(-6b^2-\frac{2b^4}{a^2}\right)+C_2\left(1+\frac{b^2}{a^2}\right)+B'\left(\frac{1}{4}-\frac{b^2}{12a^2}\right)\right]$$
$$+[C_3+C_1b^4-C_2b^2+B'(\tfrac{1}{6}b^2)]. \quad \ldots \quad (5.9)$$

Equating the coefficients of x^4, x^2 and the constant term to zero, we get C_1, C_2 and C_3. Thus g is known and consequently the expression for the temperature distribution T is known from eqn. (5.3).

Physical Interpretation and Discussion

Equation (1.4) shows that the curves of equal velocity are similar and similarly situated ellipses. Equation (2.6) shows that the temperature varies as a symmetrical function of x and y. At $x = 0$, the boundary temperature is given by Ab^4-Bb^2-D+C, where the constants A, B, C and D are found in terms of the semi-axes of the section. From eqn. (2.6) the temperature at the centre of the tube, i.e. the axial temperature, is

$$T_0+D(a^2+b^2)^2(a^4+a^2b^2+b^4)^{-1},$$

where T_0 is the boundary temperature given by eqn. (2.8). In § 3, the temperature distribution was found in terms of Mathieu functions. For small frequency of oscillations, the temperature distribution is given by eqn. (3.2) which at a given time shows that the curves of equal temperature are similar and similarly situated ellipses as in the case of velocity distribution in § 2. For large frequency of oscillations it would be difficult to put the temperature distribution in terms of the ellipse equation, as we see from eqn. (3.19). In § 4, the energy equation is solved for exponential type of flow and it is discussed in ref. 3 for heat problem and physical examples are given there. In § 5, eqn. (5.3) gives T in terms of x, y and z and at $z = 0$, we expect results similar to those given in § 2. Such flows are possible if the boundary of the tube is maintained at constant temperature which is generally done in some mechanical and chemical engineering problems.

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