TRANSFORMATIONS OF PRODUCTS OF BASIC BILATERAL HYPERGEOMETRIC SERIES

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A general transformation involving generalized basic bilateral hypergeometric functions on the base q and $q^{\frac{1}{2}}$ is deduced by considering Slater type of contour integral which in turn is used to obtain transformations of products of basic bilateral hypergeometric series on bases q and $q^{\frac{1}{2}}$. It is shown that the method of procedure is quite general and can be used with advantage for obtaining transformations of any number of series on the base q and any number of series on the base $q^{\frac{1}{2}}$. The paper is concluded by obtaining equivalence relations between infinite products on bases q and $q^{\frac{1}{2}}$.

§ 1. *Introduction and Notations.*—The transformations of products of basic bilateral hypergeometric series of different bases (q and $q^{\frac{1}{2}}$) were considered by the authors in a previous paper (*communicated*). In this paper transformations between products of two or more basic bilateral hypergeometric series of bases q and p ($= q^{\frac{1}{2}}$) are deduced. These transformations are quite general and incorporate the results of the previous paper and those of a paper due to Denis (*in press*), as special cases. Several theorems on equivalent products have also been deduced. Incidently, these generalize certain known results of Slater (1953, 1954).

The basic bilateral hypergeometric series of base is defined as

$$ {}_M\psi_M^{(x)}\left[\begin{matrix} x^{(\alpha_M)}\,;\, z \\ x^{(\beta_M)} \end{matrix}\right] \equiv {}_M\psi_M^{(x)}\left[\begin{matrix} (\alpha_M)\,;\, z \\ (\beta_M) \end{matrix}\right] = \sum_{m=-\infty}^{\infty} \frac{[x^{(\alpha_M)}\,;\, m]}{[x^{(\beta_M)}\,;\, m]} z^m, $$

$$ \left| x^{\Sigma(\beta_M)-\Sigma(\alpha_M)} \right| < |z| < 1,\ |x| < 1 $$

where $(\alpha_{M,N})$ denotes $(N-M+1)$ parameters of the type $\alpha_M, \alpha_{M+1}, \ldots, \alpha_N$. But when $M = 1$, we shall write simply (α_N) instead of $(\alpha_{1,N})$.
Also

$$ [x^{\alpha}\,;\, m] = (1-x^{\alpha})(1-x^{\alpha+1}) \ldots (1-x^{\alpha+n-1})\,;\, [x^{\alpha}\,;\, 0] = 1. $$

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Lastly, we denote by

$$\mathop{\Pi_x}^{P}\begin{bmatrix} x^{(\alpha_r)},\, x^{(a_R)}\,;\, x^y \\ x^{(\beta_s)},\, x^{(b_S)} \end{bmatrix} \equiv \mathop{\Pi_x}^{P}\begin{bmatrix} (\alpha_r),\, (a_R)\,;\, x^y \\ (\beta_s),\, (b_S) \end{bmatrix}$$

or

$$\prod_{u=0}^{P-1}\begin{bmatrix} 1-x^{\alpha_1+uy},\, 1-x^{\alpha_2+uy},\, \ldots,\, 1-x^{\alpha_r+uy},\, 1-x^{a_1+uy},\, \ldots,\, 1-x^{a_R+uy} \\ 1-x^{\beta_1+uy},\, 1-x^{\beta_2+uy},\, \ldots,\, 1-x^{\beta_s+uy},\, 1-x^{b_1+uy},\, \ldots,\, 1-x^{b_S+uy} \end{bmatrix} \quad (1)$$

the products

$$\prod_{u=0}^{P-1}\left[\frac{[1-x^{(\alpha_r)+uy}][1-x^{(a_R)+uy}]}{[1-x^{(\beta_s)+uy}][1-x^{(b_S)+uy}]}\right],$$

but when $y = 1$ and there is no chance of an ambiguity we denote the product simply by $\mathop{\Pi_x}^{P}\begin{bmatrix} (\alpha_r),\, (a_R)\,; \\ (\beta_s),\, (b_S) \end{bmatrix}$ and when the index P will be infinite it will also be dropped.

§ 2. In the first instance we shall prove that

$$\Pi_q\begin{bmatrix} (b_M),\, 1-(c_M),\, 1-x-\alpha,\, x+\alpha\,; \\ (a_M),\, 1-(a_M) \end{bmatrix}\Bigg\{\Pi_p\begin{bmatrix} 2-(k_N)+n\,; \\ 1-(c_N)+n \end{bmatrix}$$

$$\times\sum_{r=-\infty}^{\infty}\frac{[q^{(c_M)};r][p^{(k_N)-1-n};r]}{[q^{(b_M)};r][p^{(c_N)-n};r]}q^{xr}+\Pi_p\begin{bmatrix} -[2-(k_N)+n]^*\,; \\ -[1-(c_N)+n]^* \end{bmatrix}$$

$$\times\sum_{r=-\infty}^{\infty}\frac{[q^{(c_M)};r][-p^{(k_N)-n-1};r]}{[q^{(b_M)};r][-p^{(c_N)-n};r]}q^{xr}\Bigg\}$$

$$+\Pi_q\begin{bmatrix} a_1-(c_M),\, 1-a_1+(b_M),\, 2-x-\alpha-a_1,\, x+\alpha+a_1-1\,; \\ 1-a_1+(a_M)',\, a_1-(a_M)',\, 2-a_1,\, a_1-1 \end{bmatrix}\Bigg\{\Pi_p\begin{bmatrix} 1+a_1-(k_N)+n\,; \\ a_1-(c_N)+n \end{bmatrix}$$

$$\times\sum_{s=-\infty}^{\infty}\frac{[q^{1-a_1+(c_M)};r][p^{(k_N)-a_1-n};r]q^{xr}}{[q^{1-a_1+(b_M)};r][p^{1-a_1+(c_N)-n};r]}+\Pi_p\begin{bmatrix} -[1+a_1-(k_N)+n]^*\,; \\ -[a_1-(c_N)+n]^* \end{bmatrix}$$

$$\times\sum_{r=-\infty}^{\infty}\frac{[q^{1-a_1+(c_M)};r][-p^{(k_N)-a_1-n};r]q^{xr}}{[q^{1-a_1+(b_M)};r][-p^{1-a_1+(c_N)-n};r]}\Bigg\}+\text{idem }(a_1;\, a_2,\, \ldots,\, a_M) = 0, \quad (2.1)$$

where

$$p = q^{\frac{1}{2}},\ \alpha = \Sigma(c_M)-\Sigma(a_M)+\tfrac{1}{2}[\Sigma(k_N)-\Sigma(c_N)-N],\ M \geqslant N,\ \mathrm{Re}\,(x) > 0,$$

and $a_r-(a_M)'$ stands for the sequence of parameters a_r-a_1, $a_r-a_2, \ldots,$ a_r-a_{r-1}, $a_r-a_{r+1}, \ldots, a_r-a_M$ from which (a_r-a_r) is dropped, and the presence of the term $-(\alpha)^*$ (in a product or in the hypergeometric series) of base q indicates the presence of a factor of the type $[-q^{\alpha}]$.

To prove this result, consider the contour integral*

$$I_{P,R} = \frac{1}{2\pi i}\int \overset{P}{\Pi}(s)\,ds,$$

where the integral is taken round the rectangular contour $A(-2\pi i/t)$, $B(2\pi i/t)$, $C\left(2R+\frac{2\pi i}{t}\right)$, $D\left(2R-\frac{2\pi i}{t}\right)$ where $P < R$ in the s-plane and

$$\overset{P}{\Pi}(s) = \overset{P}{\Pi}_q\left[\begin{matrix}1-(c_M)-s,\ (b_M)+s,\ 1-x-\alpha+s,\ x+\alpha-s\,; \\ (a_M)+s,\ 1-(a_M)-s\end{matrix}\right]\overset{P}{\Pi}_p\left[\begin{matrix}2-(k_N)+n-s\,; \\ 1-(c_N)+n-s\end{matrix}\right],$$

where q is real, $0 < q < 1$ and $q = e^{-t}$, $t > 0$, and n is an integer. The restriction that q is real, introduced to shorten the proof, can be waived off by analytic continuation over the circle $|q| < 1$.

Now $q^{a+s} = e^{-(a+s)t}$ and has a period $2\pi i/t$ and hence $\left\{\prod_{n=0}^{\infty}[q^{a+n+s}]\right\}^{-1}$ has poles at the points $s = -a-n+2K\pi i/t$, one set in every strip of width $2\pi/t$ in the s-plane. Similarly, $\left\{\prod_{n=0}^{\infty}[p^{a+s+n}]\right\}^{-1}$ has poles at the points $s = -a-n+4K\pi i/t$, one set in every strip of width $4\pi/t$.

In the contour $ABCD$ the line AB is indented, if necessary, to ensure that all the points in the sequence $s = 1-(a_M)+r+2K\pi i/t$, $s = 1-(c_N)+r+4K\pi i/t$ for $r = 0, 1, 2, \ldots, P-1$ lie to the right of AB and those of the sequence $s = (a_M)-r+4K\pi i/t$, $r = 0, 1, 2, \ldots, P-1$ lie to the left of it. The product has then a number of poles within the contour $ABCD$ at the points $s = 1-(a_M)+r+2K\pi i/t$, two sets (corresponding to two suitably chosen values of K) and $s = 1-(c_N)+n+r+4K\pi i/t$ one set (corresponding to a suitably chosen value of K).

Then by the periodicity of the integrand

$$\int_{BC}\overset{P}{\Pi}(s)\,ds+\int_{DA}\overset{P}{\Pi}(s)\,ds = 0,$$

hence

$$I_{P,R}-\frac{1}{2\pi i}\int_{-2\pi i/t}^{2\pi i/t}\overset{P}{\Pi}(s)\,ds = \frac{1}{2\pi}\int_0^{2\pi/t}\left\{\overset{P}{\Pi}(2R+ir)-\overset{P}{\Pi}(2R-ir)\right\}dr.$$

* For a similar treatment, *see* Slater (1952*a*).

Now if

$$\overset{P}{\Pi}'(R) = \Pi_q\left[\begin{matrix}-[\mathrm{Re}\,(c_M)+2R-1]^*, -[\mathrm{Re}\,(b_M)+2R]^*, -[1-\mathrm{Re}\,(x+\alpha)+2R]^*, -[\mathrm{Re}\,(x+\alpha)+2R]^*; \\ \mathrm{Re}\,(a_M)+2R, \mathrm{Re}\,(a_M)+2R-1, 2R+1, 2R\end{matrix}\right]\overset{P}{\Pi_p}\left[\begin{matrix}-[\mathrm{Re}\,(k_N)+2R(-n-2)]^* \\ \mathrm{Re}\,(c_N)+2R-n-1\end{matrix}\right],$$

then

$$\overset{P}{\Pi}'(R) \to 1 \text{ as } R\to\infty \quad\text{and}\quad \left|\overset{P}{\Pi}(2R\pm ir)\right| \leqslant q^k\,\overset{P}{\Pi}'(R),$$

where

$$k = P\,.\,\mathrm{Re}\,(x).$$

Therefore

$$\left|I_{P,R} - \frac{1}{2\pi i}\int_{-2\pi i/t}^{2\pi i/t}\overset{P}{\Pi}(s)\,ds\right| = O\left[q^k\,\overset{P}{\Pi}'(R)\int_0^{2\pi/t}dr\right]$$

$$= O\left[q^k\,\overset{P}{\Pi}'(R)\right].$$

Also if

$$I_P = \lim_{R\to\infty} I_{P,R}$$

then

$$\left|I_P - \frac{1}{2\pi i}\int_{-2\pi i/t}^{2\pi i/t}\overset{P}{\Pi}(s)\,ds\right| = O[q^k].$$

But if I_P has residues within the strip $ABCD$ at the poles of $\overset{P}{\Pi}(s)$, then

$$I_P = \frac{1}{t}\sum_{r=0}^{P-1}\frac{\left[q^{(b_M)+r};P\right]\left[q^{1-(c_M)-r};P\right]\left[q^{1-x-\alpha+r};P\right]\left[q^{x+\alpha-r};P\right](-)^r q^{r/2(r+1)}}{\left[q^{(a_M)+r};P\right]\left[q^{1-(a_M)-r};P\right]\left[q^{1+r};P\right]\left[q;r\right]\prod_{u=0}^{P-r-2}\left[1-q^{1+u}\right]}\left\{\frac{\left[p^{2-(k_N)+n-r};P\right]}{\left[p^{1-(c_N)+n-r};P\right]}+\frac{\left[-p^{2-(k_N)+n-r};P\right]}{\left[-p^{1-(c_N)+n-r};P\right]}\right\}$$

$$+\frac{1}{t}\sum_{r=0}^{P-1}\frac{\left[q^{a_1-(c_M)-r};P\right]\left[q^{1-a_1+(b_M)+r};P\right]\left[q^{2-x-\alpha-a_1+r};P\right]\left[q^{x+\alpha+a_1-1-r};P\right]}{\left[q^{1-a_1+(a_M)+r};P\right]\left[q^{a_1-(a_M)'-r};P\right]\left[q^{2-a_1+r};P\right]}$$

$$\times \frac{(-)^r q^{\frac{1}{2}r(r+1)} \left\{ \prod_{u=0}^{P-r-2} [1-q^{1+u}] \right\}^{-1}}{[q^{a_1-1-r}; P][q; r]} \left\{ \frac{[p^{1+a_1-(k_N)+n-r}; P]}{[p^{a_1-(c_N)+n-r}; P]} + \frac{[-p^{1+a_1-(k_N)+n-r}; P]}{[-p^{a_1-(c_N)+n-r}; P]} \right\} + \text{idem } (a_1; a_2, \ldots, a_M)$$

$$+ \frac{1}{t} \sum_{r=0}^{P-1} \frac{[q^{c_1-(c_M)-n-r}; P][q^{1-c_1+(b_M)+n+r}; P][p^{1+c_1-(k_N)-r}; P][q^{2-x-\alpha-c_1+n+r}; P][q^{x+\alpha+c_1-n-r-1}; P](-)^r p^{\frac{r}{2}(r+1)}}{[q^{1-c_1+(a_M)+n+r}; P][q^{c_1-(a_M)-n-r}; P][p^{c_1-(c_N)'-r}; P][q^{2-c_1+n+r}; P][q^{c_1-n-r-1}; P][p; r] \prod_{u=0}^{P-r-2} [1-p^{1+u}]}$$

$$+ \text{idem } (c_1; c_2, \ldots, c_N) = 0.$$

Letting $P \to \infty$, and assuming that $M \geqslant N$, Re $x > 0$, and setting $\lim_{P\to\infty} I_P = I$, we get

$$I = \frac{1}{t} \Pi_q \left[\begin{matrix} (b_M), 1-(c_M), 1-x-\alpha, x+\alpha; \\ (a_M), 1-(a_M), 1, 1 \end{matrix} \right] \sum_{r=0}^{\infty} \frac{[q^{(c_M)}; r] q^{rx}}{[q^{(b_M)}; r]} \left\{ \Pi_p \left[\begin{matrix} 2-(k_N)+n; \\ 1-(c_N)+n \end{matrix} \right] \frac{[p^{(k_N)-n-1}; r]}{[p^{(c_N)-n}; r]} \right.$$

$$\left. + \Pi_p \left[\begin{matrix} -[2-(k_N)+n]^*; \\ -[1-(c_N)+n]^* \end{matrix} \right] \frac{[-p^{(k_N)-n-1}; r]}{[-p^{(c_N)-n}; r]} \right\} + \frac{1}{t} \Pi_q \left[\begin{matrix} a_1-(c_M), 1+(b_M)-a_1, 2-x-\alpha-a_1, x+\alpha+a_1-1; \\ 1-a_1+(a_M)', a_1-(a_M)', 2-a_1, a_1-1, 1, 1 \end{matrix} \right]$$

$$\times \sum_{r=0}^{\infty} \frac{[q^{1-a_1+(c_M)}; r]}{[q^{1-a_1+(b_M)}; r]} q^{rx} \left\{ \Pi_p \left[\begin{matrix} 1+a_1-(k_N)+n; \\ a_1-(c_N)+n \end{matrix} \right] \frac{[p^{(k_N)-a_1-n}; r]}{[p^{1-a_1+(c_N)-n}; r]} + \Pi_p \left[\begin{matrix} -[1+a_1-(k_N)+n]^*; \\ -[a_1-(c_N)+n]^* \end{matrix} \right] \frac{[-p^{(k_N)-a_1-n}; r]}{[-p^{1-a_1+(c_N)-n}; r]} \right\}$$

$$+ \text{idem } (a_1; a_2, \ldots, a_M). \qquad (2.2)$$

Each of the last series along with the series in idem $(c_1; c_2, \ldots, c_N)$ vanishes, since $\lim_{P\to\infty} [q^{-n-r}; P] = 0$ occurs in the numerator of all the above N series.

Similarly, evaluating the residues of $\overset{P}{\Pi}(s)$ in a strip to the left of the imaginary axis and letting $P \to \infty$, we get

$$I=\frac{1}{t}\Pi_q\begin{bmatrix}(b_M)-1,\ 2-(c_M),\ -x-\alpha,\ 1+x+\alpha\ ;\\ (a_M)-1,\ 2-(a_M),\ 1,\ 1\end{bmatrix}\sum_{r=0}^{\infty}\frac{[q^{2-(b_M)};r]}{[q^{2-(c_M)};r]}Q_1^r\left\{\Pi_p\begin{bmatrix}3-(k_N)+n\ ;\\ 2-(c_N)+n\end{bmatrix}\right.$$

$$\times\frac{[p^{2-(c_N)+n};r]}{[p^{3-(k_N)+n};r]}+\Pi_p\begin{bmatrix}-[3-(k_N)+n]^*\ ;\\ -[2-(c_N)+n]^*\end{bmatrix}\left.\frac{[-p^{2-(c_N)+n};r]}{[-p^{3-(k_N)+n};r]}\right\}+\frac{1}{t}\Pi_q\begin{bmatrix}(b_M)-a_1,\ 1+a_1-(c_M),\ 1-x-\alpha-a_1,\ x+\alpha+a_1\ ;\\ (a_M)'-a_1,\ 1+a_1-(a_M)',\ 1-a_1,\ a_1,\ 1,\ 1\end{bmatrix}$$

$$\times\sum_{r=0}^{\infty}\frac{[q^{1+a_1-(b_M)};r]}{[q^{1+a_1-(c_M)};r]}Q_1^r\left\{\Pi_p\begin{bmatrix}2+a_1-(k_N)+n\ ;\\ 1+a_1-(c_N)+n\end{bmatrix}\frac{[p^{1+a_1-(c_N)+n};r]}{[p^{2+a_1-(k_N)+n};r]}+\Pi_p\begin{bmatrix}-[2+a_1-(k_N)+n]^*\ ;\\ -[1+a_1-(c_N)+n]^*\end{bmatrix}\frac{[-p^{1+a_1-(c_N)+n};r]}{[-p^{2+a_1-(k_N)+n};r]}\right\}$$

$$+\text{idem}\ (a_1\ ;\ a_2,\ldots,a_M)=0, \qquad \ldots \qquad (2.3)$$

where

$$Q_1=q^{\Sigma(b_M)-\Sigma(a_M)-x-\alpha}.$$

Replacing r by $-r-1$ in (2.3) and combining it with (2.2) we get the required result (2.1).

An immediate consequence of (2.1) is the result

$$\Pi_q\begin{bmatrix}(b_M),\ 1-(c_M),\ 1-x-\alpha,\ x+\alpha\ ;\\ (a_M),\ 1-(a_M)\end{bmatrix}\left\{\Pi_p\begin{bmatrix}2-(k_N)\ ;\\ 1-(c_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}1-(c_N)\ ;\ z\\ 2-(k_N)\end{bmatrix}+\Pi_p\begin{bmatrix}-[2-(k_N)]^*\ ;\\ -[1-(c_N)]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[1-(c_N)]^*\ ;\ z\\ -[2-(k_N)]^*\end{bmatrix}\right\}$$

$$\times{}_M\psi_M^{(q)}\begin{bmatrix}(c_M)\ ;\ zq^{x+t}\\ (b_M)\end{bmatrix}+\Pi_q\begin{bmatrix}a_1-(c_M),\ 1-a_1+(b_M),\ 2-x-\alpha-a_1,\ x+\alpha+a_1-1\ ;\\ 1-a_1+(a_M)',\ a_1-(a_M)',\ 2-a_1,\ a_1-1\end{bmatrix}\left\{\Pi_p\begin{bmatrix}1+a_1-(k_N)\ ;\\ a_1-(c_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}a_1-(c_N)\ ;\ z\\ 1+a_1-(k_N)\end{bmatrix}\right.$$

$$+\Pi_p\begin{bmatrix}-[1+a_1-(k_N)]^*\ ;\\ -[a_1-(c_N)]^*\end{bmatrix}\left.{}_N\psi_N^{(p)}\begin{bmatrix}-[a_1-(c_N)]^*\ ;\ z\\ -[1+a_1-(k_N)]^*\end{bmatrix}\right\}{}_M\psi_M^{(q)}\begin{bmatrix}1-a_1+(c_M)\ ;\ zq^{x+t}\\ 1-a_1+(b_M)\end{bmatrix}+\text{idem}\ (a_1\ ;\ a_2,\ldots,a_M)=0, \qquad \ldots \qquad (2.4)$$

where

$t=\frac{1}{2}[\Sigma(k_N)-\Sigma(c_N)-N]$, $M\geqslant N$, $\text{Re}\,x>0$, $\left|q^{\Sigma(b_M)-\Sigma(c_M)}\right|<|zq^{x+t}|<1$, $\left|p^{N+\Sigma(c_N)-\Sigma(k_N)}\right|<|z|<1$, $\alpha=\Sigma(c_M)-\Sigma(a_M)+t$.

This can be verified easily by equating the coefficients of z^n on both the sides and using (2.1).

The result (2.1) reduces to a result due to Slater for $N=0$. The transformation given by the authors (Verma and Upadhyay, *communicated*) follows from this on setting $x=-\alpha$ and specializing the parameters suitably. Next specializing the parameters after replacing M by $M+N$ in the above result by replacing (c_N), $(c_{N+1,\,N+M})$, $(b_{M+1,\,M+N})$ and (k_N) by (e_N), (c_M), (e_N), $1+(d_N)$ and one gets the following interesting transformation between $2[M+N+1]$ basic bilateral hypergeometric series of the type ${}_M\psi_M^{(q)}\times{}_N\psi_N^{(p)}$:

$$\Pi_q\begin{bmatrix}1-(c_M),\,(b_M),\,1-(e_N),\,(e_N),\,1-x-\alpha,\,x+\alpha\,;\\ 1-(a_{M+N}),\,(a_{M+N})\end{bmatrix}\left\{\Pi_p\begin{bmatrix}1-(d_N)\,;\\ 1-(e_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}1-(e_N)\,;\,z\\ 1-(d_N)\end{bmatrix}\right.$$
$$\left.+\Pi_p\begin{bmatrix}-[1-(d_N)]^*;\\ -[1-(e_N)]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[1-(e_N)]^*;\,z\\ -[1-(d_N)]^*\end{bmatrix}\right\}{}_M\psi_M^{(q)}\begin{bmatrix}(c_M)\,;\,zq^{x+t}\\ (b_M)\end{bmatrix}$$
$$+\Pi_q\begin{bmatrix}a_1-(c_M),\,1-a_1+(b_M),\,a_1-(e_N),\,1-a_1+(e_N),\,2-x-\alpha-a_1,\,x+\alpha+a_1-1\,;\\ 1-a_1+(a_{M+N})',\,a_1-(a_{M+N})',\,2-a_1,\,a_1-1\end{bmatrix}$$
$$\times\left\{\Pi_p\begin{bmatrix}a_1-(d_N)\,;\\ a_1-(e_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}a_1-(e_N)\,;\,z\\ a_1-(d_N)\end{bmatrix}+\Pi_p\begin{bmatrix}-[a_1-(d_N)]^*;\\ -[a_1-(e_N)]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[a_1-(e_N)]^*;\,z\\ -[a_1-(d_N)]^*\end{bmatrix}\right\}{}_M\psi_M^{(q)}\begin{bmatrix}1-a_1+(c_M)\,;\,zq^{x+t}\\ 1-a_1+(b_M)\end{bmatrix}$$
$$+\text{idem}\,(a_1\,;\,a_2,\ldots,a_{M+N})=0, \qquad (2.5)$$

where

$$\alpha=\Sigma(c_M)+\tfrac{1}{2}\,\Sigma(d_N)+\tfrac{1}{2}\,\Sigma(e_N)-\Sigma(a_{M+N}),$$
$$t=\tfrac{1}{2}\,\Sigma(d_N)-\tfrac{1}{2}\,\Sigma(e_N).$$

§ 3. A survey of the literature shows that there is no general transformation between products of three or more basic hypergeometric series. In this section we proceed to deduce a transformation between $2[M+P+1]$ products of three hypergeometric series of the type ${}_M\psi_M^{(q)}\times{}_P\psi_P^{(q)}\times{}_N\psi_N^{(p)}$. To do so, we proceed as follows:

In the relation (2.4) replace M by $M+P$ and let $b_{M+r}=b_{M+r}-n$ and $c_{M+r}=c_{M+r}-n : r=1,\,2,\,\ldots,\,P$, then on some simplification we get

$$\Pi_q\begin{bmatrix}(b_{M+P}),\,1-(c_{M+P}),\,1-x-\alpha,\,x+\alpha\,;\\ (a_{M+P}),\,1-(a_{M+P})\end{bmatrix}\frac{\left[q^{1-(b_{M+1,\,M+P})}\,;\,n\right]}{\left[q^{1-(c_{M+1,\,M+P})}\,;\,n\right]}\left\{\Pi_p\begin{bmatrix}2-(k_N)\,;\\ 1-(c_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}1-(c_N)\,;\,z\\ 2-(k_N)\end{bmatrix}\right.$$
$$\left.+\Pi_p\begin{bmatrix}-[2-(k_N)]^*;\\ -[1-(c_N)]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[1-(c_N)]^*;\,z\\ -[2-(k_N)]^*\end{bmatrix}\right\}{}_{M+P}\psi_{M+P}^{(q)}\begin{bmatrix}(c_M),\,(c_{M+1,\,M+P})-n\,;\,zq^{x+t}\\ (b_M),\,(b_{M+1,\,M+P})-n\end{bmatrix}$$

$$+\Pi_q\begin{bmatrix}a_1-(c_{M+P}),\ 1-a_1+(b_{M+P}),\ 2-x-\alpha-a_1,\ x+\alpha+a_1-1\,; \\ 1-a_1+(a_{M+P})',\ a_1-(a_{M+P})',\ 2-a_1,\ a_1-1\end{bmatrix}\frac{[q^{a_1-(b_{M+1,\,M+P})}\,;\,n]}{[q^{a_1-(c_{M+1,\,M+P})}\,;\,n]}\left\{\Pi_p\begin{bmatrix}1+a_1-(k_N)\,; \\ a_1-(c_N)\end{bmatrix}\right.$$

$$\left.\times\,{}_N\psi_N^{(p)}\begin{bmatrix}a_1-(c_N)\,;\,z \\ 1+a_1-(k_N)\end{bmatrix}+\Pi_p\begin{bmatrix}-[1+a_1-(k_N)]^*\,; \\ -[a_1-(c_N)]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[a_1-(c_N)]^*\,;\,z \\ -[1+a_1-(k_N)]^*\end{bmatrix}\right\}$$

$$\times\,{}_{M+P}\psi_{M+P}^{(q)}\begin{bmatrix}1-a_1+(c_M),\ 1+(c_{M+1,\,M+P})-a_1-n\,;\,zq^{x+t} \\ 1-a_1+(b_M),\ 1+(b_{M+1,\,M+P})-a_1-n\end{bmatrix}+\text{idem}\,(a_1\,;\,a_2,\,\ldots,\,a_{M+P})=0, \qquad (3.1)$$

where

$$\alpha=\Sigma\,(c_{M+P})-\Sigma\,(a_{M+P})+\tfrac{1}{2}\,[\Sigma\,(k_N)-\Sigma\,(c_N)-N],\ t=\tfrac{1}{2}\,[\Sigma\,(k_N)-\Sigma\,(c_N)-N],$$

$$M+P\geqslant N,\ \text{Re}\,x>0,\ \left|q^{\Sigma\,(b_{M+P})-\Sigma\,(c_{M+P})}\right|<|zq^{x+t}|<1,\ |p^{-2t}|<|z|<1.$$

Hence the following transformation between products of the type ${}_M\psi_M^{(q)}\times{}_P\psi_P^{(q)}\times{}_N\psi_N^{(p)}$ can be verified easily by equating the coefficients of u^n on both the sides and using (3.1)

$$\Pi_q\begin{bmatrix}(b_{M+P}),\ 1-(c_{M+P}),\ 1-x-\alpha,\ x+\alpha\,; \\ (a_{M+P}),\ 1-(a_{M+P})\end{bmatrix}\left\{\Pi_p\begin{bmatrix}2-(k_N)\,; \\ 1-(c_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}1-(c_N)\,;\,z \\ 2-(k_N)\end{bmatrix}+\Pi_p\begin{bmatrix}-[2-(k_N)]^*\,; \\ -[1-(c_N)]^*\end{bmatrix}\right.$$

$$\left.\times\,{}_N\psi_N^{(p)}\begin{bmatrix}-[1-(c_N)]^*\,;\,z \\ -[2-(k_N)]^*\end{bmatrix}\right\}{}_M\psi_M^{(q)}\begin{bmatrix}(c_M)\,;\,zuq^{x+v} \\ (b_M)\end{bmatrix}{}_P\psi_P^{(q)}\begin{bmatrix}1-(b_{M+1,\,M+P})\,;\,u \\ 1-(c_{M+1,\,M+P})\end{bmatrix}$$

$$+\Pi_q\begin{bmatrix}a_1-(c_{M+P}),\ 1-a_1+(b_{M+P}),\ 2-x-\alpha-a_1,\ x+\alpha+a_1-1\,; \\ a_1-(a_{M+P})',\ 1-a_1+(a_{M+P})',\ a_1-1,\ 2-a_1\end{bmatrix}\left\{\Pi_p\begin{bmatrix}1+a_1-(k_N)\,; \\ a_1-(c_N)\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}a_1-(c_N)\,;\,z \\ 1+a_1-(k_N)\end{bmatrix}\right.$$

$$\left.+\Pi_p\begin{bmatrix}-[1+a_1-(k_N)]^*\,; \\ -[a_1-(c_N)]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[a_1-(c_N)]^*\,;\,z \\ -[1+a_1-(k_N)]^*\end{bmatrix}\right\}{}_M\psi_M^{(q)}\begin{bmatrix}1-a_1+(c_M)\,;\,zuq^{x+v} \\ 1-a_1+(b_M)\end{bmatrix}$$

$$\times\,{}_P\psi_P^{(q)}\begin{bmatrix}a_1-(b_{M+1,\,M+P})\,;\,u \\ a_1-(c_{M+1,\,M+P})\end{bmatrix}+\text{idem}\,(a_1\,;\,a_2,\,\ldots,\,a_{M+P})=0, \qquad (3.2)$$

where

$$\alpha=\Sigma\,(c_{M+P})-\Sigma(a_{M+P})+\tfrac{1}{2}\,[\Sigma\,(k_N)-\Sigma\,(c_N)-N],$$

$$v=\Sigma\,(c_{M+1,\,M+P})-\Sigma\,(b_{M+1,\,M+P})+\tfrac{1}{2}\,[\Sigma\,(k_N)-\Sigma\,(c_N)-N],$$

$$M+P \geqslant N,\ \mathrm{Re}\, x>0,\ \left|p^{\Sigma(c_N)-\Sigma(k_N)+N}\right|<|z|<1,$$
$$\left|q^{\Sigma(b_{M+1,\,M+P})-\Sigma(c_{M+1,\,M+P})}\right|<|u|<1,\ \left|q^{\Sigma(b_M)-\Sigma(c_M)}\right|<|zuq^{x+v}|<1.$$

This result for $N=0$ becomes [P is replaced by N]

$$\Pi_q\begin{bmatrix}1-(c_{M+N}),\ (b_{M+N}),\ 1-x-\alpha,\ x+\alpha\ ; \\ 1-(a_{M+N}),\ (a_{M+N})\end{bmatrix} {}_M\psi_M^{(q)}\begin{bmatrix}(c_M)\ ;\ zq^{x+t} \\ (b_M)\end{bmatrix} {}_N\psi_N^{(q)}\begin{bmatrix}1-(b_{M+1,\,M+N})\ ;\ z \\ 1-(c_{M+1,\,M+N})\end{bmatrix}$$
$$+\Pi_q\begin{bmatrix}a_1-(c_{M+N}),\ 1-a_1+(b_{M+N}),\ 2-x-\alpha-a_1,\ x+\alpha+a_1-1\ ; \\ a_1-(a_{M+N})',\ 1-a_1+(a_{M+N})',\ 2-a_1,\ a_1-1\end{bmatrix} {}_M\psi_M^{(q)}\begin{bmatrix}1-a_1+(c_M)\ ;\ z \\ 1-a_1+(b_M)\end{bmatrix}$$
$$\times {}_N\psi_N^{(q)}\begin{bmatrix}a_1-(b_{M+1,\,M+N})\ ;\ z \\ a_1-(c_{M+1,\,M+N})\end{bmatrix}+\text{idem}\ (a_1\,;\ a_2,\ \ldots,\ a_{M+N})=0, \qquad (3.3)$$

where

$$\alpha=\Sigma(c_{M+N})-\Sigma(a_{M+N}),\ t=\Sigma(c_{M+1,\,M+N})-\Sigma(b_{M+1,\,M+N}),$$

a result due to Denis (*in press*), which incorporates in turn the results of Shukla (1957, 1959). It may be mentioned that (3.2) also contains a result due to the authors (Verma and Upadhyay, *communicated*) as a special case.

§ 4. One may naturally like to find the transformation of product of three series with two series on the base p and one on the base q instead of having two series on the base q and one on the base p. Hence, we proceed to find a transformation between $2(M+1)$ products of basic bilateral hypergeometric series of the type ${}_N\psi_N^{(p)}\times{}_R\psi_R^{(p)}\times{}_M\psi_M^{(q)}$. To deduce this transformation, replace N by $N+R$ and replace k_{N+r} and c_{N+r} in (2.4) by $k_{N+r}+n$ and $c_{N+r}+n$: $r=1,\ 2,\ \ldots,\ R$ respectively and after a little simplification we get a transformation from which the following identity can be verified without much difficulty:

$$\Pi_q\begin{bmatrix}(b_M),\ 1-(c_M),\ 1-x-\alpha,\ x+\alpha\ ; \\ (a_M),\ 1-(a_M)\end{bmatrix}\left\{\Pi_p\begin{bmatrix}2-(k_N),\ 2-(k_{N+1,\,N+R})\ ; \\ 1-(c_N),\ 1-(c_{N+1,\,N+R})\end{bmatrix} {}_N\psi_N^{(p)}\begin{bmatrix}1-(c_N)\ ;\ zyp^{-\eta} \\ 2-(k_N)\end{bmatrix}\right.$$
$$\times {}_R\psi_R^{(p)}\begin{bmatrix}(k_{N+1,\,N+R})-1\ ;\ y \\ (c_{N+1,\,N+R})\end{bmatrix}+\Pi_p\begin{bmatrix}-[2-(k_N)]^*,\ -[2-(k_{N+1,\,N+R})]^*\ ; \\ -[1-(c_N)]^*,\ -[1-(c_{N+1,\,N+R})]^*\end{bmatrix} {}_N\psi_N^{(p)}\begin{bmatrix}-[1-(c_N)]^*\ ;\ zyp^{-\eta} \\ -[2-(k_N)]^*\end{bmatrix}$$
$$\left.\times {}_R\psi_R^{(p)}\begin{bmatrix}-[(k_{N+1,\,N+R})-1]^*\ ;\ y \\ -[(c_{N+1,\,N+R})]^*\end{bmatrix}\right\} {}_M\psi_M^{(q)}\begin{bmatrix}(c_M)\ ;\ zq^{x+t} \\ (b_M)\end{bmatrix}+\Pi_q\begin{bmatrix}a_1-(c_M),\ 1-a_1+(b_M),\ 2-x-\alpha-a_1,\ x+\alpha+a_1-1\ ; \\ a_1-(a_M)',\ 1-a_1+(a_M)',\ 2-a_1,\ a_1-1\end{bmatrix}$$

$$\times\left\{\Pi_p\begin{bmatrix}1+a_1-(k_N),\ 1+a_1-(k_{N+1,\,N+R})\,; \\ a_1-(c_N),\ a_1-(c_{N+1,\,N+R})\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}a_1-(c_N)\,;\ zyp^{-\eta} \\ 1+a_1-(k_N)\end{bmatrix}{}_R\psi_R^{(p)}\begin{bmatrix}(k_{N+1,\,N+R})-a_1\,;\ y \\ 1-a_1+(c_{N+1,\,N+R})\end{bmatrix}\right.$$

$$\left.+\Pi_p\begin{bmatrix}-[1+a_1-(k_N)]^*,\ -[1+a_1-(k_{N+1,\,N+R})]^*\,; \\ -[a_1-(c_N)]^*,\ -[a_1-(c_{N+1,\,N+R})]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[a_1-(c_N)]^*\,;\ zyp^{-\eta} \\ -[1+a_1-(k_N)]^*\end{bmatrix}{}_R\psi_R^{(p)}\begin{bmatrix}-[(k_{N+1,\,N+R})-a_1]^*\,;\ y \\ -[1-a_1+(c_{N+1,\,N+R})]^*\end{bmatrix}\right\}$$

$$\times{}_M\psi_M^{(q)}\begin{bmatrix}1-a_1+(c_M)\,;\ zq^{x+t} \\ 1-a_1+(b_M)\end{bmatrix}+\text{idem }(a_1\,;\ a_2,\ldots,a_M)=0, \qquad (4.1)$$

where

$$t=\tfrac{1}{2}[\Sigma(k_{N+R})-\Sigma(c_{N+R})-N-R],\ \alpha=\Sigma(c_M)-\Sigma(a_M)+t,$$

$$\eta=\Sigma(c_{N+1,\,N+R})-\Sigma(k_{N+1,\,N+R})+R,\ M\geqslant N+R,\ \left|q^{\Sigma(b_M)-\Sigma(a_M)}\right|<|zq^{x+t}|<1,$$

$$\left|p^{\Sigma(c_N)-\Sigma(k_N)+N}\right|<|zyp^{-\eta}|<1,\ \left|p^{\Sigma(c_{N+1,\,N+R})-\Sigma(k_{N+1,\,N+R})+R}\right|<|y|<1.$$

Similarly, if one replaces M by $M+P$ and b_{M+r} and c_{M+r} by $b_{M+r}-n$ and $c_{M+r}-n$ respectively in (4.1) and proceeds as described above one can get a transformation between $2[M+P+2]$ products of the basic bilateral hypergeometric series of the type ${}_M\psi_M^{(q)}\times{}_P\psi_P^{(q)}\times{}_N\psi_N^{(p)}\times{}_R\psi_R^{(p)}$:

$$\Pi_q\begin{bmatrix}(b_{M+P}),\ 1-(c_{M+P}),\ 1-x-\alpha,\ x+\alpha\,; \\ (a_{M+P}),\ 1-(a_{M+P})\end{bmatrix}\left\{\Pi_p\begin{bmatrix}2-(k_N),\ 2-(k_{N+1,\,N+R})\,; \\ 1-(c_{N+R})\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}1-(c_N)\,;\ zyp^{-\eta} \\ 2-(k_N)\end{bmatrix}\right.$$

$$\left.\times{}_R\psi_R^{(p)}\begin{bmatrix}(k_{N+1,\,N+R})-1\,;\ y \\ (c_{N+1,\,N+R})\end{bmatrix}+\Pi_p\begin{bmatrix}-[2-(k_{N+R})]^*\,; \\ -[1-(c_{N+R})]^*\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}-[1-(c_N)]^*\,;\ zyp^{-\eta} \\ -[2-(k_N)]^*\end{bmatrix}{}_R\psi_R^{(p)}\begin{bmatrix}-[(k_{N+1,\,N+R})-1]^*\,;\ y \\ -[(c_{N+1,\,N+R})]^*\end{bmatrix}\right\}$$

$$\times{}_M\psi_M^{(q)}\begin{bmatrix}(c_M)\,;\ zuq^{x+v} \\ (b_M)\end{bmatrix}{}_P\psi_P^{(q)}\begin{bmatrix}1-(b_{M+1,\,M+P})\,;\ u \\ 1-(c_{M+1,\,M+P})\end{bmatrix}+\Pi_q\begin{bmatrix}a_1-(c_{M+P}),\ 1-a_1+(b_{M+P}),\ 2-x-\alpha-a_1,\ x+\alpha+a_1-1\,; \\ a_1-(a_{M+P})',\ 1-a_1+(a_{M+P})',\ a_1-1,\ 2-a_1\end{bmatrix}$$

$$\times\left\{\Pi_p\begin{bmatrix}1+a_1-(k_{N+R})\,; \\ a_1-(c_{N+R})\end{bmatrix}{}_N\psi_N^{(p)}\begin{bmatrix}a_1-(c_N)\,;\ zyp^{-\eta} \\ 1+a_1-(k_N)\end{bmatrix}{}_R\psi_R^{(p)}\begin{bmatrix}(k_{N+1,\,N+R})-a_1\,;\ y \\ 1-a_1+(c_{N+1,\,N+R})\end{bmatrix}+\Pi_p\begin{bmatrix}-[1+a_1-(k_{N+R})]^*\,; \\ -[a_1-(c_{N+R})]^*\end{bmatrix}\right.$$

$$\left.\times{}_N\psi_N^{(p)}\begin{bmatrix}-[a_1-(c_N)]^*\,;\ zyp^{-\eta} \\ -[1+a_1-(k_N)]^*\end{bmatrix}{}_R\psi_R^{(p)}\begin{bmatrix}-[(k_{N+1,\,N+R})-a_1]^*\,;\ y \\ -[1-a_1+(c_{N+1,\,N+R})]^*\end{bmatrix}\right\}{}_M\psi_M^{(q)}\begin{bmatrix}1-a_1+(c_M)\,;\ zuq^{x+v} \\ 1-a_1+(b_M)\end{bmatrix}$$

$$\times{}_P\psi_P^{(q)}\begin{bmatrix}a_1-(b_{M+1,\,M+P})\,;\ u \\ a_1-(c_{M+1,\,M+P})\end{bmatrix}+\text{idem }(a_1\,;\ a_2,\ldots,a_{M+P})=0, \qquad (4.2)$$

where

$$\alpha = \Sigma(c_{M+P}) - \Sigma(a_{M+P}) + \tfrac{1}{2}[\Sigma(k_{N+R}) - \Sigma(c_{N+R}) - N - R],\ \eta = \Sigma(c_{N+1,\,N+R}) - \Sigma(k_{N+1,\,N+R}) + R,$$

$$v = \Sigma(c_{M+1,\,M+P}) - \Sigma(b_{M+1,\,M+P}) + \tfrac{1}{2}[\Sigma(k_{N+R}) - \Sigma(c_{N+R}) - N - R],$$

$$M+P \geqslant N+R,\ \mathrm{Re}\, x > 0,\ \left|p^{N+\Sigma(c_N)-\Sigma(k_N)}\right| < |zyp^{-\eta}| < 1,\ |p^\eta| < |y| < 1,\ \left|q^{\Sigma(b_M)-\Sigma(c_M)}\right|$$

$$< |uzq^{x+v}| < 1,\ \left|q^{\Sigma(b_{M+1,\,M+P})-\Sigma(c_{M+1,\,M+P})}\right| < |u| < 1.$$

It is, however, not difficult to see at this stage that the transformations of bilateral hypergeometric series of different bases with products of five or six or even more series could also be deduced by a repetition of the above procedure, with any number of hypergeometric series on the base q and the rest of them on the base p.

§ 5. Lastly, we mention, in brief, certain interesting consequences of the transformations deduced in § 2 to § 4.

Let us consider the question of exhibiting equivalences between certain sets of infinite products. A study of such a question leads us to the interrelation between some of the more recondite branches of analysis, namely the theory of elliptic functions, generalized basic hypergeometric functions, modular functions and partition functions.

In (3.3) setting $(b_{M-1}) = (c_{M-1})$, $q^{b_M} = zq^{x+t+(c_M)+b_{M+N}+1}$, $(b_{M+1,\,M+N-1}) = (c_{M+1,\,M+N-1})$ and replacing z by $zq^{\,b_{M+N}}$, and summing the ${}_1\psi_1^{(q)}[q^a;\ q^b;\ z]$ series by the known result

$$ {}_1\psi_1^{(q)}[q^a;\, q^b;\, z] = \Pi\begin{bmatrix} zq^a,\ \frac{1}{z}q^{1-a},\ q,\ q^{b-a}; \\ z,\ \frac{1}{z}q^{b-a},\ q^{1-a},\ q^b \end{bmatrix}, \qquad (5.1)$$

we get

$$\Pi\begin{bmatrix} q^{1-(c_{M+N-1})},\ q^{(c_{M+N-1})},\ zq^{x+c_M+c_{M+N}},\ q^{1-x-\alpha},\ q^{x+\alpha},\ zq,\ \frac{1}{z},\ \frac{1}{z}q^{1-x-c_M-c_{M+N}}; \\ q^{-(a_{M+N})+1},\ q^{(a_{M+N})},\ q^{1-c_M},\ q^{c_M} \end{bmatrix}$$

$$+\Pi\begin{bmatrix} q^{a_1-(c_{M+N-1})},\ q^{1-a_1+(c_{M+N-1})},\ zq^{1-a_1+x+c_M+c_{M+N}},\ q^{2-x-\alpha-a_1},\ q^{x+\alpha+a_1-1},\ \frac{1}{z}q^{1-a_1},\ \frac{1}{z}q^{a_1-x-c_M-c_{M+N}},\ zq^{a_1}; \\ q^{a_1-(a_{M+N})'},\ q^{1-a_1+(a_{M+N})'},\ q^{2-a_1},\ q^{a_1-1},\ q^{a_1-c_M},\ q^{1+c_M-a_1} \end{bmatrix}$$

$$+\ \text{idem}\ (a_1;\, a_2, \ldots, a_M) = 0, \qquad (5.2)$$

where

$$\alpha = \Sigma(c_{M+N}) - \Sigma(a_{M+N}).$$

This result reduces for $x = \alpha = 0$, $z = q^{-c_M}$ to a result due to Slater (1954).

Next, we proceed to deduce another result on equivalent products which is different in nature from (5.2). In (2.5) replace (b_{M-1}), (d_{N-1}), q^{b_M} and p^{-d_N} by (c_{M-1}), (e_{N-1}), $zq^{1+x+t+c_M}$, and zp^{1-e_N} respectively and summing the ${}_1\psi_1{}^{(\alpha)}[q^a; q^b; z]$ series by (5.1), we get

$$\Pi\left[\begin{matrix} q^{1-(c_M)}, q^{(c_{M-1})}, q^{1-(e_{N-1})}, q^{(e_{N-1})}, zq^{x+t+c_M}, q^{1-x-\alpha}, q^{x+\alpha}, \frac{1}{z}q^{1-x-t-c_M}; q \\ q^{1-(a_{M+N})}, q^{(a_{M+N})}, q^{1-(c_M)} \end{matrix}\right]$$

$$\times\left\{\Pi\left[zp^{1-e_N}, \frac{1}{z}p^{e_N}, -p^{1-e_N}, -p^{e_N}; p\right] + \Pi\left[p^{1-e_N}, p^{e_N}, -\frac{1}{z}p^{e_N}, -zp^{1-e_N}; p\right]\right\}$$

$$+\Pi\left[\begin{matrix} q^{a_1-(c_M)}, q^{1-a_1+(c_{M-1})}, q^{1-a_1+(e_{N-1})}, q^{a_1-(e_{N-1})}, zq^{1-a_1+x+t+c_M}, q^{2-x-\alpha-a_1}, q^{x+\alpha+a_1-1}, \frac{1}{z}q^{a_1-x-t-c_M}; \\ q^{1-a_1+(a_{M+N})'}, q^{a_1-(a_{M+N})'}, q^{2-a_1}, q^{a_1-1}, q^{a_1-c_M} \end{matrix}\right]$$

$$\left\{\Pi\left[zp^{a_1-e_N}, \frac{1}{z}p^{1-a_1+e_N}, -p^{a_1-e_N}, -p^{1-a_1+e_N}; p\right] + \Pi\left[-zp^{a_1-e_N}, -\frac{1}{z}p^{1-a_1+e_N}, p^{a_1-e_N}, p^{1-a_1+e_N}; p\right]\right\}$$

$$+ \text{idem } (a_1; a_2, \ldots, a_{M+N}) = 0. \qquad \ldots \quad (5.3)$$

This relation connects products on bases $q^{\frac{1}{2}}$ and q. This result also reduces for $N = 0$ and $z = 1$ to the aforesaid result due to Slater (1954).

If we start from (3.2) or (4.1) we get results which generalize the result (5.3), but they are too complicated to be mentioned here.

Finally, we deduce another result on equivalent products different from (5.3). To do so, we set

$$c_2 = c_1 + \frac{1}{M}, c_3 = c_1 + \frac{2}{M}, \ldots, c_M = c_1 + 1 - \frac{1}{M},$$

$$b_2 = b_1 + \frac{1}{M}, b_3 = b_1 + \frac{2}{M}, \ldots, b_M = b_1 + 1 - \frac{1}{M},$$

$$c_{M+2} = c_{M+1} - \frac{1}{N},\ c_{M+3} = c_{M+2} - \frac{2}{N},\ \ldots,\ c_{M+N} = c_{M+1} - 1 + \frac{1}{N},$$

$$b_{M+2} = b_{M+1} - \frac{1}{N},\ b_{M+3} = b_{M+1} - \frac{2}{N},\ \ldots,\ b_{M+N} = b_{M+1} - 1 + \frac{1}{N}$$

and then replace z and q^x by $(-)^N z q^{\frac{1}{2}+Nb_{M+1}}$ and $(-)^{M+N}\frac{1}{z}q^{x-Mc_1-Nc_{M+1}}$, respectively, and then using the formula:

$$\left[q^{a_1};\ Mn\right]_{q^{\frac{1}{M}}} = \left[q^a;\ n\right]\left[q^{a+\frac{1}{M}};\ n\right]\ldots\left[q^{a+1-\frac{1}{M}};\ n\right],$$

and letting q^{c_1} and $q^{c_{M+1}}$ tend to infinity and q^{b_1}, $q^{b_{M+1}}$ tend to zero, and summing the bilateral series involved by the Jacobi's formula

$$\sum_{n=-\infty}^{\infty} x^n q^{an^2} = \prod_{n=1}^{\infty}\left[q^{2an},\ -\frac{1}{x}q^{a(2n-1)},\ -xq^{a(2n-1)};\right],$$

we get the following result

$$\Pi\left[\begin{matrix} zq^{1-\frac{M-N}{2}-x+\Sigma(a_{M+N})},\ \frac{1}{z}q^{\frac{M-N}{2}+x-\Sigma(a_{M+N})};\ q \\ q^{1-(a_{M+N})},\ q^{(a_{M+N})} \end{matrix}\right]\Pi\left[-zq^{\frac{3N}{2}},\ -\frac{1}{z}q^{-\frac{N}{2}};\ q^N\right]$$

$$\times\Pi\left[-q^{\frac{M}{2}+x},\ -q^{\frac{M}{2}-x};\ q^M\right]+\Pi\left[\begin{matrix} zq^{2-\frac{M-N}{2}+\Sigma(a_{2,\ M+N})-x},\ \frac{1}{z}q^{\frac{M-N}{2}+x-1-\Sigma(a_{2,\ M+N})};\ q \\ q^{a_1-(a_{M+N})'},\ q^{1-a_1+(a_{M+N})'},\ q^{2-a_1},\ q^{a_1-1} \end{matrix}\right]$$

$$\times\Pi\left[-zq^{\frac{N}{2}+Na_1},\ -\frac{1}{z}q^{\frac{N}{2}-Na_1};\ q^N\right]\Pi\left[-q^{\frac{3M}{2}+x-Ma_1},\ -q^{Ma_1-\frac{M}{2}-x};\ q^M\right] + \text{idem}\ (a_1;\ a_2,\ \ldots,\ a_{M+N}) = 0.$$

More general equivalent products can be obtained if one starts from a transformation of products of three or even more basic bilateral hypergeometric series by adopting a similar procedure.

In conclusion we may remark that the transformations between basic bilateral hypergeometric series on the bases q and $q^{\frac{1}{M}}$ (M being an integer) could also be developed by modifying the contour suitably and following exactly the procedure described in the above sections. But even in the simplest cases of transformations between basic bilateral hypergeometric series on the bases q and $q^{\frac{1}{2}}$ the results become so cumbersome that they cease to be of any practical utility and hence they are not mentioned here.

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