ON A UNIVERSE FIELD WITH INCOHERENT MATTER AND ELECTROMAGNETIC RADIATION

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A solution of Einstein's field equations with incoherent matter and electromagnetic radiation has been presented. It is the generalized form of a solution previously obtained by Ozsvath. The universe represented by it has zero expansion and shear but non-zero rotation.

1. Introduction

Construction of world models is an important problem in general relativity. But the solution of field equations corresponding to the most general case is a very complicated problem and, therefore, it is advisable to construct more restricted models. Recently, Ozsvath (1966) obtained a solution of field equations representing an universe filled with dust and electromagnetic radiation. In this paper we obtain a more general solution than that of Ozsvath with a different technique. We make use of tetrad formalism to construct a metric which satisfies Einstein's field equations. The model has zero expansion and shear but non-zero rotation.

The Einstein-Maxwell equations corresponding to this problem are

$$R_{jk}-\tfrac{1}{2}g_{jk}R = -\mu u_j u_k-\lambda g_{jk}+F_{jk} \quad \ldots \quad (1.1)$$

$$F_{jk} \equiv 2f_{jl}f^{l}_{k} \quad \ldots \quad (1.2)$$

$$f^{jk}_{;k} = 0, \; {}^{*}f^{jk}_{;k} = 0 \quad \ldots \quad (1.3)$$

where μ is the rest energy of the matter and λ is cosmological constant. u_j is unit velocity vector field tangent to the paths of the dust particles and satisfies the following equation

$$u_j u^j = 1 \quad \ldots \quad (1.4)$$

f^{jk} is the electromagnetic field tensor. Here and in what follows the Latin letters $j, k \ldots$, etc., shall be used as tensor indices and $a, b \ldots$, etc., as labels.*

Suppose that our operational space is the normal hyperbolic Riemannian

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* Small Latin indices $a, b \ldots.$ assume the values from 1 to 4 while $j, k \ldots.$ run over I to IV. Both follow the summation convention.

manifold V_4 with signature $- - - +$. In V_4 a set of four mutually perpendicular covariant vectors e^a_j can always be defined, such that

$$e^a_j e^k_a = \delta^k_j, \quad e^a_j e^j_b = \delta^a_b. \qquad (1.5)$$

Using e^a_j and e^j_a as tetrad vectors we may obtain the tetrad components of any tensor (say, R_{jk}) in V_4 with the help of the relation

$$R_{ab} = e^j_a e^k_b R_{jk}, \quad R_{jk} = R_{ab} e^a_j e^b_k. \qquad (1.6)$$

The metric tensor g_{jk} is then obtained as

$$g_{jk} = e^a_j e^b_k g_{ab}. \qquad (1.7)$$

We may choose g_{ab} as

$$g_{ab} = g^{ab} = (-1, -1, -1, 1). \qquad (1.8)$$

The labels may be raised (lowered) with the help of g^{ab} (g_{ab}). Thus we solve the field equations (1.1)–(1.3) in tetrad form.

2. The Metric and the Field Equations

Let the line element be chosen in the following form:

$$ds^2 = -(dx^{\mathrm{I}})^2 - Y^2(dx^{\mathrm{II}})^2 + 2X dx^{\mathrm{II}} dx^{\mathrm{III}} + (dx^{\mathrm{III}})^2 - (dx^{\mathrm{IV}})^2 \qquad (2.1)$$

where X and Y are functions of x^{I} only. For this line element the non-vanishing components of the Christoffel symbols Γ^i_{kl} are

$$\left.\begin{aligned}
&\Gamma^{\mathrm{I}}_{\mathrm{II\,II}} = -YY_{\mathrm{I}}, && \Gamma^{\mathrm{II}}_{\mathrm{I\,III}} = -\tfrac{1}{2}\frac{X_{\mathrm{I}}}{X^2+Y^2} \\
&\Gamma^{\mathrm{I}}_{\mathrm{II\,III}} = \tfrac{1}{2}X_{\mathrm{I}}, && \Gamma^{\mathrm{III}}_{\mathrm{I\,II}} = \tfrac{1}{2}\frac{Y^2X_{\mathrm{I}} - 2XYY_{\mathrm{I}}}{X^2+Y^2} \\
&\Gamma^{\mathrm{II}}_{\mathrm{I\,II}} = \tfrac{1}{2}\frac{XX_{\mathrm{I}} + 2YY_{\mathrm{I}}}{X^2+Y^2}, && \Gamma^{\mathrm{III}}_{\mathrm{I\,III}} = \tfrac{1}{2}\frac{XX_{\mathrm{I}}}{X^2+Y^2}.
\end{aligned}\right\} \qquad (2.2)$$

The subscripts denote the differentiation with respect to x^{I}. The components of the contracted curvature tensor R_{ij} may now easily be computed. These are found to be

$$\left.\begin{aligned}
R_{\mathrm{II}} &= \tfrac{1}{2}\frac{X^2_{\mathrm{I}} + 2Y^2_{\mathrm{I}} + 2XX_{\mathrm{II}} + 2YY_{\mathrm{II}}}{X^2+Y^2} - \left(\frac{XX_{\mathrm{I}} + YY_{\mathrm{I}}}{X^2+Y^2}\right)^2 \\
R_{\mathrm{II\,II}} &= YY_{\mathrm{II}} + \tfrac{1}{2}\frac{2Y^2_{\mathrm{I}}X^2 + Y^2X^2_{\mathrm{I}} - 2XYX_{\mathrm{I}}Y_{\mathrm{I}}}{X^2+Y^2} \\
R_{\mathrm{II\,III}} &= \tfrac{1}{2}\left[-X_{\mathrm{II}} + \frac{X_{\mathrm{I}}YY_{\mathrm{I}}}{X^2+Y^2}\right] \\
R_{\mathrm{III\,III}} &= -\tfrac{1}{2}\frac{X^2_{\mathrm{I}}}{X^2+Y^2}.
\end{aligned}\right\} \qquad (2.3)$$

Corresponding to the line element (2.1) the tetrad vectors may be chosen as

$$\left.\begin{aligned} e^1_j &= \left(-\cos x,\ \sqrt{X^2+Y^2}\sin x,\ 0,\ 0\right) \\ e^2_j &= \left(\sin x,\ \sqrt{X^2+Y^2}\cos x,\ 0,\ 0\right) \\ e^3_j &= \frac{1}{2\sqrt{-p}}\left(0,\ (p+1)X,\ (p+1),\ (p-1)\right) \\ e^4_j &= \frac{1}{2\sqrt{-p}}\left(0,\ (p-1)X,\ (p-1),\ (p+1)\right) \end{aligned}\right\} \quad \ldots \quad (2.4)$$

and

$$\left.\begin{aligned} e^j_1 &= \left(-\cos x,\ \frac{\sin x}{\sqrt{X^2+Y^2}},\ \frac{-X\sin x}{\sqrt{X^2+Y^2}},\ 0\right) \\ e^j_2 &= \left(\sin x,\ \frac{\cos x}{\sqrt{X^2+Y^2}},\ \frac{-X\cos x}{\sqrt{X^2+Y^2}},\ 0\right) \\ e^j_3 &= \frac{1}{2\sqrt{-p}}\left(0,\ 0,\ -(p+1),\ (p-1)\right) \\ e^j_4 &= \frac{1}{2\sqrt{-p}}\left(0,\ 0,\ (p-1),\ -(p+1)\right) \end{aligned}\right\} \quad \ldots \quad (2.5)$$

where x is a function of coordinates to be specified by Maxwell equations and p is a constant such that $p \leqslant -1$. We further express the electromagnetic field tensor f_{jk} as

$$f_{jk} \equiv k_{[j}l_{k]} \quad \ldots \quad (2.6)$$

with

$$k_j = q\left(-e^3_j+e^4_j\right) = \frac{q}{\sqrt{-p}}(0,\ -X,\ -1,\ 1) \quad \ldots \quad (2.7)$$

and

$$l_j = \left(\sin x,\ \sqrt{X^2+Y^2}\cos x,\ 0\ ,\ 0\right) \quad \ldots \quad (2.8)$$

where q is a constant to be determined later.

We may solve for the functions X and Y satisfying (1.1)–(1.3) in the coordinate frame of (2.4) and (2.5). For this purpose we need the tetrad components of k_j, u_j, l_j and f_{jk}. Thus we easily obtain

$$\left.\begin{aligned} ka &= k_j e^j_a = q(0,\ 0,\ -1,\ 1) \\ la &= l_j e^j_a = (0,\ 1,\ 0,\ 0) \\ u_a &= u_j e^j_a = (0,\ 0,\ 0,\ 1) \\ f_{ab} &= k_{[a}l_{b]}. \end{aligned}\right\} \quad \ldots \quad (2.9)$$

The equations which must be satisfied are

$$R_{11}+\tfrac{1}{2}R = \lambda \quad \ldots \quad (2.10)$$

$$R_{12} = R_{13} = R_{14} = R_{23} = R_{24} = 0 \quad .. \quad (2.11)$$

$$R_{22}+\tfrac{1}{2}R = \lambda \quad .. \quad .. \quad .. \quad .. \quad (2.12)$$

$$R_{33}+\tfrac{1}{2}R = \lambda-\tfrac{1}{2}q^2 \quad .. \quad .. \quad .. \quad .. \quad (2.13)$$

$$R_{34} = \tfrac{1}{2}q^2 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.14)$$

$$R_{44}-\tfrac{1}{2}R = -\mu-\lambda-\tfrac{1}{2}q^2. \quad .. \quad .. \quad (2.15)$$

The functional form of R_{jk}'s may be substituted in eqns. (2.10)–(2.15) to obtain differential equations for the functions X and Y. From (2.11) we at once obtain

$$R_{\text{II III}}-XR_{\text{III III}} = 0 \quad .. \quad .. \quad .. \quad (2.16)$$

which is equivalent to the following in view of eqns. (1.6), (2.3), (2.4), (2.5) and (2.9):

$$X_{\text{I}}^2 = K^2(X^2+Y^2) \quad .. \quad .. \quad .. \quad .. \quad (2.17)$$

where K is constant of integration. The same equation is also obtained from eqns. (2.10) and (2.12). The constant q is determined from eqn. (2.14) as

$$q^2 = -\frac{(p+1)(p-1)}{4p}K^2. \quad .. \quad .. \quad .. \quad (2.18)$$

Further, eqns. (2.10) and (2.13) combine to give

$$R_{11}-R_{33} = \tfrac{1}{2}q^2 = R_{34}$$

which on substitution yields

$$X_{\text{III}}-K^2\frac{3p+1}{4p}X_{\text{I}} = 0. \quad .. \quad .. \quad .. \quad (2.19)$$

Equation (2.10) is integrated to give

$$X = A\exp(ax^{\text{I}})+B\exp(-ax^{\text{I}})+C \quad .. \quad .. \quad (2.20)$$

where A, B and C are integration constants and $a^2 = K^2\dfrac{3p+1}{4p}$. In view of eqns. (2.20) and (2.17) we easily obtain

$$Y^2 = \frac{3p+1}{4p}\left[A\exp(ax^{\text{I}})-B\exp(-ax^{\text{I}})\right]^2-X^2 \quad .. \quad (2.21)$$

we also have

$$\lambda = -\tfrac{1}{4}K^2 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.22)$$

and

$$\left.\begin{aligned} \mu &= -\tfrac{1}{2}pK^2 \\ R &= \mu+4\lambda = -\frac{2p+1}{2p}K^2. \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (2.23)$$

Thus we obtain the appropriate metric in the form

$$ds^2 = -(dx^{\mathrm{I}})^2 - \left[\frac{3p+1}{4p}\{A\exp(ax^{\mathrm{I}}) - B\exp(-ax^{\mathrm{I}})\}^2 - \{A\exp(ax^{\mathrm{I}}) + B\exp(-ax^{\mathrm{I}}) + c\}^2\right](dx^{\mathrm{II}})^2$$
$$+2\{A\exp(ax^{\mathrm{I}}) + B\exp(-ax^{\mathrm{I}}) + c\}\,dx^{\mathrm{II}}\,dx^{\mathrm{III}}$$
$$+(dx^{\mathrm{III}})^2 - (dx^{\mathrm{IV}})^2. \quad \ldots \quad (2.24)$$

However, it has yet to be seen that with the above choice of the functions X and Y the Maxwell equations are satisfied. The non-vanishing components of the electromagnetic field tensor are

$$\left.\begin{aligned} f_{\mathrm{I\,II}} &= \tfrac{1}{2}\frac{q}{\sqrt{-p}}X\sin x \\ f_{\mathrm{I\,III}} &= \tfrac{1}{2}\frac{q}{\sqrt{-p}}\sin x \\ f_{\mathrm{I\,IV}} &= -\tfrac{1}{2}\frac{q}{\sqrt{-p}}\sin x \\ f_{\mathrm{II\,III}} &= \tfrac{1}{2}\frac{q}{\sqrt{-p}}\sqrt{X^2+Y^2}\cos x \\ f_{\mathrm{II\,IV}} &= -\tfrac{1}{2}\frac{q}{\sqrt{-p}}\sqrt{X^2+Y^2}\cos x \end{aligned}\right] \quad \ldots \quad (2.25)$$

and

$$\left.\begin{aligned} f^{\mathrm{I\,III}} &= -\tfrac{1}{2}\frac{q}{\sqrt{-p}}\sin x \\ f^{\mathrm{I\,IV}} &= -\tfrac{1}{2}\frac{q}{\sqrt{-p}}\sin x \\ f^{\mathrm{II\,III}} &= -\tfrac{1}{2}\frac{q}{\sqrt{-p}}\frac{\cos x}{\sqrt{X^2+Y^2}} \\ f^{\mathrm{II\,IV}} &= -\tfrac{1}{2}\frac{q}{\sqrt{-p}}\frac{\cos x}{\sqrt{X^2+Y^2}} \\ f^{\mathrm{III\,IV}} &= \tfrac{1}{2}\frac{q}{\sqrt{-p}}\frac{X\cos x}{\sqrt{X^2+Y^2}}. \end{aligned}\right\} \quad \ldots \quad (2.26)$$

It is easily seen that all the eight Maxwell equations (1.3) are satisfied provided we choose x such that

$$\left.\begin{aligned} \frac{\partial x}{\partial x^{\mathrm{III}}} &= -\frac{\partial x}{\partial x^{\mathrm{IV}}} = -K\frac{3p+1}{4p} \\ \frac{\partial x}{\partial x^{\mathrm{I}}} &= \frac{\partial x}{\partial x^{\mathrm{II}}} = 0. \end{aligned}\right\} \quad \ldots \quad (2.27)$$

We, therefore, make the following choice for x:

$$x = -K\frac{3p+1}{4p}(x^{\mathrm{III}}-x^{\mathrm{IV}}). \qquad \ldots \quad \ldots \quad \ldots \quad (2.28)$$

3. Some Properties of the Solution

The kinematical behaviour of a congruence is locally described by the skewsymmetric tensor of rotation ω_{jk}, the symmetric and traceless tensor σ_{jk} of shear, the scalar of expansion θ and by the vector of acceleration $\dot{u}_j \equiv u_{j;\,k}u^k$. Since ω_{jk} is a simple bi-vector, it may be replaced by a space-like vector given by

$$\omega^j \equiv \tfrac{1}{2}\eta^{jklm}\omega_{kl}u_m.$$

We then obtain for the null vector k_j the following relation:

$$\left.\begin{aligned} \sigma(k) &= \{\tfrac{1}{2}k_{(i;j)}k^{(i;j)}\}^{\frac{1}{2}} = 0 \\ \theta(k) &= k^j{}_{;j} = 0 \\ \dot{k}_j &= k_{j;k}k^k = 0 \\ \omega(k) &= \{\tfrac{1}{2}k_{[j;k]}k^{[j;k]}\}^{\frac{1}{2}} = \frac{q}{2\sqrt{-p}}K. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Therefore, the rays are geodesic, shear-free, expansion-free but not hypersurface orthogonal. Similarly, for the kinematical behaviour of dust particles we have

$$\left.\begin{aligned} \sigma(u) &= \{\tfrac{1}{2}u_{(j;k)}u^{(j;k)}\}^{\frac{1}{2}} = 0 \\ \theta(u) &= u^j{}_{;j} = 0 \\ \dot{u}_j &= u_{j;k}u^k = 0 \\ \omega(u) &= (\omega_j\omega^j)^{\frac{1}{2}} = \{\tfrac{1}{2}u_{[j;k]}u^{[j;k]}\}^{\frac{1}{2}} = \frac{p-1}{4\sqrt{-p}}K. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.2)$$

Thus the worldliness of the dust particles are expansion-free, geodesic, shear-free. The vector of rotation is obtained as

$$\omega^a = \left(0,\, 0,\, \frac{p-1}{4\sqrt{-p}}K,\, 0\right) \qquad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

Further, the non-vanishing components of the Weyl tensor are

$$\left.\begin{aligned} c_{12\,12} &= -c_{34\,34} = -\frac{p-1}{12p}K^2 \\ c_{13\,13} &= c_{23\,23} = -c_{14\,14} = -c_{24\,24} = \frac{p-1}{24p}K^2. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.4)$$

The Riemannian space characterized by eqn. (2.24) is, therefore, Petrov type I degenerate.

The group properties of this solution may be studied by defining the operators $A_a \equiv e^j_a \frac{\partial}{\partial x^j}$ which satisfy the following equation (Heckmann and Schücking 1962):

$$[A_a, A_b] = (A_a, A_b - A_b A_a) = C^f_{ab} A_f \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

C^f_{ab} are known as structure constants of the group and are defined as

$$C^f_{ab} \equiv (e^f_{j,k} - e^f_{k,j}) e^j_a e^k_b. \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

Using eqns. (2.4), (2.5), (2.20), (2.21), (3.5) and (3.6), eqn. (2.24) forms a four-parametric group given by

$$\left.\begin{aligned}
[A_1, A_2] &= \frac{K}{2\sqrt{-p}} \{(p+1)A_3 + (p-1)A_4\} \\
[A_2, A_3] &= \frac{K}{2\sqrt{-p}} \frac{3p+1}{2p} A_1 \\
[A_3, A_1] &= \frac{K}{2\sqrt{-p}} \frac{3p+1}{2p} A_2 \\
[A_1, A_4] &= -\frac{K}{2\sqrt{-p}} \frac{3p+1}{2p} A_2 \\
[A_2, A_4] &= \frac{K}{2\sqrt{-p}} \frac{3p+1}{2p} A_1.
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.7)$$

Further group properties of (2.24) may be studied by Ozsvath's method (1966).

4. Discussion and Conclusion

Our solution shows the gravitational interaction between incoherent matter and electromagnetic radiation. If we choose $p = -1$, the electromagnetic field vanishes [$q = 0$ from eqn. (2.18)] and the solution reduces to a solution of the field equations obtained by Wright (1965). Assuming that $B = C = 0$ and $A = a = 1$ one obtains Gödel (1949) metric. It is easier to understand the properties of Gödel's universe if one studies this general solution. If we choose $p < -1$ and $B = C = 0$ and $A = a = 1$, our solution is equivalent to that of Ozsvath. Therefore, the metric (2.24) is a general form of the solution discussed by Ozsvath (1966).

Ozsvath (1966) has stated the following theorem: 'There exist only two homogeneous solutions of (1.1) (obtained by him) with vanishing shear and non-vanishing density of matter.' Evidently the theorem should be considered to include more general metric (2.24).

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