STRESSES DUE TO A NUCLEUS OF THERMOELASTIC STRAIN IN A COMPOSITE SHELL

by P. R. Ghosh, *Department of Mathematics, Vidyāsagar Evening College, 39 Sankar Ghosh Lane, Calcutta 6*

(Communicated by R. S. Varma, F.N.I.)

(*Received 14 August 1968*)

Stresses in a hollow composite sphere composed of two different materials, due to a nucleus of thermoelastic strain, have been found in series form. The radial and the shearing stresses have been calculated numerically for some points on the sphere.

Nomenclature

The following nomenclature is used in the paper:

r, θ, ϕ = spherical polar coordinates of a point

u_r, u_θ, u_ϕ = components of displacement in the case of the inner material in spherical polar coordinates

u'_r, u'_θ, u'_ϕ = components of displacement in the case of the outer material

$\left.\begin{matrix}\epsilon_r, \epsilon_\theta, \epsilon_\phi \\ \gamma_{r\theta}, \gamma_{\theta\phi}, \gamma_{\phi r}\end{matrix}\right\}$ = components of the thermal strain-tensor in the case of the inner material

$\left.\begin{matrix}\epsilon'_r, \epsilon'_\theta, \epsilon'_\phi \\ \gamma'_{r\theta}, \gamma'_{\theta\phi}, \gamma'_{\phi r}\end{matrix}\right\}$ = components of the thermal strain-tensor in the case of the outer material

$\left.\begin{matrix}\sigma_r, \sigma_\theta, \sigma_\phi \\ \tau_{r\theta}, \tau_{\theta\phi}, \tau_{\phi r}\end{matrix}\right\}$ = components of the thermal stress-tensor in the case of the inner material

$\left.\begin{matrix}\sigma'_r, \sigma'_\theta, \sigma'_\phi \\ \tau'_{r\theta}, \tau'_{\theta\phi}, \tau'_{\phi r}\end{matrix}\right\}$ = components of the thermal stress-tensor in the case of the outer material

α = coefficient of linear expansion of the inner material

α' = coefficient of linear expansion of the outer material

ν = Poisson's ratio of the inner material

ν' = Poisson's ratio of the outer material

μ = coefficient of elasticity in shear in the case of the inner material

μ' = coefficient of elasticity in shear in the case of the outer material

$p = \cos\theta$

$\bar{p} = \sin\theta$

$$P'_n = \frac{d}{dp} P_n(p)$$

Introduction

The thermal stress problem of an infinite elastic solid at zero temperature except for a heated region has been solved by Goodier (1937). Stresses due to a nucleus of thermoelastic strain in an infinite solid with spherical cavity and in a solid elastic sphere have been obtained by Sharma (1957). Nowacki (1962) has extended the results to the case of a hollow sphere, i.e. a shell.

In this paper the hollow sphere is considered to be composed of two different elastic materials. In the first case the nucleus is taken in the 'inner material' while in the second case it is taken in the 'outer material'. We consider the axisymmetric character of the problem.

The radial and the shearing stresses have been calculated numerically at some points on the sphere and are shown in graphs. In the numerical calculations, wrought iron has been taken as the 'inner material', while steel has been taken as the 'outer material'.

Solution

The origin is taken at the centre of the shell. Let the outer radius of the hollow sphere be a and the inner radius of the hollow sphere be b. The shell is made of two different materials, i.e. the material of $b \leqslant r \leqslant d$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ is different from that of $d \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ where a, b and d have some finite values. We call the first material as the 'inner material' and the second as the 'outer material'.

Let us consider that the shell is at zero temperature while an element of volume $d\Omega$ at $(0, 0, c)$ is heated to a temperature T. In this case, the displacement is given by the gradient of the function χ where

$$\chi = \frac{A}{2\mu} \cdot \frac{1}{\sqrt{r^2 - 2rc\cos\theta + c^2}} \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

in which

$$A = -\frac{m\mu}{2\pi} T\, d\Omega \text{ and } m = \frac{1+\nu}{1-\nu}\alpha.$$

It may be noted that a nucleus in the form of a centre of dilatation produces similar effect.

The components of displacement and stress in space polar coordinates due to χ are given by

$$\left.\begin{aligned} \bar{u}_r &= \frac{\partial \chi}{\partial r} = -\frac{A(r - c\cos\theta)}{2\mu(r^2 - 2rc\cos\theta + c^2)^{3/2}} \\ \bar{u}_\theta &= \frac{1}{r}\frac{\partial \chi}{\partial \theta} = -\frac{Ac\sin\theta}{2\mu(r^2 - 2rc\cos\theta + c^2)^{3/2}} \\ \bar{u}_\phi &= 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and

$$\left.\begin{aligned}
\bar{\sigma}_r &= 2\mu\frac{\partial \bar{u}_r}{\partial r} = A\,\frac{2r^2+3c^2\cos^2\theta-4rc\cos\theta-c^2}{(r^2-2rc\cos\theta+c^2)^{\frac{5}{2}}}\\
\bar{\sigma}_\theta &= \frac{2\mu}{r}\left(\bar{u}_r+\frac{\partial \bar{u}_\theta}{\partial r}\right) = A\left\{\frac{3c^2\sin^2\theta}{(r^2-2rc\cos\theta+c^2)^{\frac{5}{2}}}-\frac{1}{(r^2-2rc\cos\theta+c^2)^{\frac{3}{2}}}\right\}\\
\bar{\sigma}_\phi &= \frac{2\mu}{r}(\bar{u}_r+\bar{u}_\theta\cot\theta) = -\frac{A}{(r^2-2rc\cos\theta+c^2)^{\frac{3}{2}}}\\
\bar{\tau}_{r\theta} &= \mu\left(\frac{\partial \bar{u}_\theta}{\partial r}-\frac{\bar{u}_\theta}{r}+\frac{1}{r}\frac{\partial \bar{u}_r}{\partial \theta}\right) = A\,.\,\frac{3c\sin\theta(r-c\cos\theta)}{(r^2-2rc\cos\theta+c^2)^{\frac{5}{2}}}\\
\bar{\tau}_{r\phi} &= \bar{\tau}_{\theta\phi} = 0.
\end{aligned}\right\} \qquad (3)$$

We denote this displacement and stress field derived from χ by $[\bar{S}]$.

The stresses in the solid being known we have to think in what way we can free the spherical surface of traction. The problem can be attacked by any one of the various stress function approaches to the axisymmetric problems of elasticity. Here we utilize the approach originated by Boussinesq according to which the general solution of the displacement equation of equilibrium in case of torsion-free rotational symmetry and the absence of body forces, is represented as the sum of two displacement fields,

$$2\mu(u_x, u_y, u_z) = \text{grad}\ \phi \qquad \ldots \quad (4)$$

$$2\mu(u_x, u_y, u_z) = \text{grad}\ (z\psi)-4(1-\nu)(0, 0, \psi) \qquad \ldots \quad (5)$$

where

$$\nabla^2\phi = \nabla^2\psi = 0.$$

In order to get the particular solutions, we introduce the interior and exterior spherical harmonics of integral order as generating stress functions ϕ and ψ (Sternberg *et al.* 1951).

We consider the harmonic stress functions

$$\phi_n(r, \theta) = r^{-n-1}P_n(p), \quad \psi_n(r, \theta) = r^{-n-1}P_n(p) \qquad \ldots \quad (6)$$

and

$$\phi_n'(r, \theta) = r^nP_n(p), \qquad \psi_n'(r, \theta) = r^nP_n(p) \qquad \ldots \quad (7)$$

where P_n denotes the Legendre polynomial of degree n. The stress functions in (6) represent exterior spherical harmonics while those in (7) represent the interior ones. We shall distinguish the particular solutions generated by ϕ_n, ψ_n, ϕ_n', ψ_n' by $[A_n]$, $[E_n]$, $[C_n]$ and $[F_n]$ respectively for the inner material and by $[A_n']$, $[E_n']$, $[C_n']$ and $[F_n']$ respectively for the outer material. For simplification, we replace the solutions $[E_n]$, $[F_n]$ and $[E_n']$, $[F_n']$ by the following linear combinations:

$$[B_n] = (2n+1)\,[E_n]-(n+4-4\nu)\,[A_{n-1}] \qquad \ldots \quad (8)$$

$$[D_n] = (2n+1)\,[F_n]-(n-3+4\nu)\,[C_{n+1}] \qquad \ldots \quad (9)$$

$$[B_n'] = (2n+1)\,[E_n']-(n+4-4\nu')\,[A_{n-1}'] \qquad \ldots \quad (8a)$$

$$[D_n'] = (2n+1)[F_n']-(n-3+4\nu')\,[C_{n+1}'] \qquad \ldots \quad (9a)$$

The displacement and stress-fields corresponding to the solutions $[A_n]$, $[B_n]$, $[C_n]$, $[D_n]$ and $[A'_n]$, $[B'_n]$, $[C'_n]$, $[D'_n]$, with the help of some recurrence relations in P_n (Nowacki 1962), are as follows:

$$\text{For } [A_n]: \left\{ \begin{aligned} 2\mu\, \bar{\bar{u}}_r &= -\frac{n+1}{r^{n+2}} P_n \\ 2\mu\, \bar{\bar{u}}_\theta &= -\frac{1}{r^{n+2}} \bar{p} P'_n \\ \bar{\bar{u}}_\phi &= 0 \\ \bar{\bar{\sigma}}_r &= \frac{(n+1)(n+2)}{r^{n+3}} P_n \\ \bar{\bar{\sigma}}_\theta &= \frac{P'_{n+1}-(n+1)(n+2)}{r^{n+3}} P_n \\ \bar{\bar{\sigma}}_\phi &= -\frac{P'_{n+1}}{r^{n+3}} \\ \bar{\bar{\tau}}_{r\theta} &= \frac{n+2}{r^{n+3}} \bar{p} P'_n \\ \bar{\bar{\tau}}_{r\phi} &= \bar{\bar{\tau}}_{\theta\phi} = 0 \end{aligned} \right. \qquad (10)$$

$$\text{For } [B_n]: \left\{ \begin{aligned} 2\mu\, \bar{\bar{u}}_r &= -\frac{(n+1)(n+4-4\nu)}{r^{n+1}} P_{n+1} \\ 2\mu\, \bar{\bar{u}}_\theta &= -\frac{n-3+4\nu}{r^{n+1}} \bar{p} P'_{n+1} \\ \bar{\bar{u}}_\phi &= 0 \\ \bar{\bar{\sigma}}_r &= \frac{(n+1)\{(n+1)(n+4)-2\nu\}}{r^{n+2}} P_{n+1} \\ \bar{\bar{\sigma}}_\theta &= -\frac{1}{r^{n+2}}\{(n+1)(n^2-n+1-2\nu) P_{n+1}-(n-3+4\nu) P'_n\} \\ \bar{\bar{\sigma}}_\phi &= -\frac{1}{r^{n+2}}\{(1-2\nu)(n+1)(2n+1) P_{n+1}+(n-3+4\nu) P'_n\} \\ \bar{\bar{\tau}}_{r\theta} &= \frac{(n^2+2n-1+2\nu)}{r^{n+2}} \bar{p} P'_{n+1} \\ \bar{\bar{\tau}}_{r\phi} &= \bar{\bar{\tau}}_{\theta\phi} = 0 \end{aligned} \right. \qquad (11)$$

$$\text{For } [C_n]: \left\{ \begin{aligned} 2\mu\, \bar{\bar{u}}_r &= n r^{n-1} P_n \\ 2\mu\, \bar{\bar{u}}_\theta &= -r^{n-1} \bar{p} P'_n \\ \bar{\bar{u}}_\phi &= 0 \\ \bar{\bar{\sigma}}_r &= n(n-1) r^{n-2} P_n \\ \bar{\bar{\sigma}}_\theta &= r^{n-2}\{P'_{n+1}-(n^2+n+1) P_n\} \\ \bar{\bar{\sigma}}_\phi &= -r^{n-2}\{P'_{n+1}-(2n+1) P_n\} \\ \bar{\bar{\tau}}_{r\theta} &= -(n-1) r^{n-2} \bar{p} P'_n \\ \bar{\bar{\tau}}_{r\phi} &= \bar{\bar{\tau}}_{\theta\phi} = 0 \end{aligned} \right. \qquad (12)$$

$$\text{For } [D_n]:\left\{\begin{aligned}
2\mu\,\bar{\bar{u}}_r &= n(n-3+4\nu)\,r^n P_{n-1}\\
2\mu\,\bar{\bar{u}}_\theta &= -(n+4-4\nu)\,r^n\,\bar{p}\,P'_{n-1}\\
\bar{\bar{u}}_\phi &= 0\\
\bar{\bar{\sigma}}_r &= [n^2(n-3)-2\nu n]\,r^{n-1} P_{n-1}\\
\bar{\bar{\sigma}}_\theta &= r^{n-1}\{(n+4-4\nu)\,P'_n - n(n^2+3n+3-2\nu)\,P_{n-1}\}\\
\bar{\bar{\sigma}}_\phi &= -r^{n-1}\{(n+4-4\nu)\,P'_n - n(2n+1)(1-2\nu)\,P_{n-1}\}\\
\bar{\bar{\tau}}_{r\theta} &= (2-n^2-2\nu)\,r^{n-1}\,\bar{p}\,P'_{n-1}\\
\bar{\bar{\tau}}_{r\phi} &= \bar{\bar{\tau}}_{\theta\phi} = 0
\end{aligned}\right. \qquad \text{.. (13)}$$

$$\text{For } [A'_n]:\left\{\begin{aligned}
2\mu'\,\bar{\bar{u}}'_r &= -\frac{n+1}{r^{n+2}}\,P_n\\
2\mu'\,\bar{\bar{u}}'_\theta &= -\frac{1}{r^{n+2}}\,\bar{p}\,P'_n\\
\bar{\bar{u}}'_\phi &= 0\\
\bar{\bar{\sigma}}' &= \frac{(n+1)(n+2)}{r^{n+3}}\,P_n\\
\bar{\bar{\sigma}}'_\theta &= \frac{1}{r^{n+3}}\left[P'_{n+1}-(n+1)(n+2)\,P_n\right]\\
\bar{\bar{\sigma}}'_\phi &= -\frac{P'_{n+1}}{r^{n+3}}\\
\bar{\bar{\tau}}'_{r\theta} &= \frac{n+2}{r^{n+3}}\,\bar{p}\,P'_n\\
\bar{\bar{\tau}}'_{r\phi} &= \bar{\bar{\tau}}'_{\theta\phi} = 0
\end{aligned}\right. \qquad \text{.. .. .. (10a)}$$

$$\text{For } [B'_n]:\left\{\begin{aligned}
2\mu'\,\bar{\bar{u}}'_r &= -\frac{(n+1)(n+4-4\nu')}{r^{n+1}}\,P_{n+1}\\
2\mu'\,\bar{\bar{u}}'_\theta &= -\frac{n-3+4\nu'}{r^{n+1}}\,\bar{p}\,P'_{n+1}\\
\bar{\bar{u}}'_\phi &= 0\\
\bar{\bar{\sigma}}'_r &= \frac{(n+1)\{(n+1)(n+4)-2\nu'\}}{r^{n+2}}\,P_{n+1}\\
\bar{\bar{\sigma}}'_\theta &= -\frac{1}{r^{n+2}}\{(n+1)(n^2-n+1-2\nu')\,P_{n+1}-(n-3+4\nu')\,P'_n\}\\
\bar{\bar{\sigma}}'_\phi &= -\frac{1}{r^{n+2}}\{(1-2\nu')(n+1)(2n+1)\,P_{n+1}+(n-3+4\nu')\,P'_n\}\\
\bar{\bar{\tau}}'_{r\theta} &= \frac{n^2+2n-1+2\nu'}{r^{n+2}}\,\bar{p}\,P'_{n+1}\\
\bar{\bar{\tau}}'_{r\phi} &= \bar{\bar{\tau}}'_{\theta\phi} \doteq 0
\end{aligned}\right. \qquad \text{.. (11a)}$$

$$\text{For } [C_n']: \begin{cases} 2\mu'\bar{\bar{u}}_r' = n\,r^{n-1}P_n \\ 2\mu'\bar{\bar{u}}_\theta' = -r^{n-1}\bar{p}\,P_n' \\ \bar{\bar{u}}_\phi' = 0 \\ \bar{\bar{\sigma}}_r' = n(n-1)\,r^{n-2}P_n \\ \bar{\bar{\sigma}}_\theta' = r^{n-2}\left[P_{n+1}' - (n^2+n+1)\,P_n\right] \\ \bar{\bar{\sigma}}_\phi' = -r^{n-2}\left\{P_{n+1}' - (2n+1)\,P_n\right\} \\ \bar{\bar{\tau}}_{r\theta}' = -(n-1)r^{n-2}\bar{p}\,P_n' \\ \bar{\bar{\tau}}_{r\phi}' = \bar{\bar{\tau}}_{\theta\phi}' = 0 \end{cases} \qquad \ldots \quad (12a)$$

$$\text{For } [D_n']: \begin{cases} 2\mu'\bar{\bar{u}}_r' = n(n-3+4\nu')\,r^n P_{n-1} \\ 2\mu'\bar{\bar{u}}_\theta' = -(n+4-4\nu')\,r^n\bar{p}\,P_{n-1}' \\ \bar{\bar{u}}_\phi' = 0 \\ \bar{\bar{\sigma}}_r' = \{n^2(n-3) - 2\nu' n\}\,r^{n-1}P_{n-1} \\ \bar{\bar{\sigma}}_\theta' = r^{n-1}\left\{(n+4-4\nu')\,P_n' - n(n^2+3n+3-2\nu')\,P_{n-1}\right\} \\ \bar{\bar{\sigma}}_\phi' = -r^{n-1}\left\{(n+4-4\nu')\,P_n' - n(2n+1)(1-2\nu')\,P_{n-1}\right\} \\ \bar{\bar{\tau}}_{r\theta}' = (2-n^2-2\nu')\,r^{n-1}\bar{p}\,P_{n-1}' \\ \bar{\bar{\tau}}_{r\phi}' = \bar{\bar{\tau}}_{\theta\phi}' = 0 \end{cases} \qquad \ldots \quad (13a)$$

Case I—The nucleus of thermoelastic strain is in the inner material.

The heated nucleus is at $(0, 0, c)$ such that $b < c < d$.

The solution of the problem under consideration is represented in the form

$$[S] = [\bar{S}] + [\bar{\bar{S}}]$$

$$= [\bar{S}] + \sum_{n=0}^{\infty} \{a_n[A_n] + b_n[B_{n-1}] + c_n[C_n] + d_n[D_{n+1}]\} \qquad \ldots \quad (14)$$

for the inner material

and $$[S'] = [\bar{\bar{S}}'] = \sum_{n=0}^{\infty} \{a_n'[A_n'] + b_n'[B_{n-1}'] + c_n'[C_n'] + d_n'[D_{n+1}']\} \qquad \ldots \quad (14a)$$

for the outer material

where a_n, b_n, c_n, d_n and a_n', b_n', c_n', d_n' are arbitrary functions of n to be determined from the boundary conditions of the problem.

As the spherical surfaces are free from tractions, the boundary conditions are

$$\sigma_r = \tau_{r\theta} = 0 \quad \text{on} \quad r = b,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi < 2\pi \qquad \ldots \quad (15)$$

$$\sigma_r' = \tau_{r\theta}' = 0 \quad \text{on} \quad r = a,\ 0 < \theta \leqslant \pi,\ 0 < \phi < 2\pi \qquad \ldots \quad (15a)$$

Because of the continuity of the stresses and displacements on the middle sphere, i.e. on $r = d$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, we have the boundary conditions

$$\left.\begin{aligned} \sigma_r &= \sigma'_r \\ \tau_{r\theta} &= \tau'_{r\theta} \\ u_r &= u'_r \\ u_\theta &= u'_\theta \end{aligned}\right\} \quad \text{on } r = d,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi \qquad \ldots \qquad (16)$$

Now from (3),

$$[\bar{\sigma}_r]_{r=b} = A \cdot \frac{2b^2+3c^2\cos^2\theta-4bc\cos\theta-c^2}{(b^2-2bc\cos\theta+c^2)^{\frac{5}{2}}}$$

$$= A \sum_{n=0}^{\infty} \frac{n(n-1)\,P_n(p)}{c^{n+1}}\, b^{n-2} \qquad (\text{because } b < c)$$

and

$$[\bar{\tau}_{r\theta}]_{r=b} = A \cdot \frac{3c\sin\theta\,(b-c\cos\theta)}{(b^2-2bc\cos\theta+c^2)^{\frac{5}{2}}}$$

$$= -A\,\bar{p} \sum_{n=0}^{\infty} \frac{(n-1)\,P'_n(p)}{c^{n+1}}\, b^{n-2} \qquad (\text{because } b < c).$$

Hence from the boundary conditions (15) and (15a) we get by (14) and (14a),

$$\left.\begin{aligned} &A\,\frac{n(n-1)b^{n-2}}{c^{n+1}} + \frac{(n+1)(n+2)}{b^{n+3}}\,a_n + \frac{n(n^2+3n-2\nu)}{b^{n+1}}\,b_n \\ &\quad + n(n-1)b^{n-2}c_n + \{(n+1)(n-2)-2\nu\}(n+1)b^n d_n = 0 \\ &A\,\frac{(n-1)b^{n-2}}{c^{n+1}} - \frac{(n+2)}{b^{n+3}}\,a_n - \frac{n^2+2\nu-2}{b^{n+1}}\,b_n + (n-1)b^{n-2}c_n \\ &\quad -(1-2n-n^2-2\nu)b^n d_n = 0 \\ &\frac{(n+1)(n+2)}{a^{n+3}}\,a'_n + \frac{n(n^2+3n-2\nu')}{a^{n+1}}\,b'_n + n(n-1)\,a^{n-2}\,c'_n \\ &\quad + (n+1)\{(n+1)(n-2)-2\nu'\}a^n d'_n = 0 \\ \text{and}\quad &\frac{n+2}{a^{n+3}}\,a'_n + \frac{n^2-2+2\nu'}{a^{n+1}}\,b'_n - (n-1)a^{n-2}\,c'_n \\ &\quad + (1-2n-n^2-2\nu')a^n d'_n = 0 \end{aligned}\right\} \quad \ldots \quad (17)$$

From (2) and (3) we obtain for $c < d$,

$$[\bar{\sigma}_r]_{r=d} = A \sum_{n=0}^{\infty} \frac{(n+1)(n+2)\,c^n}{d^{n+3}}\, P_n(p)$$

$$[\bar{\tau}_{r\theta}]_{r=d} = A\,\bar{p} \sum_{n=0}^{\infty} \frac{(n+2)\,c^n}{d^{n+3}}\, P'_n(p)$$

$$[\bar{u}_r]_{r=d} = -\frac{A}{2\mu} \sum_{n=0}^{\infty} \frac{(n+1)\,c^n}{d^{n+2}}\, P_n(p)$$

$$[\bar{u}_\theta]_{r=d} = -\frac{A}{2\mu}\,\bar{p} \sum_{n=0}^{\infty} \frac{c^n\,P'_n(p)}{d^{n+2}}.$$

Hence from the boundary conditions (16) we get by (14) and (14a),

$$
\left.
\begin{aligned}
&A\cdot\frac{(n+1)(n+2)}{d^{n+3}}c^n+\frac{(n+1)(n+2)}{d^{n+3}}a_n+\frac{n(n^2+3n-2\nu)}{d^{n+1}}b_n\\
&\qquad+n(n-1)d^{n-2}c_n+(n+1)\{(n+1)(n-2)-2\nu\}d^n d_n\\
&=\frac{(n+1)(n+2)}{d^{n+3}}a_n'+\frac{n(n^2+3n-2\nu')}{d^{n+1}}b_n'+n(n-1)d^{n-2}c_n'\\
&\qquad+(n+1)\{(n+1)(n-2)-2\nu'\}d^n d_n',\\
&A\cdot\frac{(n+2)c^n}{d^{n+3}}+\frac{n+2}{d^{n+3}}a_n+\frac{n^2-2+2\nu}{d^{n+1}}b_n-(n-1)d^{n-2}c_n\\
&\qquad+(1-2n-n^2-2\nu)d^n d_n\\
&=\frac{n+2}{d^{n+3}}a_n'+\frac{n^2-2+2\nu'}{d^{n+1}}b_n'-(n-1)d^{n-2}c_n'+(1-2n-n^2-2\nu')d^n d_n',\\
&A\cdot\frac{(n+1)c^n}{d^{n+2}}+\frac{n+1}{d^{n+2}}a_n+\frac{n(n+3-4\nu)}{d^n}b_n-n\,d^{n-1}c_n-(n+1)(n-2+4\nu)d^{n+1}d_n\\
&=\frac{\mu}{\mu'}\left[\frac{n+1}{d^{n+2}}a_n'+\frac{n(n+3-4\nu')}{d^n}b_n'-n\,d^{n-1}c_n'-(n+1)(n-2+4\nu')d^{n+1}d_n'\right]\\
&\text{and}\\
&\frac{Ac^n}{d^{n+2}}+\frac{a_n}{d^{n+2}}+\frac{n-4+4\nu}{d^n}b_n+d^{n-1}c_n+(n+5-4\nu)d^{n+1}d_n\\
&=\frac{\mu}{\mu'}\left[\frac{a_n'}{d^{n+2}}+\frac{n-4+4\nu'}{d^n}b_n'+d^{n-1}c_n'+(n+5-4\nu')d^{n+1}d_n'\right]
\end{aligned}
\right\}\quad\ldots\ (18)
$$

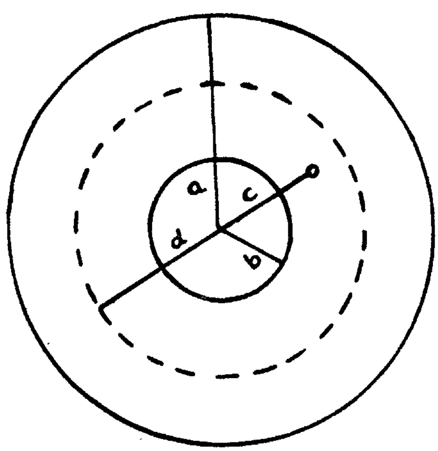

FIG. 1. Nucleus in the inner material.

Solving (17) and (18) by matrices we get the values of a_n, b_n, c_n, d_n and a_n', b_n', c_n', d_n' for $n = 0, 1, 2, \ldots$. The values of these coefficients together with (14), (14a), (2), (3), (10)–(13) and (10a)–(13a) constitute the complete solution of the problem.

Numerical results (in C.G.S. units)

Taking $b = 1$, $c = 1{\cdot}5$, $d = 2$, $a = 3$, and wrought iron as the inner material and steel as the outer material, so that $\nu = 0{\cdot}275$, $\mu = 7{\cdot}69\times10^{11}$ and $\nu' = 0{\cdot}31$, $\mu' = 8{\cdot}19\times10^{11}$, the values of some coefficients are obtained numerically.

c_1 and c_1' cannot be calculated separately because there remain two equations, both of which are identical with $8{\cdot}19c_1 - 7{\cdot}69c_1' = 0{\cdot}15339733A$ to determine c_1 and c_1'. This is due to a rigid body displacement. Thus the solution of the problem is not unique but is arbitrary to the extent of rigid body displacements. The undetermined values of c_1 and c_1' do not affect the calculation of the radial and the shearing stresses, because the coefficients of c_1 and c_1' in σ_r and $\tau_{r\theta}$ are separately zero. If one wishes to obtain c_1 and c_1' completely, an additional condition (e.g. an assumption that the interface suffers no rigid body displacements along x-axis, etc.) is required.

The values of $\frac{1}{A}[\sigma_r]_{r=d}$ and $\frac{1}{A}[\tau_{r\theta}]_{r=d}$ are shown in Figs. 2 and 3 respectively, where

$$A = -\frac{\mu T\alpha}{2\pi}\cdot\frac{1+\nu}{1-\nu}\,d\Omega.$$

Fig. 2 shows that $[\sigma_r]_{r=d}$ is maximum at $\theta = 0$ and as θ increases $[\sigma_r]_{r=d}$ decreases very rapidly and ultimately tends to zero. It is interesting to note from Fig. 3 that $[\tau_{r\theta}]_{r=d}$ is zero at $\theta = 0$ and as θ increases, $[\tau_{r\theta}]_{r=d}$ increases rapidly, reaches a maximum (when θ is nearly 7°) and then continually decreases and ultimately tends to zero.

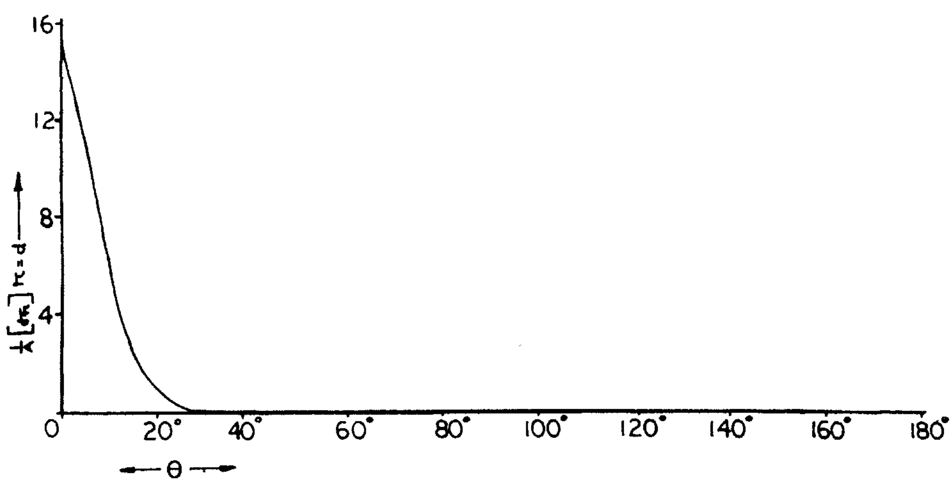

FIG. 2. Radial stress on the middle sphere.

Case II—The nucleus of thermoelastic strain is in the outer material.

The heated nucleus in this case is at $(0, 0, c)$ such that $d < c < a$. Due to the freeness of the spherical surfaces from tractions, we have

$$\sigma_r = \tau_{r\theta} = 0 \quad \text{on } r = b,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi \quad \ldots \quad (19)$$

$$\sigma_r' = \tau_{r\theta}' = 0 \quad \text{on } r = a,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi \quad \ldots \quad (19a)$$

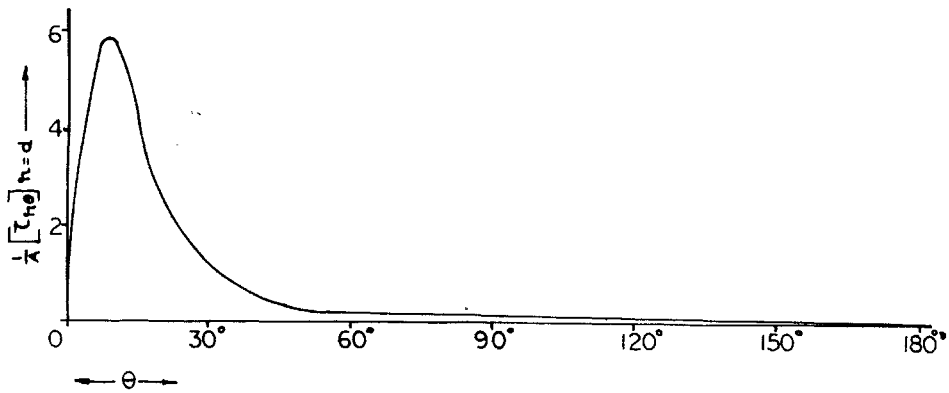

FIG. 3. Shearing stress on the middle sphere.

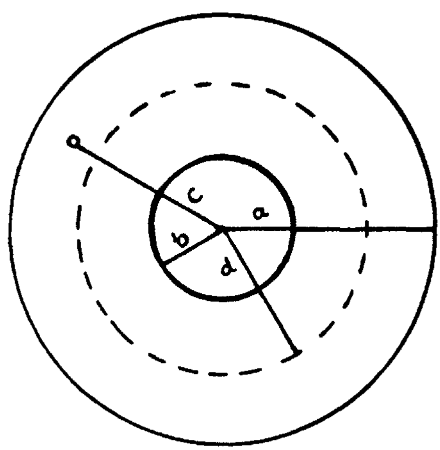

FIG. 4. Nucleus in the outer material.

Because of the continuity of the stresses and the displacements on the middle sphere, i.e. on $r = d$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, we have the boundary conditions

$$\left.\begin{aligned} \sigma_r &= \sigma_r' \\ \tau_{r\theta} &= \tau_{r\theta}' \\ u_r &= u_r' \\ u_\theta &= u_\theta' \end{aligned}\right\} \quad \text{on } r = d,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi \quad \ldots \quad (20)$$

Remembering that all the components of displacement and stress must be finite inside the shell and zero at infinity and satisfying the boundary conditions (19), (19*a*), (20), the solution of the problem under consideration is taken in the form

$$[S] = [\bar{\bar{S}}] = \sum_{n=0}^{\infty} \{e_n[A_n]+f_n[B_{n-1}]+g_n[C_n]+h_n[D_{n+1}]\} \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

for the inner material

and

$$[S'] = [\bar{S}']+[\bar{\bar{S}}'] = [\bar{S}']+ \sum_{n=0}^{\infty} \{e'_n[A'_n]+f'_n[B'_{n-1}]+g'_n[C'_n]+h'_n[D'_{n+1}]\} \qquad (21a)$$

for the outer material

where e_n, f_n, g_n, h_n, e'_n, f'_n, g'_n and h'_n are arbitrary functions of n and may be determined from the boundary conditions (19), (19*a*) and (20).

Now as in (3), we may have easily

$$\left[\bar{\sigma}'_r\right]_{r=a} = A' \frac{2a^2+3c^2 \cos^2\theta-4ac\cos\theta-c^2}{(a^2-2ac\cos\theta+c^2)^{\frac{5}{2}}}$$

$$= A' \sum_{n=0}^{\infty} \frac{(n+1)(n+2)c^n}{a^{n+3}} P_n(p) \quad (\text{because } a > c)$$

where

$$A' = -\frac{\mu' T\alpha'}{2\pi} \cdot \frac{1+\nu'}{1-\nu'} d\Omega$$

and

$$\left[\bar{\tau}'_{r\theta}\right]_{r=a} = A' \frac{3c\sin\theta(a-c\cos\theta)}{(a^2-2ac\cos\theta+c^2)^{\frac{5}{2}}}$$

$$= A'\bar{p} \sum_{n=0}^{\infty} \frac{(n+2)c^n}{a^{n+3}} P'_n(p) \qquad (\text{because } a > c) \quad \ldots \quad (22)$$

Hence from the boundary conditions (19) and (19*a*), we get by (21) and (21*a*)

$$\left.\begin{aligned}
&\frac{(n+1)(n+2)}{b^{n+3}} e_n+\frac{n(n^2+3n-2\nu)}{b^{n+1}} f_n+n(n-1)b^{n-2}g_n+\{(n+1)(n-2)-2\nu\} \\
&\qquad \times(n+1)b^n h_n = 0 \\
&\frac{n+2}{b^{n+3}} e_n+\frac{n^2-2+2\nu}{b^{n+1}} f_n-(n-1)b^{n-2}g_n+(1-2n-n^2-2\nu)b^n h_n = 0 \\
&A'\frac{(n+1)(n+2)c^n}{a^{n+3}}+\frac{(n+1)(n+2)}{a^{n+3}} e'_n+\frac{n(n^2+3n-2\nu')}{a^{n+1}} f'_n+n(n-1)a^{n-2}g'_n \\
&\qquad +(n+1)\{(n+1)(n-2)-2\nu'\}a^n h'_n = 0 \\
&A'\frac{n+2}{a^{n+3}} c^n+\frac{n+2}{a^{n+3}} e'_n+\frac{n^2-2+2\nu'}{a^{n+1}} f'_n-(n-1)a^{n-2}g'_n+(1-2n-n^2-2\nu')a^n h'_n = 0
\end{aligned}\right\}$$

$$\ldots \quad (23)$$

From (2) and (3) we obtain for $c > d$,

$$[\bar{\sigma}'_r]_{r=d} = A' \sum_{n=0}^{\infty} \frac{n(n-1)P_n}{c^{n+1}} d^{n-2}$$

$$[\bar{\tau}'_{r\theta}]_{r=d} = -A'\tilde{p} \sum_{n=0}^{\infty} \frac{(n-1)\, P'_n}{c^{n+1}} d^{n-2}$$

$$[\bar{u}'_r]_{r=d} = \frac{A'}{2\mu'} \sum_{n=0}^{\infty} \frac{n\, P_n}{c^{n+1}} d^{n-1}$$

$$[\bar{u}'_\theta]_{r=d} = -\frac{A'}{2\mu'} \tilde{p} \sum_{n=0}^{\infty} \frac{P'_n}{c^{n+1}} d^{n-1}.$$

Hence from the boundary conditions (20), we get with the help of (21) and (21a),

$$\left.\begin{aligned}
&\frac{(n+1)(n+2)}{d^{n+3}} e_n + \frac{n(n^2+3n-2\nu)}{d^{n+1}} f_n + n(n-1)d^{n-2}g_n + (n+1)\{(n+1)(n-2)-2\nu\}d^n h_n \\
&= A' \frac{n(n-1)d^{n-2}}{c^{n+1}} + \frac{(n+1)(n+2)}{d^{n+3}} e'_n + \frac{n(n^2+3n-2\nu')}{d^{n+1}} f'_n + n(n-1)d^{n-2}g'_n \\
&\qquad + (n+1)\{(n+1)(n-2)-2\nu'\}d^n h'_n, \\
&\frac{n+2}{d^{n+3}} e_n + \frac{n^2-2+2\nu}{d^{n+1}} f_n - (n-1)d^{n-2}g_n + (1-2n-n^2-2\nu)d^n h_n \\
&= -A' \frac{(n-1)d^{n-2}}{c^{n+1}} + \frac{n+2}{d^{n+3}} e'_n + \frac{n^2-2+2\nu'}{d^{n+1}} f'_n - (n-1)d^{n-2}g'_n + (1-2n-n^2-2\nu')d^n h'_n, \\
&\frac{n+1}{d^{n+2}} e_n + \frac{n(n+3-4\nu)}{d^n} f_n - n\, d^{n-1}g_n - (n+1)(n-2+4\nu)d^{n+1}h_n \\
&= \frac{\mu}{\mu'}\left[-A \cdot \frac{n\, d^{n-1}}{c^{n+1}} + \frac{n+1}{d^{n+2}} e'_n + \frac{n(n+3-4\nu')}{d^n} f'_n - n\, d^{n-1}g'_n - (n+1)(n-2+4\nu')d^{n+1}h'_n\right] \\
&\text{and} \\
&\frac{e_n}{d^{n+2}} + \frac{n-4+4\nu}{d^n} f_n + d^{n-1}g_n + (n+5-4\nu)d^{n+1}h_n \\
&= \frac{\mu}{\mu'}\left[A' \frac{d^{n-1}}{c^{n+1}} + \frac{e'_n}{d^{n+2}} + \frac{n-4+4\nu'}{d^n} f'_n + d^{n-1}g'_n + (n+5-4\nu')d^{n+1}h'_n\right]
\end{aligned}\right\} \quad (24)$$

Solving (23) and (24) by matrices we get the values of e_n, f_n, g_n, h_n and e'_n, f'_n, g'_n, h'_n for $n = 0, 1, 2, \ldots$. The values of these coefficients together with the eqns. (21), (21a), (10)–(13), (10a)–(13a) constitute the complete solution of the problem.

Numerical results [*in C.G.S. units*]

Taking $b = 1$, $d = 2$, $c = 2{\cdot}5$, $a = 3$ and as in the first case taking wrought iron as the inner material and steel as the outer material so that $\nu = 0{\cdot}275$,

$\mu = 7{\cdot}69 \times 10^{11}$ and $\nu' = 0{\cdot}31$, $\mu' = 8{\cdot}19 \times 10^{11}$, the values of some coefficients are obtained numerically.

g_1 and g_1' cannot be calculated separately. The equation $8{\cdot}19g_1 - 7{\cdot}69g_1' = 1{\cdot}26521753A'$ is only left for the calculation of g_1 and g_1'. This is due to a rigid body displacement and the remarks made in the previous case may be extended to this case also. The undetermined values of g_1 and g_1' do not disturb to calculate the radial and shearing stresses, because the coefficients of g_1 and g_1' in σ_r or $\tau_{r\theta}$ are separately zero.

The values of $\frac{[\sigma_r]_{r=d}}{A'}$, $\frac{[\tau_{r\theta}]_{r=d}}{A'}$ are shown in Figs. 5 and 6 respectively

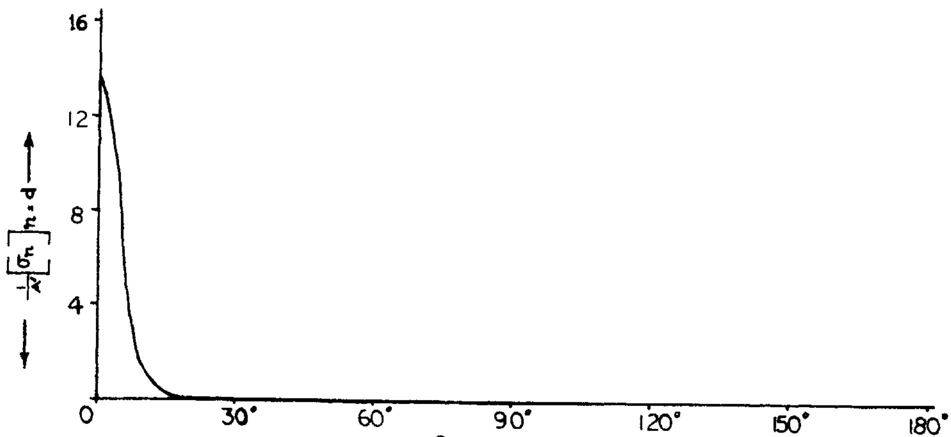

FIG. 5. Radial stress on the middle sphere.

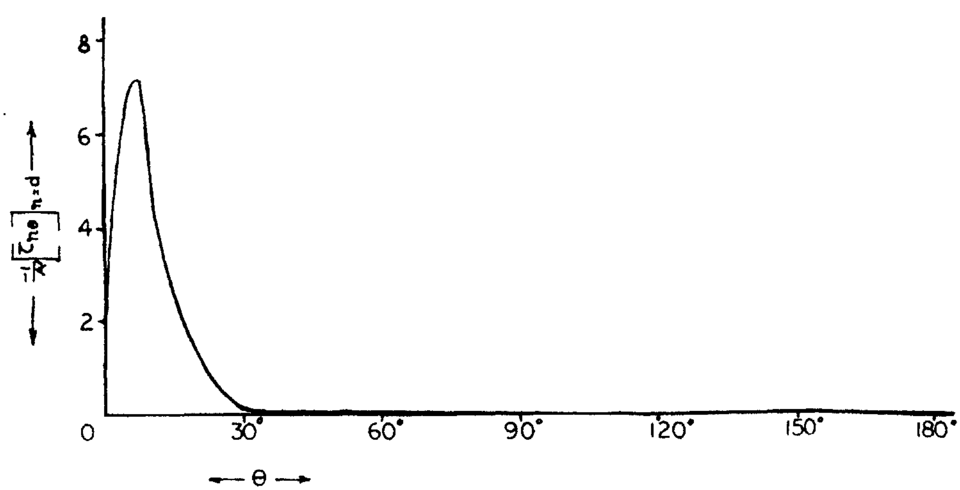

FIG. 6. Shearing stress on the middle sphere.

where

$$A' = -\frac{\mu' T' \alpha'}{2\pi} \cdot \frac{1+\nu'}{1-\nu'} d\Omega.$$

From Fig. 5, it is evident that σ_r is maximum at $\theta = 0$ and as θ increases, σ_r decreases uniformly and rapidly and ultimately tends to zero as in the first case. Fig. 6 is almost identical with Fig. 3 showing that $\tau_{r\theta}$ is maximum when $\theta = 7°$ (nearly) and finally tends to zero.

Acknowledgements

The author expresses his respectful thanks to Dr. P. P. Chattarji, Department of Applied Mathematics, University of Calcutta, for his kind help and guidance in the preparation of this paper. He also expresses his grateful thanks to Dr. A. Ghosh of the Indian Statistical Institute for his kind help in the numerical calculations by a computer of the Institute.

References

Goodier, J. N. (1937). On the integration of the thermoelastic equations. *Phil. Mag.*, **23**, 1017–1032.

Nowacki, W. (1962). Thermoelasticity. Pergamon Press, 93.

Sharma, B. D. (1957). Stress due to a nucleus of thermoelastic strain (i) in an infinite solid with spherical cavity and (ii) in a solid elastic sphere. *Z. angew. Math. Phys.*, 8, 142–150.

Sternberg, E., Eubanks, R. A., and Sadowsky, M. A. (1951). On the axisymmetrical problems of elasticity for region bounded by two concentric spheres. *Proc. First Nat. Cong. Appl. Mech., June 1951* (*ASME*, 1952), 209–16.