ON DISTURBANCES IN A PIEZOELECTRIC BAR

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The response in a N-electrode piezoelectric bar under suitable electrical and thermal excitations is obtained, employing the equations of mechanical motion, equations of Maxwell, the equation of steady heat flow and the piezoelectric constitutive equations in this paper.

INTRODUCTION

The studies, in piezoelectric transducer from the standpoint of mechanics of continuous media, that have been initiated by Redwood (1961) is of recent origin. The relevant problems which are extremely important in view of their applicability in various branches of physics and technology, particularly in ultrasonics and acoustic engineering, have hitherto been considered from the point of view of circuit theory, *vide* Mason (1948). Among the recent studies on these topics, the papers of Sinha (1963–67), Giri (1966, 1967), Das (1967, 1969) and Roy (1967) may be referred to. These problems provide the situations where the two fields, mechanical and electrical, interact with each other and certainly the studies become more interesting if the above interaction is coupled with a thermal field (Sinha 1964, 1967; Giri 1967; Das 1969). The mechanical response owing to suitable thermal and electrical conditions of a N-electrode piezoelectric bar, as conceived by Holland (1966), is investigated in the present study. As in Holland (1966) the solution of the problem has been achieved by taking recourse to Green's function.

PROBLEM, FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

The problem considered here is a N-electrode piezoelectric bar such that with the choice of the origin, the coordinate axes and the application of the mechanical force, it is assumed that the cross-sectional dimension (ω, τ) are small so that T_1 (stress) and P_1 (electric polarization) are functions of x and t only.

The bar is subjected to a time-decaying polarization gradient (Sinha 1966) and a constant flow of heat. Our object is to determine the mechanical response that stems from the interaction of electrical and thermal fields. The fundamental equations are, therefore, the equation of mechanical motion, the equations of electricity and the equation of the heat flow. The equation of mechanical motion is given by

$$\frac{\partial T_1}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2} \quad \ldots \quad (1)$$

where T_1 is the stress, ξ the displacement in the x-direction and ρ the material density.

The piezoelectric equations due to Mindlin (1961) are given by

$$T_1 = c_{11}S_1 + e_{11}E_1 + \lambda_1\theta \quad \ldots \quad (1a)$$

$$P_1 = e_{11}S_1 + k_{11}E_1 + p_1\theta \quad \ldots \quad (1b)$$

where S_1 is the normal strain $\partial\xi/\partial x$, E_1 the electric field strength, P_1 the electric polarization, c_{11} the elastic stress coefficient, k_{11} the electromechanical coupling factor, e_{11} the piezoelectric stress constant, θ the heat flow along the bar, p_1 the pyroelectric constant, λ_1 the temperature coefficient, and α the decaying factor.

In accordance with our assumption

$$\left.\begin{aligned} -\frac{\partial P_1}{\partial x} &= \frac{P_0}{\tau} \text{ under electrode region} \\ &= 0 \text{ under unelectroded region} \end{aligned}\right\} \quad \ldots \quad (2a)$$

$$\left.\begin{aligned} -\frac{\partial \theta}{\partial x} &= \frac{\theta_0}{\tau} \text{ under electrode region} \\ &= 0 \text{ under unelectroded region} \end{aligned}\right\} \quad \ldots \quad (2b)$$

P_0 and θ_0 being constants,
and

$$-\frac{\partial P_1}{\partial x} = \sum_{p=1}^{N} \frac{P_0}{\tau}\{\delta(x-l'_p) - \delta(x-l''_p)\}e^{-\alpha t} \quad \ldots \quad (3a)$$

$$-\frac{\partial \theta}{\partial x} = \sum_{p=1}^{N} \frac{\theta_0}{\tau}\{\delta(x-l'_p) - \delta(x-l''_p)\} \quad \ldots \quad (3b)$$

suffix in l and dashes in l indicate the pth electrode and ends of the same respectively.

Solution of the Problem

From eqns. (1a), (1b) with the aid of equation of motion we get

$$\frac{\partial^2 \xi}{\partial x^2}\left(c_{11} - \frac{e_{11}^2}{k_{11}}\right) - \rho\frac{\partial^2 \xi}{\partial t^2} = -\left(\lambda_1 - \frac{e_{11}p_1}{k_{11}}\right)\frac{\partial \theta}{\partial x} - \frac{e_{11}}{k_{11}}\frac{\partial P_1}{\partial x}. \quad \ldots \quad (4)$$

We now introduce the Laplace transform of parameter p. The Laplace transform of eqns. (4), (3a) and (3b) give

$$\frac{\partial^2 \bar{\xi}}{\partial x^2} + \frac{\rho p^2 \bar{\xi}}{\left(\frac{e_{11}^2}{k_{11}} - c_{11}\right)} = \frac{\left(\lambda_1 - \frac{e_{11}p_1}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}} - c_{11}\right)} \cdot \frac{\partial \bar{\theta}}{\partial x} + \frac{e_{11}}{k_{11}\left(\frac{e_{11}^2}{k_{11}} - c_{11}\right)} \cdot \frac{\partial \bar{P}_1}{\partial x} \quad \ldots \quad (5)$$

where

$$-\frac{\partial P_1}{\partial x} = \frac{1}{p+\alpha} \sum_{p=1}^{N} \frac{P_0}{\tau} \left\{\delta(x-l_p') - \delta(x-l_p'')\right\} \quad \ldots \quad \ldots \quad (6a)$$

$$-\frac{\partial \theta_1}{\partial x} = \frac{1}{p} \sum_{p=1}^{N} \frac{\theta_0}{\tau} \left\{\delta(x-l_p') - \delta(x-l_p'')\right\}. \quad \ldots \quad \ldots \quad (6b)$$

It will be convenient to integrate eqn. (5) utilizing Green's function.

As in Holland (1966) let a force of value $\omega\tau\left(c_{11} - \frac{e_{11}^2}{k_{11}}\right)$ be applied to the bar in the x-direction at some point x_2. Let the bar have no voltage and no other forces applied and let the ends be free. We define Green's function evaluated at some other point x_1 to be the resulting particle displacement at that point.

Mathematically, this means Green's function $G(x_1/x_2)$ obeys the differential equation

$$\left(\frac{\delta^2}{\partial x_1^2} + \frac{\rho p^2}{\frac{e_{11}^2}{k_{11}} - c_{11}}\right) G(x_1/x_2) = -\delta(x_1 - x_2)$$

with the boundary conditions

$$0 = \frac{T_1}{c_{11} - \frac{e_{11}^2}{k_{11}}} + \frac{e_{11} P_1}{k_{11}\left(\frac{e_{11}^2}{k_{11}} - c_{11}\right)} + \frac{\left(\lambda_1 - \frac{p_1 e_{11}}{k_{11}}\right)\theta}{\left(\frac{e_{11}^2}{k_{11}} - c_{11}\right)}\Bigg|_{x_1 = 0,\, l}$$

$$= S_1\Big|_{x_1 = 0,\, l} = \frac{\partial}{\partial x_1} G(x_1/x_2)\Big|_{x_1 = 0,\, l}. \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

A Fourier expansion of solution for $G(x_1/x_2)$ may be found quite easily.

Let us assume

$$G(x_1/x_2) = A_0(x_2) + \sum_{n=1}^{\infty} A_n(x_2) \cos n\pi x_1/l. \quad \ldots \quad \ldots \quad (8)$$

Therefore

$$G(x_1/x_2) = -\frac{1}{\rho l p^2 \Big/ \left(\frac{e_{11}^2}{k_{11}} - c_{11}\right)} - \sum_{n=1}^{\infty} \frac{\frac{2}{l} \cos n\pi x_1/l \cos n\pi x_2/l}{\frac{p^2 \rho}{\frac{e_{11}^2}{k_{11}} - c_{11}} - \frac{n^2 \pi^2}{l^2}}. \quad \ldots \quad (9)$$

Here we note that $G(x_1/x_2)$ is symmetric in x_1 and x_2. In spite of the previous definition of x_1 and x_2 we shall have x_1 the source coordinate and x_2 the observer coordinate. This convention makes subsequent results more convenient.

The wave equation can be integrated for $\bar{\xi}_1$. Multiplying eqn. (7) by $\bar{\xi}_1(x_1)$ and eqn. (5) by $G(x_1/x_2)$ and subtracting second from first, let us integrate the result from ϵ to $l-\epsilon$ where ϵ is an infinitesimal quantity.

$$\int_\epsilon^{l-\epsilon}\left[\bar{\xi}_1(x_1)\left\{\frac{\partial^2 G(x_1/x_2)}{\partial x_1^2}+\frac{\rho p^2}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}G(x_1/x_2)\right\}\right.$$
$$\left.-G(x_1/x_2)\left\{\frac{\partial^2\bar{\xi}_1(x_1)}{\partial x_1^2}+\frac{\rho p^2\bar{\xi}_1(x_1)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\right\}\right]dx_1$$
$$=\int_\epsilon^{l-\epsilon}\left[\bar{\xi}_1(x_1)\{-\delta(x_1-x_2)\}-G(x_1/x_2)\left\{\frac{\left(\lambda_1-\frac{e_{11}p_1}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{\theta}}{\partial x_1}+\frac{e_{11}}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{P}_1}{\partial x_1}\right\}\right]dx_1.$$

Therefore

$$\bar{\xi}_1(x_2)=-\int_\epsilon^{l-\epsilon}G(x_1/x_2)\left\{\frac{\left(\lambda_1-\frac{e_{11}p_1}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{\theta}}{\partial x_1}+\frac{e_{11}}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{P}_1}{\partial x_1}\right\}dx_1$$
$$-\int_\epsilon^{l-\epsilon}\left[\bar{\xi}_1(x_1)\frac{\partial^2 G(x_1/x_2)}{\partial x_1^2}-G(x_1/x_2)\frac{\partial^2\bar{\xi}_1(x_1)}{\partial x_1^2}\right]dx_1$$
$$=-\int_\epsilon^{l-\epsilon}G(x_1/x_2)\left\{\frac{\left(\lambda_1-\frac{e_{11}p_1}{k_{11}}\right)\frac{\partial\bar{\theta}}{\partial x_1}}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}+\frac{e_{11}}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{P}_1}{\partial x_1}\right\}dx_1$$
$$-\left[\bar{\xi}_1(x_1)\frac{\partial G(x_1/x_2)}{\partial x_1}-G(x_1/x_2)\frac{\partial\bar{\xi}_1(x_1)}{\partial x_1}\right]_\epsilon^{l-\epsilon}. \qquad \ldots \quad \ldots \quad (10)$$

For the boundary condition in (7), the first term within the bracket vanishes. The second term may be expressed as

$$G(x_1/x_2)\frac{\partial\bar{\xi}_1(x_1)}{\partial x_1}\bigg|_\epsilon^{l-\epsilon}=G(x_1/x_2)\bar{S}_1\bigg|_\epsilon^{l-\epsilon}.$$

Again since

$$\bar{S}_1=\frac{\bar{T}_1}{c_{11}-\frac{e_{11}^2}{k_{11}}}+\frac{e_{11}\bar{P}_1}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}+\frac{\left(\lambda_1-\frac{p_1e_{11}}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\bar{\theta},$$

therefore

$$G(x_1/x_2)\,\bar{S}_1\Big|_{\epsilon}^{l-\epsilon}$$

$$= G(x_1/x_2)\left\{\frac{\bar{T}_1}{c_{11}-\frac{e_{11}^2}{k_{11}}}+\frac{e_{11}\bar{P}_1}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}+\frac{\left(\lambda_1-\frac{p_1 e_{11}}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\bar{\theta}\right\}\Bigg|_{\epsilon}^{l-\epsilon}$$

$$= \frac{1}{\omega\tau\left(c_{11}-\frac{e_{11}^2}{k_{11}}\right)}\left[F_l G(l/x_2)-F_0 G(0/x_2)\right]$$

$$+\left[\left\{\frac{e_{11}\bar{P}_1}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}+\frac{\left(\lambda_1-\frac{p_1 e_{11}}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\bar{\theta}\right\}G(x_1/x_2)\right]_{\epsilon}^{l-\epsilon}$$

$$= \frac{1}{\omega\tau\left(c_{11}-\frac{e_{11}^2}{k_{11}}\right)}\left[F_l G(l/x_2)-F_0 G(0/x_2)\right]$$

$$-\left(\int_{-\epsilon}^{\epsilon}+\int_{l-\epsilon}^{l+\epsilon}\right)G(x_1/x_2)\left\{\frac{e_{11}}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{P}_1}{\partial x_1}+\frac{\left(\lambda_1-\frac{p_1 e_{11}}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{\theta}}{\partial x_1}\right\}dx_1$$

i.e.

$$\bar{\xi}_1(x_2) = -\int_{-\epsilon}^{l+\epsilon} G(x_1/x_2)\left\{\frac{e_{11}}{k_{11}\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{P}_1}{\partial x_1}+\frac{\left(\lambda_1-\frac{p_1 e_{11}}{k_{11}}\right)}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}\frac{\partial\bar{\theta}}{\partial x_1}\right\}dx_1$$

$$+\frac{1}{\omega\tau\left(c_{11}-\frac{e_{11}^2}{k_{11}}\right)}\left\{F_l G(l/x_2)-F_0 G(0/x_2)\right\}$$

$$= \frac{1}{\omega\tau\rho l}\,\frac{(F_l-F_0)}{p^2}+\frac{2}{\rho l\omega\tau}\sum_{n=1}^{\infty}\frac{\{(-1)^n F_l-F_0\}\cos(n\pi x_2/l)}{p^2-\frac{n^2\pi^2}{l^2}\Big/\frac{\rho}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}}$$

$$+\frac{2e_{11}}{l\tau\rho k_{11}}\,\frac{1}{(p+\alpha)}\sum_{p=1}^{N}\sum_{n=1}^{\infty}P_0\,\frac{\left[\cos\left(\frac{n\pi l_p''}{l}\right)-\cos\left(\frac{n\pi l_p'}{l}\right)\right]\cos\left(\frac{n\pi x_2}{l}\right)}{p^2-\frac{n^2\pi^2}{l^2}\Big/\frac{\rho}{\left(\frac{e_{11}^2}{k_{11}}-c_{11}\right)}}+$$

$$+\frac{2\left(\lambda_1-\dfrac{e_{11}p_1}{k_{11}}\right)}{l\tau\rho}\cdot\frac{1}{p}\sum_{p=1}^{N}\sum_{n=1}^{\infty}\theta_0\frac{\left[\cos\left(\dfrac{n\pi l_p''}{l}\right)-\cos\left(\dfrac{n\pi l_p'}{l}\right)\right]\cos\left(\dfrac{n\pi x_2}{l}\right)}{p^2-\dfrac{n^2\pi^2}{l^2}\Bigg/\dfrac{\rho}{\left(\dfrac{e_{11}^2}{k_{11}}-c_{11}\right)}}. \quad \ldots (11)$$

Let us now distinguish two cases.

Case I: $\left(\dfrac{e_{11}^2}{k_{11}}-c_{11}\right)>0$

Then

$$\bar{\xi}_1(x_2)=\frac{1}{\omega\tau\rho l}(F_l-F_0)t+\frac{2}{\omega\tau\rho l}\sum_{n=1}^{\infty}\{F_l(-1)^n-F_0\}\cos\left(\frac{n\pi x_2}{l}\right)\frac{\sinh\beta t}{\beta}$$

$$+\frac{2}{\rho l\tau}\left[\theta_0\left(\lambda_1-\frac{e_{11}p_1}{k_{11}}\right)\sum_{p=1}^{N}\sum_{n=1}^{\infty}\left\{\cos\left(\frac{n\pi l_p''}{l}\right)-\cos\left(\frac{n\pi l_p'}{l}\right)\right\}\cos\frac{n\pi x_2}{l}\frac{\cosh\beta t}{\beta^2}\right.$$

$$+\frac{e_{11}p_0}{k_{11}}\sum_{p=1}^{N}\sum_{n=1}^{\infty}\left(\cos\frac{n\pi l_p''}{l}-\cos\frac{n\pi l_p'}{l}\right)\cos\frac{n\pi x_2}{l}$$

$$\times\left\{\frac{(e^{\beta t}-e^{-\alpha t})}{2\beta(\alpha+\beta)}-\frac{(e^{-\beta t}-e^{-\alpha t})}{2\beta(\alpha-\beta)}\right\} \quad \ldots (12)$$

where

$$\beta^2=\frac{n^2\pi^2}{l^2}\Bigg/\frac{\rho}{\left(\dfrac{e_{11}^2}{k_{11}}-c_{11}\right)}.$$

The mechanical response given out by the bar is partly linear and partly hyperbolic in time t.

Case II: $\left(\dfrac{e_{11}^2}{k_{11}}-c_{11}\right)<0$

$$\xi_1(x_2)=\frac{1}{\omega\tau\rho l}(F_l-F_0)t+\frac{2}{\omega\tau\rho l}\sum_{n=1}^{\infty}\left[\{F_l(-1)^n-F_0\}\cdot\frac{\sin\beta t}{\beta}\cos\frac{n\pi x_2}{l}\right]$$

$$+\frac{2}{\rho l\tau}\left[\theta_0\left(\lambda_1-\frac{e_{11}p_1}{k_{11}}\right)\sum_{p=1}^{N}\sum_{n=1}^{\infty}\left\{\cos\frac{n\pi l_p''}{l}-\cos\frac{n\pi l_p'}{l}\right\}\right.$$

$$\times\cos\frac{n\pi x_2}{l}\left(\frac{e^{-\alpha t}-\cos\beta t+\sin\beta t}{\alpha^2+\beta^2}\right)$$

$$+\frac{e_{11}p_0}{k_{11}}\sum_{p=1}^{N}\sum_{n=1}^{\infty}\left(\cos\frac{n\pi l_p''}{l}-\cos\frac{n\pi l_p'}{l}\right)\cos\frac{n\pi x_2}{l}\frac{1}{\beta^2}(1-\cos\beta t) \quad \ldots (13)$$

where

$$\beta^2 = \frac{n^2\pi^2}{l^2} \Bigg/ \frac{\rho}{\left(c_{11} - \frac{e_{11}^2}{k_{11}}\right)}.$$

Thus we get the mechanical response given by $\xi_1(x_2)$ at time t in terms of prescribed electrical polarization constant, and the dimensions of the bar. It reveals that the response is partly linear in t, partly transient and partly periodic in time t.

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