CALCULATION OF THERMAL BOUNDARY LAYER ON THE AXIALLY SYMMETRIC LAMINAR VISCOUS FLOW DUE TO ROTATING BODIES OF REVOLUTION

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The heat transfer in the laminar boundary layer flow due to rotating bodies of revolution is investigated. The cases of the sphere, prolate and oblate spheroids and paraboloids are considered separately. The flow functions are determined by assuming a sixth degree profile for the radial velocity and a fifth degree profile for the transverse velocity. Using the results of the flow functions, the temperature distribution is obtained as a power series expansion. It is found that, for all these bodies, the Nusselt number increases with the Prandtl number.

1. INTRODUCTION

Frössling (1940) carried out calculations on the temperature distribution in the laminar boundary layer about a body of arbitrary shape for the two-dimensional and axially symmetrical cases. In his calculations, in which friction and compression work were neglected throughout, he assumed a power series for the potential velocity distribution around the body expanded in terms of the length of the arc (Blasius series).

Hoskin (1955) has investigated the velocity distribution in a laminar boundary layer of a rotating sphere in an incompressible fluid having uniform flow. His results have been utilized by Julius Siekmann (1962) in determining the thermal boundary layer of a rotating sphere in an axial stream.

In the present paper detailed analysis has been made for the solution of the equation of energy in the case of an axially symmetric viscous flow in the boundary layer due to rotating bodies of revolution, like sphere, spheroids and paraboloids. The problems connected with laminar boundary layer flow due to the above rotating bodies of revolution in Newtonian fluid have been considered by several authors like Howarth (1951), Nigam (1954), Fadnis (1954) and Verma (1962). All the above authors have expressed the velocity components in terms of a series in powers of θ or $\sin\theta$, θ being measured from the axis of rotation along a meridian. The coefficients in the series are purely functions of the normal distance from the surface of the body of revolution. These unknown functions are determined by the Karman-Pohlhausen method assuming a fourth degree polynomial. Rajeswari (1962) has studied the flow past rotating bodies of revolution in a non-Newtonian fluid.

In the present paper, we assume a sixth and a fifth degree polynomial for the flow functions. With a suitable choice of the conditions at the edge of the boundary layer, approximate solution for the flow functions has been obtained. Utilizing these results for the flow functions, the temperature functions are determined by using the Galerkin's technique and the temperature distribution is obtained as a power series expansion.

2. Flow due to a Rotating Sphere

In the following treatment r, θ and ϕ represent the spherical polar co-ordinates with r measured radially outwards from the centre of the sphere, the polar angle θ measured from the axis of rotation and ϕ the azimuthal angle; also w, u, v denote the velocities in the directions of r, θ, ϕ increasing respectively. Omitting azimuthal variation, the boundary layer equations for a sphere of radius a are (Nigam 1954)

$$\frac{u}{a}\frac{\partial u}{\partial \theta}+w\frac{\partial u}{\partial r}-\frac{v^2}{a}\cot\theta=\nu\frac{\partial^2 u}{\partial r^2}, \quad (2.1)$$

$$\frac{u}{a}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial r}+\frac{uv}{a}\cot\theta=\nu\frac{\partial^2 v}{\partial r^2}, \quad (2.2)$$

$$\frac{1}{a}\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial r}+\frac{u}{a}\cot\theta=0. \quad (2.3)$$

We assume the approximate solutions in the form

$$u=a\Omega\cos\theta(\sin\theta F_1+\sin^3\theta F_3+\sin^5\theta F_5+\ldots), \quad (2.4)$$

$$v=a\Omega(\sin\theta G_1+\sin^3\theta G_3+\sin^5\theta G_5+\ldots), \quad (2.5)$$

$$w=\left(\frac{\nu\Omega}{4}\right)^{\frac{1}{2}}(3\cos^2\theta-1)(H_1+\sin^2\theta H_3+\sin^4\theta H_5+\ldots), \quad (2.6)$$

where F_1, G_1, H_1, ... are functions of

$$z=\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}(r-a) \quad (2.7)$$

and Ω is the angular velocity.

Substituting these velocities in eqns. (2.1) to (2.3), we get the following equations for the functions F_i, G_i and H_i ($i=1, 3, 5$):

$$F_1''=F_1^2+H_1F_1'-G_1^2, \quad (2.8)$$

$$G_1''=2F_1G_1+H_1G_1', \quad (2.9)$$

$$2F_1+H_1'=0, \quad (2.10)$$

$$F_3''=4F_1F_3+F_1'H_3+H_1F_3'-2G_1G_3-2F_1^2-\tfrac{3}{2}H_1F_1', \quad (2.11)$$

$$G_3''=2F_3G_1+4F_1G_3+H_3G_1'+H_1G_3'-2F_1G_1-\tfrac{3}{2}H_1G_1', \quad (2.12)$$

$$4F_3+H_3' = 0, \quad \ldots \quad (2.13)$$

$$F_5'' = 6F_1F_5-6F_1F_3+3F_3^2+H_1F_5'+H_3F_3'+H_5F_1' -\tfrac{3}{2}(H_1F_3'+H_3F_1')-G_3^2-2G_1G_5, \quad \ldots \quad (2.14)$$

$$G_5'' = 6F_1G_5+2G_1F_5+4F_3G_3-4F_1G_3-2F_3G_1+H_1G_5' -\tfrac{3}{2}(H_1G_3'+H_3G_1')+H_3G_3'+H_5G_1', \quad \ldots \quad (2.15)$$

$$F_3+6F_5+H_5' = 0. \quad \ldots \quad (2.16)$$

The boundary conditions are

$$\left.\begin{array}{ll} F_1 = F_3 = F_5 = 0\,; \quad G_1 = 1,\ G_3 = G_5 = 0 = H_1 = H_3 = H_5 & \text{at } z = 0 \\ F_1 \to 0,\ F_3 \to 0,\ F_5 \to 0\,; \quad G_1 \to 0,\ G_3 \to 0,\ G_5 \to 0 & \text{as } z \to \infty \end{array}\right\}. \quad (2.17)$$

We shall solve the above boundary value problem using Karman-Pohlhausen method.

We assume expansions for the functions F_1 and G_1 as

$$F_1(z) = a_1\left[\frac{z}{\delta}-5\left(\frac{z}{\delta}\right)^2+10\left(\frac{z}{\delta}\right)^3-10\left(\frac{z}{\delta}\right)^4+5\left(\frac{z}{\delta}\right)^5-\left(\frac{z}{\delta}\right)^6\right], \quad \ldots \quad (2.18)$$

$$G_1(z) = 1-5\left(\frac{z}{\delta}\right)+10\left(\frac{z}{\delta}\right)^2-10\left(\frac{z}{\delta}\right)^3+5\left(\frac{z}{\delta}\right)^4-\left(\frac{z}{\delta}\right)^5, \quad \ldots \quad (2.19)$$

where $\delta = \sqrt{\frac{\Omega}{\nu}}\,\delta_R$, δ_R being the thickness of the boundary layer and F_1, G_1 have been chosen to satisfy the following conditions:

$$\left.\begin{array}{l} F_1(0) = F_1(\delta) = F_1'(\delta) = F_1''(\delta) = F_1'''(\delta) = F_1^{\text{iv}}(\delta) = 0 \\ G_1(0) = 1\,; \quad G_1(\delta) = G_1'(\delta) = G_1''(\delta) = G_1'''(\delta) = G_1^{\text{iv}}(\delta) = 0 \end{array}\right\}. \quad \ldots \quad (2.20)$$

Integrating (2.8) and (2.9) with respect to z between the limits 0 and δ and making use of (2.10), we get

$$\int_0^\delta (3F_1^2-G_1^2)\,dz+F_1'(0) = 0, \quad \ldots \quad (2.21)$$

and

$$4\int_0^\delta F_1G_1\,dz+G_1'(0) = 0. \quad \ldots \quad (2.22)$$

Substituting for F_1 and G_1 from (2.18) and (2.19), we get two equations for the two unknowns a_1 and δ:

$$a_1^2\delta^2-26\delta^2+286a_1 = 0, \quad \ldots \quad (2.23)$$

and

$$a_1\delta^2 = 165. \quad \ldots \quad (2.24)$$

Solving these two equations, we get

$$\left.\begin{array}{r} a_1 = 3{\cdot}084 \\ \text{and} \qquad \delta = 7{\cdot}314 \end{array}\right\}. \quad \ldots \quad (2.25)$$

Similarly we assume that

$$F_3(z) = a_3\left[\frac{z}{\delta}-5\left(\frac{z}{\delta}\right)^2+10\left(\frac{z}{\delta}\right)^3-10\left(\frac{z}{\delta}\right)^4+5\left(\frac{z}{\delta}\right)^5-\left(\frac{z}{\delta}\right)^6\right], \quad .. \quad (2.26)$$

$$G_3(z) = B_3\left[\frac{z}{\delta}-4\left(\frac{z}{\delta}\right)^2+6\left(\frac{z}{\delta}\right)^3-4\left(\frac{z}{\delta}\right)^4+\left(\frac{z}{\delta}\right)^5\right], \quad .. \quad .. \quad (2.27)$$

$$F_5(z) = a_5\left[\frac{z}{\delta}-5\left(\frac{z}{\delta}\right)^2+10\left(\frac{z}{\delta}\right)^3-10\left(\frac{z}{\delta}\right)^4+5\left(\frac{z}{\delta}\right)^2-\left(\frac{z}{\delta}\right)^6\right], \quad .. \quad (2.28)$$

and

$$H_5(z) = B_5\left[\frac{z}{\delta}-4\left(\frac{z}{\delta}\right)^2+6\left(\frac{z}{\delta}\right)^3-4\left(\frac{z}{\delta}\right)^4+\left(\frac{z}{\delta}\right)^5\right] \quad .. \quad .. \quad (2.29)$$

satisfying the corresponding boundary conditions. Integrating the eqns. (2.11), (2.12) after making use of eqns. (2.13) and substituting the expressions (2.26), (2.27) for F_3 and G_3 in these equations we obtain two equations for determining the constants a_3, B_3. By using a similar procedure for the eqns. (2.14), (2.15) and (2.16) we obtain the equations to determine a_5, B_5. The solutions of the two sets of equations are obtained as

$$\left.\begin{array}{ll} a_3 = 1{\cdot}3952 & a_5 = 0{\cdot}9623 \\ B_3 = 1{\cdot}1429 & B_5 = 0{\cdot}5163 \end{array}\right\}. \quad .. \quad .. \quad .. \quad (2.30)$$

Hence we have that

$$\left.\begin{array}{l} H_1 = -2a_1\delta K_1 \\ H_3 = -4a_3\delta K_1 \\ H_5 = -(a_3+6a_5)\delta K_1 \end{array}\right\}, \quad .. \quad .. \quad .. \quad (2.31)$$

where

$$K_1 = \left[\frac{1}{2}\left(\frac{z}{\delta}\right)^2-\frac{5}{3}\left(\frac{z}{\delta}\right)^3+\frac{5}{2}\left(\frac{z}{\delta}\right)^4-2\left(\frac{z}{\delta}\right)^5+\frac{5}{6}\left(\frac{z}{\delta}\right)^6-\frac{1}{7}\left(\frac{z}{\delta}\right)^7\right].$$

3. Thermal Boundary Layer on the Rotating Sphere

Formulation of the Problem

Using the thermal boundary layer approximations and omitting the dissipation terms we have the energy equation as

$$w\frac{\partial T}{\partial r}+\frac{u}{a}\frac{\partial T}{\partial \theta} = \frac{\nu}{P_r}\frac{\partial^2 T}{\partial r^2}, \quad .. \quad .. \quad .. \quad (3.1)$$

where T is the temperature of the fluid and P_r is the Prandtl number.

We assume the temperature distribution in the form

$$T = T_a+(T_\infty-T_a)[M_1(z)+M_3(z)\sin^2\theta+M_5(z)\sin^4\theta+\ldots] \quad .. \quad (3.2)$$

where T_a and T_∞ are the temperatures on the surface of the sphere and of the free stream respectively.

Substituting u and w from eqns. (2.4) and (2.6) and the temperature function T from (3.2) in eqn. (3.1), we get the following system of ordinary differential equations:

$$\frac{1}{P_r} M_1'' - H_1 M_1' = 0, \quad \text{(3.3)}$$

$$\frac{1}{P_r} M_3'' - H_1 M_3' - 2F_1 M_3 = (H_3 - \tfrac{3}{2}H_1) M_1', \quad \text{(3.4)}$$

$$\frac{1}{P_r} M_5'' - H_1 M_5' - 4F_1 M_5 = 2(F_3 - F_1) M_3 + (H_3 - \tfrac{3}{2}H_1) M_3' + (H_5 - \tfrac{3}{2}H_3) M_1', \quad \text{(3.5)}$$

where a prime denotes differentiation with respect to z.

The boundary conditions are

$$\left.\begin{array}{ll} T = T_a = \text{constant} & \text{for } z = 0 \\ T = T_\infty = \text{constant} & \text{for } z \to \infty \end{array}\right\}. \quad \text{(3.6)}$$

The boundary conditions on M_1, M_3, M_5 become

$$\left.\begin{array}{l} M_1(0) = 0 = M_3(0) = M_5(0) \\ M_1(\infty) = 1,\ M_3(\infty) = 0 = M_5(\infty) \end{array}\right\}. \quad \text{(3.7)}$$

Method of Solution

We introduce a new variable s such that $\frac{z}{\delta_T} = s$ where δ_T is the thickness of the thermal boundary layer and s varies from 0 to 1. Eqn. (3.3) in terms of the new variable becomes

$$\frac{1}{P_r} M_1''(s) + 2a_1\delta^2(\tfrac{1}{2}\alpha^3 s^2 - \tfrac{5}{3}\alpha^4 s^3 + \tfrac{5}{2}\alpha^5 s^4 - 2\alpha^6 s^5 + \tfrac{5}{6}\alpha^7 s^6 - \tfrac{1}{7}\alpha^8 s^7) M_1'(s) = 0, \quad \text{(3.8)}$$

where $\alpha = \frac{\delta_T}{\delta}$ is the ratio of the two boundary layer thicknesses. Using Galerkin's method eqn. (3.8) is solved for M_1 subject to the boundary conditions

$$\left.\begin{array}{l} M_1(0) = 0 \\ M_1(1) = 1 \end{array}\right\}, \quad \text{(3.9)}$$

we assume an approximate solution for $M_1(s)$ satisfying the boundary conditions (3.9) in the form

$$M_1(s) = s + A_1(s - s^2), \quad \text{(3.10)}$$

where A_1 is a free parameter.

Substituting (3.10) in (3.8) and orthogonalizing the defect functions with respect to the constituent functions s and $(s-s^2)$, we have the two equations (after performing the necessary integrations):

$$A_1\left(-\frac{1}{12}\alpha^3 + \frac{1}{4}\alpha^4 - \frac{1}{3}\alpha^5 + \frac{5}{21}\alpha^6 - \frac{15}{168}\alpha^7 + \frac{1}{72}\alpha^8 - \frac{1}{\lambda P_r}\right)$$
$$= -\frac{1}{6}\alpha^3 + \frac{5}{12}\alpha^4 - \frac{1}{2}\alpha^5 + \frac{1}{3}\alpha^6 - \frac{5}{42}\alpha^7 + \frac{1}{56}\alpha^8, \quad \text{(3.11)}$$

and

$$A_1\left(-\frac{1}{120}\alpha^3+\frac{1}{42}\alpha^4-\frac{5}{168}\alpha^5+\frac{5}{252}\alpha^6-\frac{1}{144}\alpha^7+\frac{1}{990}\alpha^8-\frac{1}{6\lambda P_r}\right)$$

$$=-\frac{1}{40}\alpha^3+\frac{1}{18}\alpha^4-\frac{5}{84}\alpha^5+\frac{1}{28}\alpha^6-\frac{5}{432}\alpha^7+\frac{1}{630}\alpha^8, \quad \ldots (3.12)$$

where

$$\lambda = a_1 . \delta^2 = 165.$$

Eliminating A_1 from (3.11) and (3.12) we get the equation governing α, and solving this resulting equation for different values of P_r we have a unique value for α in the range 0 to 1.

In the case when $P_r = 1$, the solution of the equations are $\alpha = 0{\cdot}2525$ and $A_1 = 0{\cdot}2134$.

When $P_r = 0{\cdot}7$, we have $\alpha = 0{\cdot}275$ and $A_1 = 0{\cdot}1851$.

In a similar manner, eqns. (3.4) and (3·5) are transformed in terms of the new variable s. The boundary conditions on $M_3(s)$ and $M_5(s)$ are

$$\left.\begin{array}{l} M_3(0) = 0 = M_5(0) \\ M_3(1) = 0 = M_5(1) \end{array}\right\}. \quad \ldots \quad \ldots \quad \ldots (3.13)$$

We assume approximate solutions for M_3 and M_5 in the form

$$M_3(s) = L_1(s-s^2)+L_2(s^2-s^3), \quad \ldots \quad \ldots \quad \ldots (3.14)$$

$$M_5(s) = N_1(s-s^2)+N_2(s^2-s^3). \quad \ldots \quad \ldots \quad \ldots (3.15)$$

The unknown parameters L_1, L_2, N_1, N_2 involved in the above equations are determined as before by Galerkin's method and the following results are obtained:

P_r	L_1	L_2	N_1	N_2
1·0	−0·05889	−0·09906	−0·02483	−0·03013
0·7	−0·05974	−0·07273	−0·03245	−0·01147

Heat Transfer Parameter

In order to calculate the rate of heat transfer, we have, from eqn. (3.2),

$$\left(\frac{\partial T}{\partial r}\right)_{r=a} = (T_\infty - T_a)\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\left[M_1'(z)+M_3'(z)\sin^2\theta+M_5'(z)\sin^4\theta+\ldots\right]_{z=0}. \quad (3.16)$$

We define a Nusselt number Nu by

$$Q = \frac{\mathrm{Nu}\, k\, S(T_a - T_\infty)}{d}, \quad \ldots \quad \ldots \quad \ldots (3.17)$$

where Q is the quantity of heat transferred in unit time across area S, k is the thermal conductivity and d a characteristic length. The rate of heat transfer per unit area is also given by

$$\frac{Q}{S} = -k\left(\frac{\partial T}{\partial r}\right)_{r=a}. \quad \ldots \quad \ldots \quad \ldots (3.18)$$

From (3.17) and (3.18), using (3.16), we get

$$\frac{\mathrm{Nu}\,k\,(T_a-T_\infty)}{d} = -k(T_\infty-T_a)\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}$$
$$\times\left[M_1'(z)+M_3'(z)\sin^2\theta+M_5'(z)\sin^4\theta+\ldots\right]_{z=0}. \quad (3.19)$$

Putting $d = a$ in this equation we get

$$\frac{\mathrm{Nu}}{\sqrt{\mathrm{Re}}} = \left[M_1'(z)+M_3'(z)\sin^2\theta+M_5'(z)\sin^4\theta+\ldots\right]_{z=0}, \quad (3.20)$$

where

$$\mathrm{Re} = \frac{a^2\Omega}{\nu}. \quad (3.21)$$

In terms of the new variable s, the above equation becomes

$$\frac{\mathrm{Nu}}{\sqrt{\mathrm{Re}}} = \left[M_1'(0)+\sin^2\theta\, M_3'(0)+\sin^4\theta\, M_5'(0)+\ldots\right]\frac{1}{\delta_T}, \quad (3.22)$$

where prime denotes differentiation with respect to s. Numerical values of the Nusselt numbers are obtained and the following conclusions are made from the results.

Discussion of the Results

The values of the flow functions and temperature functions M_i, M_i' ($i = 1, 3, 5$) for $P_r = 0{\cdot}7$, $1{\cdot}0$ have been computed and the following features are noted:

(i) The function M_1 is positive and is increasing in value with increase of the Prandtl number.

(ii) The functions M_3 and M_5 are negative in the entire range and are increasing in absolute value with increase of the Prandtl number.

(iii) The absolute values of M_1, M_3, M_5 are in decreasing order.

A graphical presentation of the non-dimensional temperature distributions for the values $P_r = 0{\cdot}7$, $1{\cdot}0$ and the Nusselt number $\frac{\mathrm{Nu}}{\sqrt{\mathrm{Re}}}$ is plotted in figures. From the graph, we see that the ratio $\frac{\mathrm{Nu}}{\sqrt{\mathrm{Re}}}$ increases with the Prandtl number in the entire range. This observation is quite in agreement with that made by Siekmann (1962) who has calculated the thermal boundary layer on a rotating sphere using a two-dimensional non-rotating coordinate system with origin at the forward stagnation point of the body.

4. Flow and Heat Transfer due to Rotating Spheroids and Paraboloids

We now consider the boundary layer equations in a set of coordinates formed by orthogonal curvilinear coordinates ξ and η on the surface of the body and ζ along the normal to the surface. The velocity components in ξ, η and ζ directions are denoted by u, v and w respectively. Omitting azimuthal variation, the steady boundary layer equations in orthogonal curvilinear coordinates for incompressible fluids are (Fadnis 1954)

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{w}{h_3}\frac{\partial u}{\partial \zeta}-\frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}=\frac{\nu}{h_3^2}\frac{\partial^2 u}{\partial \zeta^2}, \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{w}{h_3}\frac{\partial v}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi}=\frac{\nu}{h_3^2}\frac{\partial^2 v}{\partial \zeta^2}, \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

$$\frac{1}{h_1}\frac{\partial u}{\partial \xi}+\frac{1}{h_3}\frac{\partial w}{\partial \zeta}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial \xi}+\frac{u}{h_1 h_3}\frac{\partial h_3}{\partial \xi}=0. \quad \ldots \quad \ldots \quad (4.3)$$

Neglecting the viscous dissipation, and assuming constant thermal conductivity, the energy equation reduces to

$$\rho c_p\left[\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{w}{h_3}\frac{\partial T}{\partial \zeta}\right]=\frac{k}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial \xi}\right)+\frac{\partial}{\partial \zeta}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial \zeta}\right)\right], \quad \ldots \quad (4.4)$$

where T is the temperature of the fluid, ρ the density, c_p is the specific heat at constant pressure and k is the coefficient of thermal conductivity.

(a) The Case of a Prolate Spheroid

For a prolate spheroid, we introduce a system of coordinates defined by

$$z+ir=c\cosh(\zeta+i\xi), \quad \phi=\eta, \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

where r, z, ϕ are the cylindrical coordinates and ζ and ξ are the elliptic coordinates in the meridian plane and η is the azimuth. After substituting $h_1=h_3=c\sqrt{\sinh^2\zeta+\sin^2\xi}$, $h_2=c\sinh\zeta\sin\xi$ and writing $\zeta=\zeta_0$ for a given prolate spheroid, eqns. (4.1) to (4.3) become

$$u\frac{\partial u}{\partial \xi}+w\frac{\partial u}{\partial \zeta}-v^2\cot\xi=\frac{\nu}{cK_1}\frac{\partial^2 u}{\partial \zeta^2}, \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

$$u\frac{\partial v}{\partial \xi}+w\frac{\partial v}{\partial \zeta}+uv\cot\xi=\frac{\nu}{cK_1}\frac{\partial^2 v}{\partial \zeta^2}, \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

$$\frac{\partial u}{\partial \xi}+\frac{\partial w}{\partial \zeta}+u\left(\cot\xi+\frac{\sin\xi\cos\xi}{K_1^2}\right)=0, \quad \ldots \quad \ldots \quad (4.8)$$

where

$$K_1=\sqrt{\sinh^2\zeta_0+\sin^2\xi}.$$

Substituting the values of h_1, h_2, h_3 and using the boundary layer approximations mentioned earlier, the energy equation for a given prolate spheroid reduces to

$$u\frac{\partial T}{\partial \xi}+w\frac{\partial T}{\partial \zeta}=\frac{\nu}{cP_r K_1}\frac{\partial^2 T}{\partial \zeta^2}, \quad \ldots \quad \ldots \quad \ldots \quad (4.9)$$

where P_r is the Prandtl number and K_1 has the same meaning. In order to solve the eqns. (4.6) to (4.8) we assume the velocity functions in the form

$$u = c\Omega \sinh \zeta_0 \cos \xi (F_1 \sin \xi + F_3 \sin^3 \xi + F_5 \sin^5 \xi + \ldots), \quad \ldots (4.10)$$

$$v = c\Omega \sinh \zeta_0 \sin \xi (G_1 + G_3 \sin^2 \xi + G_5 \sin^4 \xi + \ldots), \quad \ldots \quad \ldots (4.11)$$

$$w = \left(\frac{\nu\Omega}{4}\right)^{\frac{1}{2}} (3 \cos^2 \xi - 1)(H_1 + H_3 \sin^2 \xi + H_5 \sin^4 \xi + \ldots), \quad \ldots (4.12)$$

where $F_1, G_1, H_1, \ldots$ are functions of

$$s = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} c \sinh \zeta_0 (\zeta - \zeta_0).$$

Further, we assume the temperature distribution in the form

$$T = T_{\zeta_0} + (T_\infty - T_{\zeta_0})[M_1 + M_3 \sin^2 \xi + M_5 \sin^4 \xi + \ldots], \quad \ldots (4.13)$$

where M_1, M_3, M_5 are functions of z. T_{ζ_0} and T_∞ are the temperatures on the surface of the spheroid and the free stream respectively.

(b) Flow and Heat Transfer due to an Oblate Spheroid

In the case of an oblate spheroid we introduce a new system of coordinates defined by

$$z + ir = c \sinh (\zeta + i\xi), \quad \phi = \eta \quad \ldots \quad \ldots \quad \ldots (4.14)$$

where r, z, ϕ have the meanings mentioned in the earlier section and h_1, h_2, h_3 have the values

$$h_1 = h_3 = c\sqrt{\cosh^2 \zeta - \sin^2 \xi}, \quad h_2 = c \cosh \zeta \sin \xi.$$

Using the above system of coordinates, the general boundary layer equations in the case of a given oblate spheroid $\zeta = \zeta_0$ become

$$u \frac{\partial u}{\partial \xi} + w \frac{\partial u}{\partial \zeta} - v^2 \cot \xi = \frac{\nu}{cE} \frac{\partial^2 u}{\partial \xi^2}, \quad \ldots \quad \ldots \quad \ldots (4.15)$$

$$u \frac{\partial v}{\partial \xi} + w \frac{\partial v}{\partial \zeta} + uv \cot \xi = \frac{\nu}{cE} \frac{\partial^2 v}{\partial \zeta^2}, \quad \ldots \quad \ldots \quad \ldots (4.16)$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial \zeta} + u \left(\cot \xi + \frac{\sin \xi \cos \xi}{E^2}\right) = 0, \quad \ldots \quad \ldots (4.17)$$

where

$$E = \sqrt{\cosh^2 \zeta_0 - \sin^2 \xi}.$$

The thermal boundary layer equation in this case becomes

$$u \frac{\partial T}{\partial \xi} + w \frac{\partial T}{\partial \zeta} = \frac{\nu}{cP_r E} \frac{\partial^2 T}{\partial \zeta^2}. \quad \ldots \quad \ldots \quad \ldots (4.18)$$

The velocity functions in this case are assumed as

$$u = c\Omega \cosh \zeta_0 \cos \xi (F_1 \sin \xi + F_3 \sin^3 \xi + F_5 \sin^5 \xi + \ldots), \quad \ldots (4.19)$$

$$v = c\Omega \cosh \zeta_0 \sin \xi (G_1 + G_3 \sin^2 \xi + G_5 \sin^4 \xi + \ldots), \quad \ldots \quad \ldots (4.20)$$

$$w = \left(\frac{\nu\Omega}{4}\right)^{\frac{1}{2}} (3 \cos^2 \xi - 1)(H_1 + H_3 \sin^2 \xi + H_5 \sin^4 \xi + \ldots), \quad \ldots (4.21)$$

where F_1, G_1 and H_1 are functions of the new variable

$$z = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} c \cosh \zeta_0(\zeta-\zeta_0).$$

The temperature distribution is taken as the same expression (4.13) where M_1, M_3, M_5 are now functions of the new variable z.

(c) *Flow and Heat Transfer due to a Rotating Paraboloid*

For a paraboloid of revolution, we introduce a system of coordinates defined by

$$z+ir = \tfrac{1}{2}l(\zeta+i\xi)^2, \quad \phi = \eta \qquad \text{.. .. .. (4.22)}$$

and h_1, h_2, h_3 have now the values

$$h_1 = h_3 = l\sqrt{\zeta^2+\xi^2}, \quad h_2 = l\xi\zeta.$$

For a given paraboloid of revolution ($\zeta = \zeta_0$), we have boundary layer equations as

$$u\frac{\partial u}{\partial \xi}+w\frac{\partial u}{\partial \zeta}-\frac{v^2}{\xi} = \frac{\nu}{lE_1}\frac{\partial^2 u}{\partial \zeta^2}, \qquad \text{.. .. .. (4.23)}$$

$$u\frac{\partial v}{\partial \xi}+w\frac{\partial v}{\partial \zeta}+\frac{uv}{\xi} = \frac{\nu}{lE_1}\frac{\partial^2 v}{\partial \zeta^2}, \qquad \text{.. .. .. (4.24)}$$

$$\frac{\partial u}{\partial \xi}+\frac{\partial w}{\partial \zeta}+\frac{u}{\xi}+\frac{u\xi}{E_1^2} = 0, \qquad \text{.. .. .. .. (4.25)}$$

where

$$E_1 = \sqrt{\xi^2+\zeta_0^2}.$$

The thermal boundary layer equation reduces to

$$u\frac{\partial T}{\partial \xi}+w\frac{\partial T}{\partial \zeta} = \frac{\nu}{P_r l E_1}\frac{\partial^2 T}{\partial \zeta^2}. \qquad \text{.. .. .. (4.26)}$$

Correspondingly in this case, we assume the velocity components in the form

$$u = l\Omega\zeta_0\xi[F_1+F_3\xi^2+F_5\xi^4+\ldots], \qquad \text{.. .. .. (4.27)}$$

$$v = l\Omega\zeta_0\xi[G_1+G_3\xi^2+G_5\xi^4+\ldots], \qquad \text{.. .. .. (4.28)}$$

$$w = \left(\frac{\nu\Omega}{4}\right)^{\frac{1}{2}}(2-3\xi^2)[H_1+H_3\xi^2+H_5\xi^4+\ldots], \qquad \text{.. .. (4.29)}$$

where F_1, G_1, H_1 are functions of the new variable

$$Z = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} l\zeta_0(\zeta-\zeta_0).$$

The temperature distribution is assumed to be the same as the expression (4.13), where the temperature functions M_1, M_3, M_5 are now functions of the variable z.

5. Solution of the Boundary Layer Equations

Due to the no-slip conditions on the boundary, the boundary conditions in all the above cases become

$$\left.\begin{array}{ll} F_1 = 0 = F_3 = F_5\,; \quad G_1 = 1,\, G_3 = 0 = G_5 = H_1 = H_3 = H_5 & \text{at } z = 0 \\ F_1 \to 0,\, F_3 \to 0,\, F_5 \to 0\,; \quad G_1 \to 0,\, G_3 \to 0,\, G_5 \to 0 & \text{as } z \to \infty \end{array}\right\} . \quad (5.1)$$

The conditions on M_1, M_3, M_5 are

$$\left.\begin{array}{ll} M_1 = M_3 = M_5 = 0 & \text{at } z = 0 \\ M_1 = 1,\, M_3 = M_5 = 0 & \text{as } z \to \infty \end{array}\right\} . \quad \ldots \quad \ldots \quad (5.2)$$

Substituting the velocity functions (4.10) to (4.12) in eqns. (4.6) to (4.9) and comparing the coefficients, we obtain a set of ordinary differential equations governing the functions F_1, G_1, H_1. Similarly, in the case of an oblate spheroid and paraboloid we obtain two different sets of ordinary differential equations governing the corresponding flow functions. These sets of equations are to be solved, satisfying the boundary conditions (5.1). In all the three cases it is found that the equations governing F_1, G_1 are the same as in the case of a rotating sphere and hence the same solutions for F_1, G_1 and H_1 as in the case of a sphere will hold good. (The equations governing the flow functions are omitted here as they are lengthy). To evaluate F_3, G_3; F_5, G_5 we assume the same expressions as (2.26) to (2.29) for these functions, satisfying the boundary conditions (5.1). Integrating the differential equations governing these functions F_3, G_3 . . . by a similar procedure (as in the case of a rotating sphere) and substituting the expressions for these functions, we obtain three different sets (in the three cases) of equations for determining the constants a_3, B_3; a_5, B_5. To facilitate further discussion, a particular value for ζ_0 is taken. In the case of prolate and oblate spheroids we take $\sinh^2 \zeta_0 = 10$ and in the case of a paraboloid we take $\zeta_0^2 = 10$.

The solutions of the three sets of equations are obtained as

	a_3	B_3	a_5	B_5
Prolate spheroid	1·3818	1·0058	0·9816	0·5741
Oblate spheroid	1·4070	1·2677	0·9787	0·6054
Paraboloid	−0·0132	−0·1372	0·00106	0·00798

and the corresponding flow functions H_3 and H_5 are prolate spheroid,

$$H_3(\zeta) = -\left(4a_3 + \frac{a_1}{\sinh^2 \zeta_0}\right)\delta k_1, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

$$H_5(\zeta) = -\left[\left(1 + \frac{1}{\sinh^2 \zeta_0}\right)a_3 + \left(\frac{1}{2\sinh^2 \zeta_0} - \frac{1}{\sinh^4 \zeta_0}\right)a_1 + 6a_5\right]\delta k_1, \quad (5.4)$$

oblate spheroid,

$$H_3(\zeta) = \left[-4a_3+\frac{a_1}{\cosh^2 \zeta_0}\right]\delta k_1, \quad \ldots \quad (5.5)$$

$$H_5(\zeta) = \left[\frac{1}{2\cosh^2 \zeta_0}\left(1+\frac{2}{\cosh^2 \zeta_0}\right)a_1-\tanh^2 \zeta_0 \,.\, a_3-6a_5\right]\delta k_1, \quad \ldots \quad (5.6)$$

and paraboloid of revolution,

$$H_3(\zeta) = -\left[4a_3+\left(3+\frac{1}{\zeta_0^2}\right)a_1\right]\delta k_1, \quad \ldots \quad (5.7)$$

$$H_5(\zeta) = -\left[6a_5+\left(6+\frac{1}{\zeta_0^2}\right)a_3+\left(\frac{9}{2}+\frac{3}{2\zeta_0^2}-\frac{1}{\zeta_0^4}\right)a_1\right]\delta k_1, \quad \ldots \quad (5.8)$$

where

$$k_1 = \left[\frac{1}{2}\left(\frac{z}{\delta}\right)^2-\frac{5}{3}\left(\frac{z}{\delta}\right)^3+\frac{5}{2}\left(\frac{z}{\delta}\right)^4-2\left(\frac{z}{\delta}\right)^5+\frac{5}{6}\left(\frac{z}{\delta}\right)^6-\frac{1}{7}\left(\frac{z}{\delta}\right)^7\right],$$

and z has a different meaning in each of the cases as mentioned earlier.

Solution of Thermal Boundary Layer Equations

Substituting the expressions for u, w and T in eqns. (4.9), (4.18), (4.26) and comparing the coefficients, we get three sets of ordinary differential equations governing the temperature functions M_1, M_3, M_5. In the case of a prolate spheroid the equations are given as

$$\frac{1}{P_r}M_1''-H_1M_1' = 0, \quad \ldots \quad (5.9)$$

$$\frac{1}{P_r}M_3''-H_1M_3'-2F_1M_3 = \left[H_3+\frac{1}{2}\left(\frac{1}{\sinh^2 \zeta_0}-3\right)H_1\right]M_1', \quad \ldots \quad (5.10)$$

$$\begin{aligned}\frac{1}{P_r}M_5''-H_1M_5'-4F_1M_5 = &\left[2F_3+\left(\frac{1}{\sin^2 \zeta_0}-2\right)F_1\right]M_3 \\ &+\left[H_3+\left(\frac{1}{z\sin^2 \zeta_0}-\frac{3}{2}\right)H_1\right]M_3' \\ &+\left[H_5+\left(\frac{1}{2\sin^2 \zeta_0}-\frac{3}{2}\right)H_3-\frac{1+6\sinh^2 \zeta_0}{8\sinh \zeta_0}H\right]M_1',\end{aligned} \quad \ldots \quad (5.11)$$

where a prime denotes differentiation with respect to z. Similar expressions are obtained for the other two cases. Also, eqn. (5.9) is same in all the cases including the case of a rotating sphere. Hence the solution of M_1 will be the same as in the case of a sphere. Introducing the variable $s = \frac{z}{\delta_T}$ (where δ_T is the thermal boundary layer thickness) we transform the eqns. (5.9) to (5.11) and solve them by Galerkin's technique. We get the solution for M_1 as

$$M_1(s) = s+A_1(s-s^2) \quad \ldots \quad (5.12)$$

where A_1 has the same values as in the case of a sphere. Further, we take the approximate solution of M_3 and M_5 satisfying the boundary conditions (5.2) as

$$M_3(s) = L_1(s-s^2)+L_2(s^2-s^3), \quad \text{.. .. ..} \quad (5.13)$$

$$M_5(s) = N_1(s-s^2)+N_2(s^2-s^3). \quad \text{.. .. ..} \quad (5.14)$$

Using Galerkin's technique, the values of the constants L_1, L_2, N_1, N_2 are obtained as:

P_r		L_1	L_2	N_1	N_2
	Prolate spheroid	−0·04987	−0·08395	−0·02423	−0·03817
1·0	Oblate spheroid	−0·06710	−0·11290	−0·01637	−0·03329
	Paraboloid	0·00904	0·01527	0·00095	−0·00246
	Prolate spheroid	−0·05064	−0·06146	−0·03318	−0·01761
0·7	Oblate spheroid	−0·06816	−0·08275	−0·02190	−0·01797
	Paraboloid	0·00918	0·01116	−0·00047	−0·00035

It is to be noted here that $\sinh^2 \zeta_0$ is given the value 10 in the first two cases and ζ_0^2 is given the value 10 in the case of a paraboloid in the above numerical computations.

The Nusselt number Nu can be defined as

$$\frac{\text{Nu}}{\sqrt{\text{Re}}} = [M_1'(0)+M_3'(0)\sin^2\xi+\sin^4\xi\, M_5'(0)+\dots]\frac{1}{\delta_T}, \quad \text{..} \quad (5.15)$$

where the Reynold's number Re is now given by

$$\text{Re} = \frac{c^2\Omega}{\nu}\sinh^2\zeta_0 \quad \text{(prolate spheroid)},$$

$$\text{Re} = \frac{c^2\Omega}{\nu}\cosh^2\zeta_0 \quad \text{(oblate spheroid)}.$$

In the case of a paraboloid of revolution, the Nusselt number is defined as

$$\frac{\text{Nu}}{\sqrt{\text{Re}}} = [M_1'(0)+\xi^2 M_3'(0)+\xi^4 M_5'(0)+\dots]\frac{1}{\delta_T}, \quad \text{..} \quad (5.16)$$

where primes denote differentiation with respect to s and

$$\text{Re} = \frac{l^2\Omega}{\nu}\zeta_0^2.$$

6. Discussion of the Results

Numerical results for the flow functions and the temperature functions are obtained for different values of the Prandtl number ($P_r = 1{\cdot}0,\ 0{\cdot}7$). They are exhibited in the graphs.

For the cases of oblate and prolate spheroids.—The behaviour of the functions M_1, M_3, M_5 is qualitatively the same as in the case of the sphere. The graphical representation of the non-dimension temperature and Nusselt numbers for different values of P_r are plotted in figures. It can be easily observed from the figures that the ratio $\frac{\text{Nu}}{\sqrt{\text{Re}}}$ increases with the Prandtl number in the entire range considered for sin ξ. (The range considered is 0 to 1).

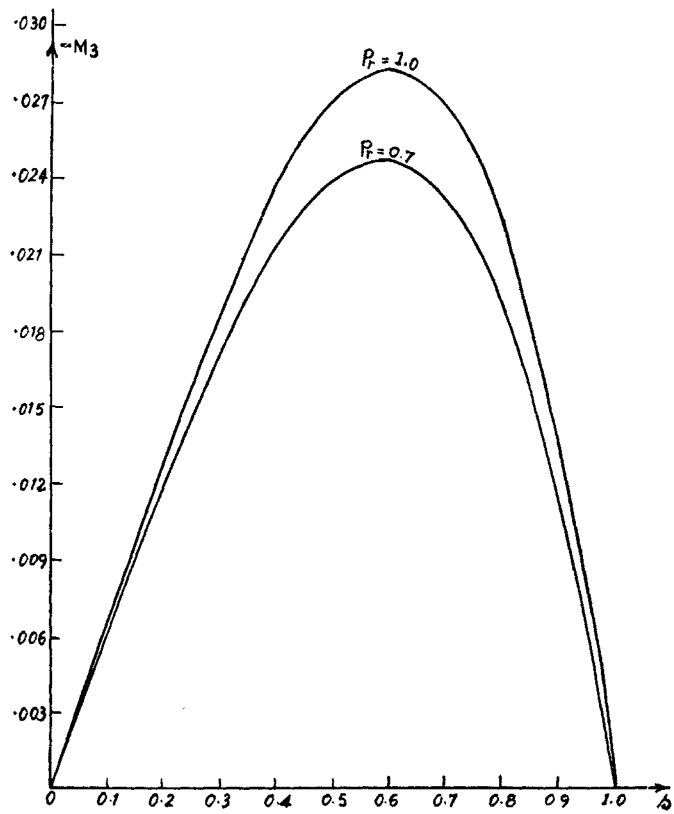

Fig. 1. Temperature profile M_3 *vs* s for the sphere for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

The case of paraboloid of revolution.—In this case also the values of the functions M_1, M_3, M_5 are computed for $P_r = 1{\cdot}0$ and $0{\cdot}7$, graphical presentations of the temperature distribution and Nusselt numbers are given in figures.

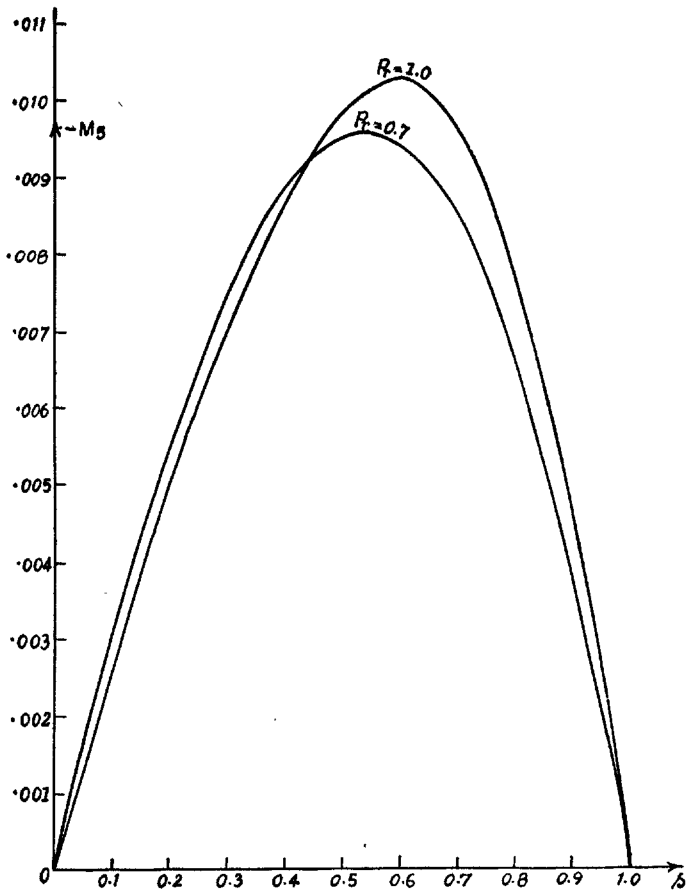

Fig. 2. Temperature profile M_5 *vs* s for the sphere for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

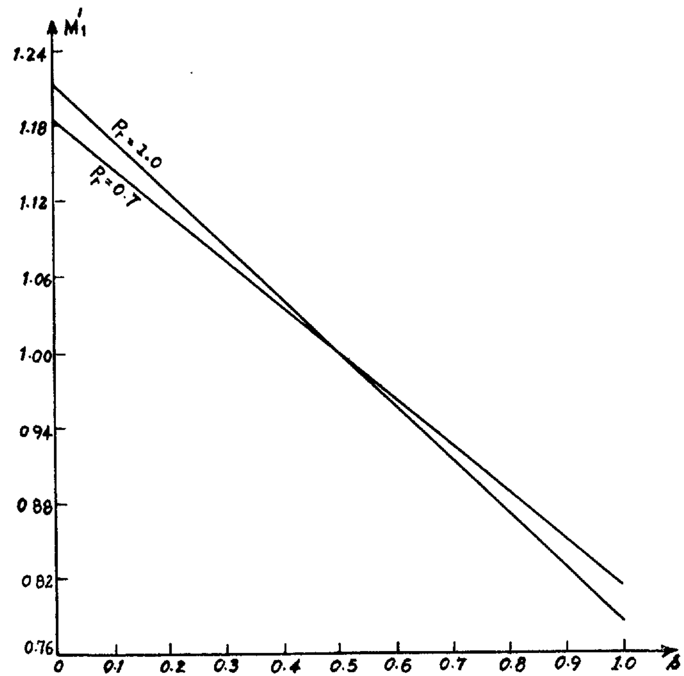

Fig. 3. Plot of M_1' for all the revolving bodies for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

The following conclusions are made from the results obtained:

(i) M_1 and M_3 are both positive in the entire range and increase with increase of Prandtl number.

(ii) M_5 is negative in the entire range for $P_r = 0{\cdot}7$.

(iii) The absolute values of M_1, M_3, M_5 are in decreasing order of magnitude.

(iv) The ratio of the Nusselt number and the Reynolds number [Nu/$\sqrt{\text{Re}}$] increases with the Prandtl number in the entire range.

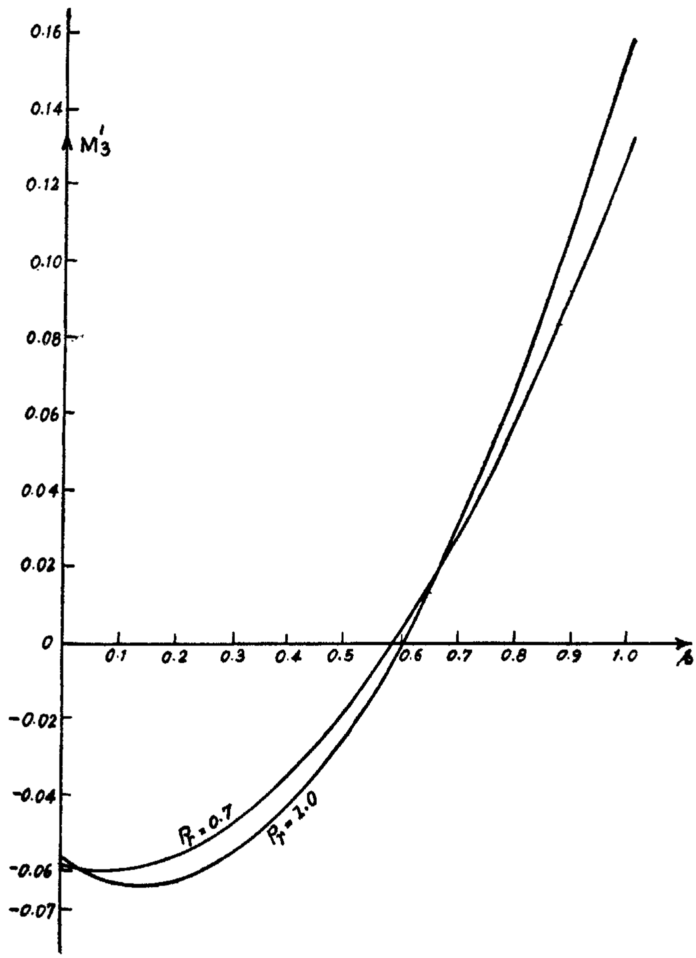

FIG. 4. Plot of M_3' for the sphere for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

Here it may be noted that the results obtained in this paper are comparable with the earlier results only in the case of the sphere. It can be seen that the values for the flow functions obtained here in the case of sphere are better than the results of the earlier authors.

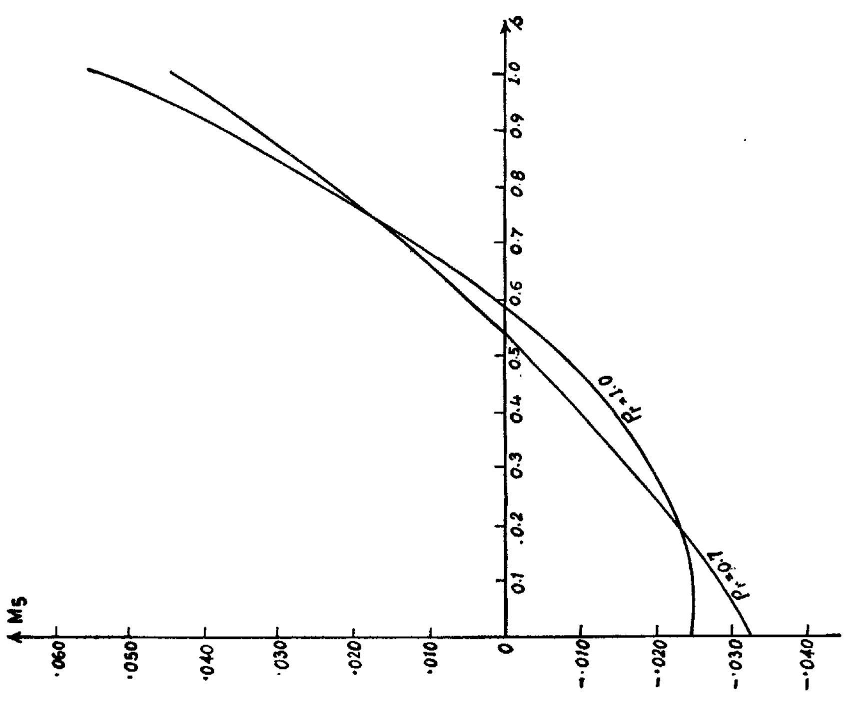

FIG. 5. Plot of M_5' for the sphere for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

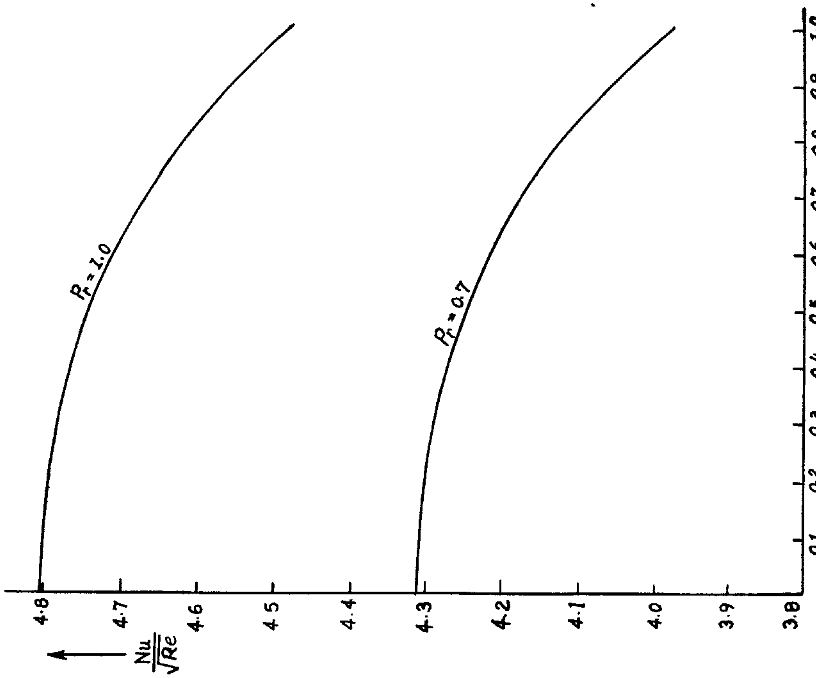

FIG. 6. Variation of $\frac{\mathrm{Nu}}{\sqrt{\mathrm{Re}}}$ with Prandtl number (sphere).

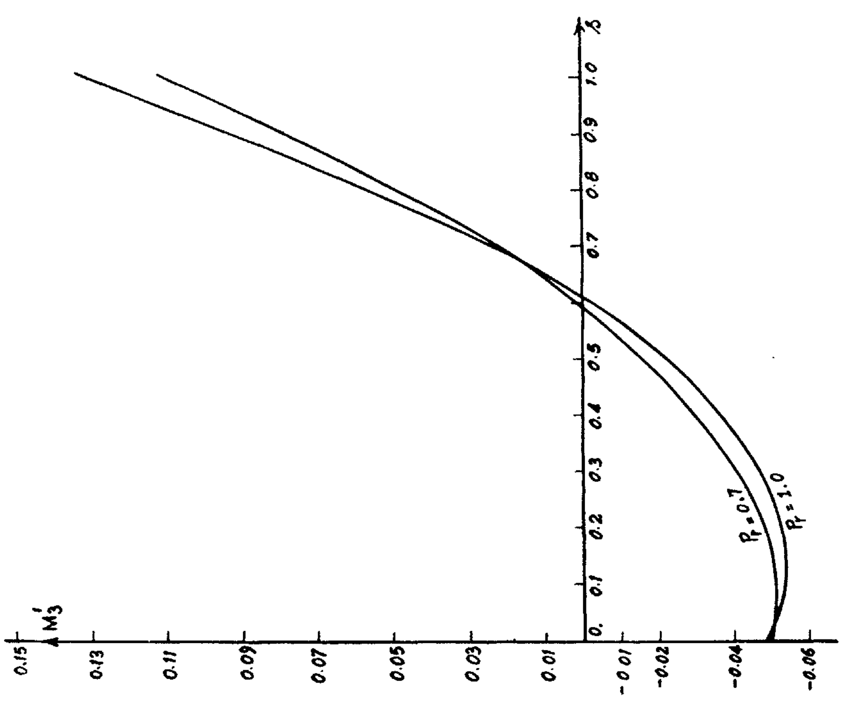

FIG. 7. Plot of M_3' for the prolate spheroid for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

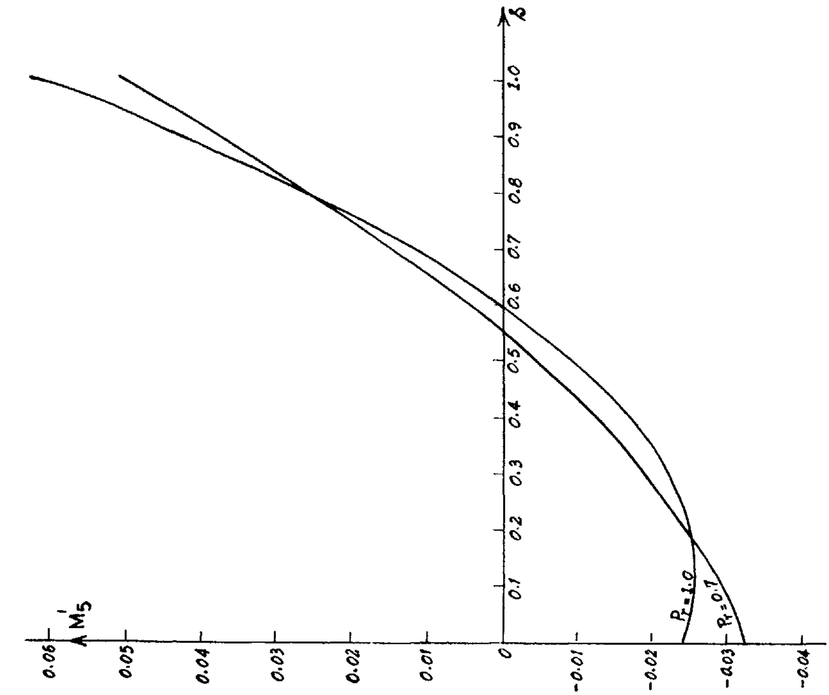

FIG. 8. Plot of M_5' for the prolate spheroid for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

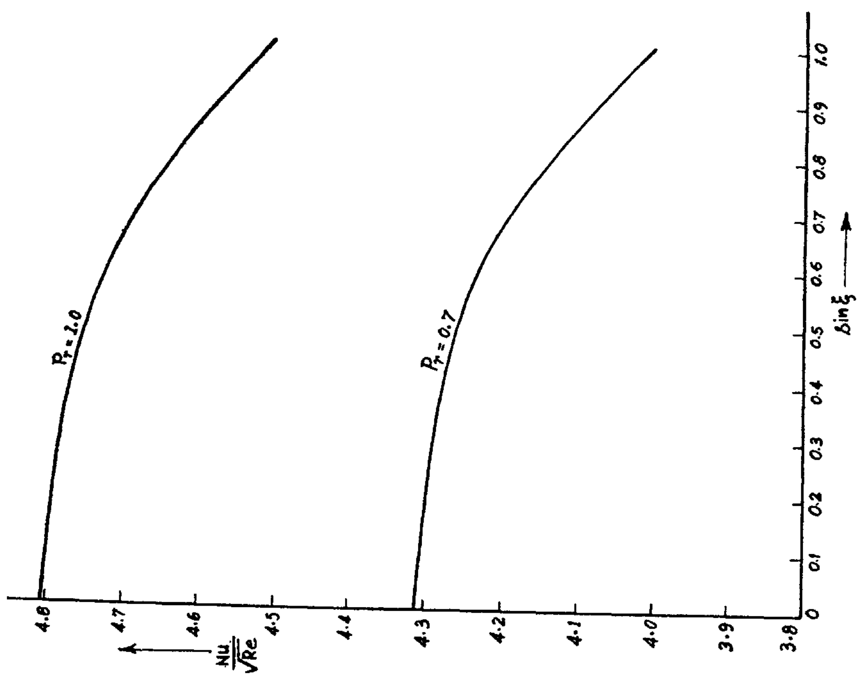

FIG. 9. Variation of $\frac{\text{Nu}}{\sqrt{\text{Re}}}$ with Prandtl number (prolate spheroid).

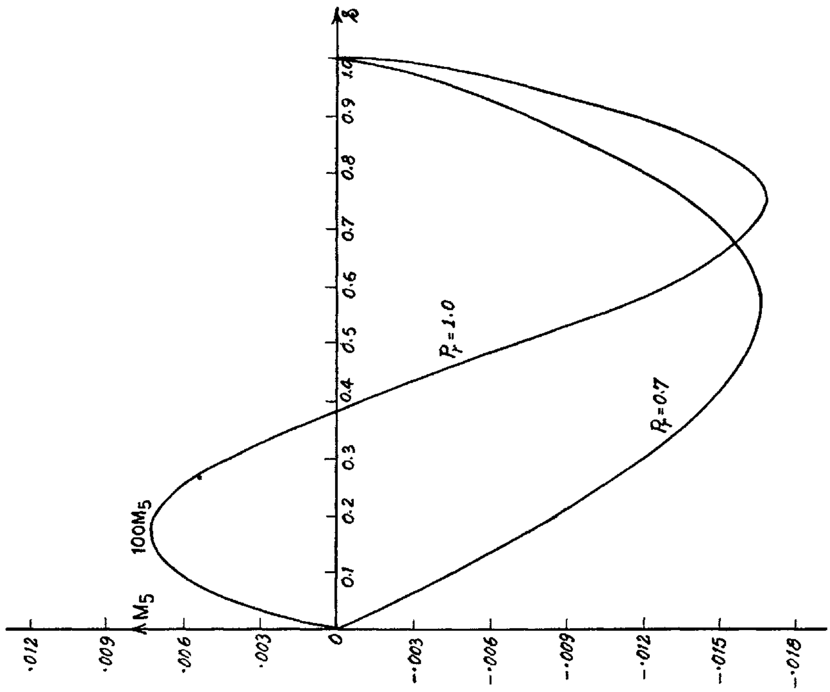

FIG. 10. Temperature profile M_5 *vs* s for the paraboloid for $P_r = 0{\cdot}7, 1{\cdot}0$.

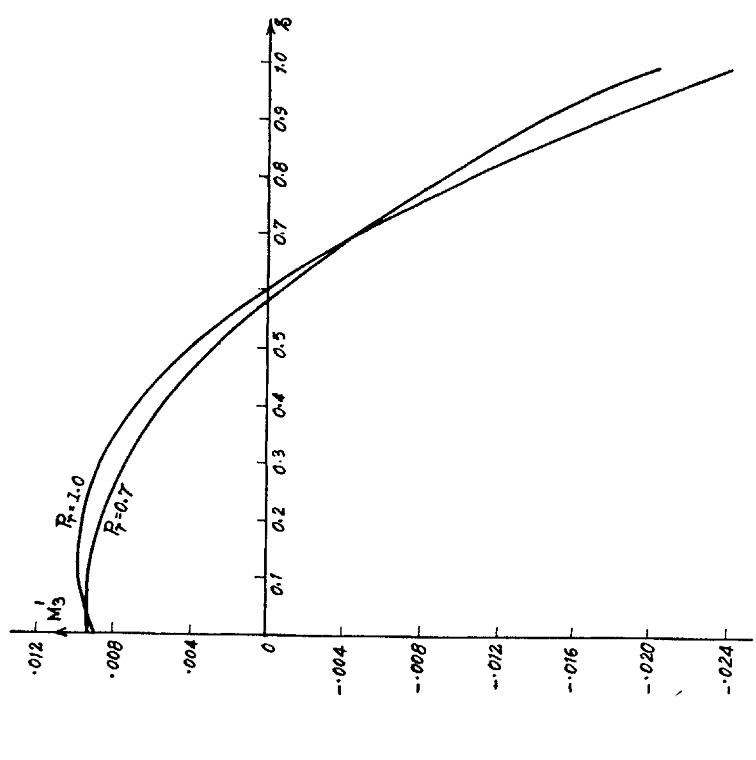

FIG. 11. Plot of M_3' for the paraboloid for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

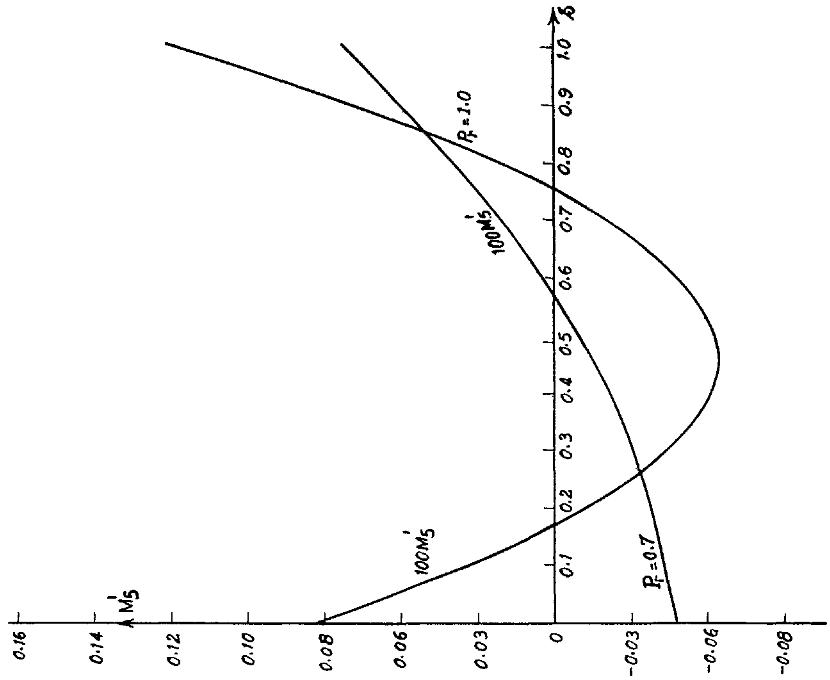

FIG. 12. Plot of M_5' for the paraboloid for $P_r = 0{\cdot}7,\ 1{\cdot}0$.

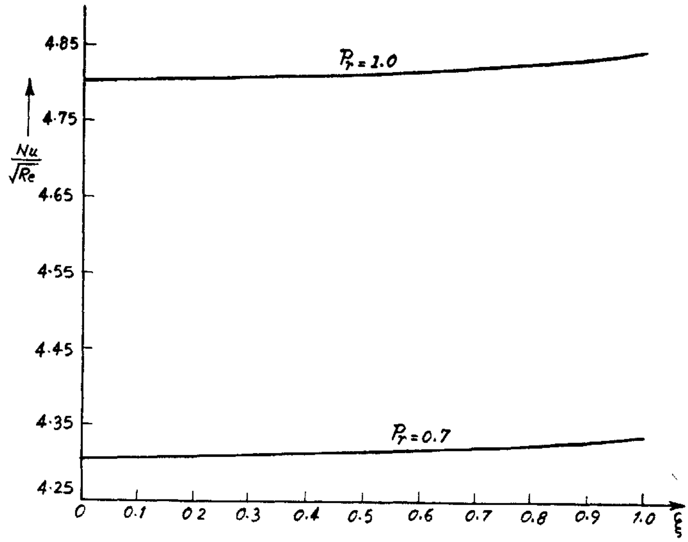

FIG. 13. Variation of $\frac{\text{Nu}}{\sqrt{\text{Re}}}$ with Prandtl number (paraboloid).

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