TRITON BINDING ENERGY IN UPA WITH HARD-CORE SPIN-DEPENDENT POTENTIALS

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Following the earlier work on the 'Unitary Pole Expansion' for hard-core potentials, a calculation for the binding energy of a three-particle system interacting according to Herzfeld potential (hard-core with square well shape) is done in the 'Unitary Pole Approximation'. The value of the binding energy so obtained agrees well with that of Kim and Tubis (1970) from an exact three-body calculation. Subsequently, a UPA calculation for the triton binding energy is done using spin-dependent local potential with hard core and exponential shape. The results are compared with those of a variational calculation for the same potential done earlier.

INTRODUCTION

As shown by Mitra (1962) and Sitenko and Kharchenko (1963), a considerable simplification results in the three-body equations if the input two-body t-matrix is either separable or is a finite sum of separable terms. Whereas the t-matrix is automatically separable if non-local separable potentials are used, it is not so for local potentials. To this effect Harms (1970) developed a convenient separable expansion of the t-matrix for local potentials which, apart from conforming to the requirement of 'unitarity' in all orders and desired behaviour near the bound state pole, showed a reasonable covergence in the case of local potentials with and without soft cores. This expansion is referred to as the 'Unitary Pole Expansion (UPE)'. For the local potentials with hard cores this kind of expansion was recently worked out by Mathur *et al.* (1972) who found the convergence of the expansion in this case to be even better : a one-term approximation [referred to as 'Unitary Pole Approximation' by Fuda (1968) and Lovelace (1964)] being good enough. Besides, UPE t-matrix elements for a hard-core potential of exponential shape were found to be in good agreement with the corresponding exact t-matrix elements computed by Laughlin and Scott (1968).

As regards three-body calculations with hard-core potentials, Fuda (1969) did such calculations using Weinberg (1963) type of expansion for the t-matrix to make the relevant equations solvable. Meanwhile Kim and Tubis (1970) did a more involved calculation without using a separable approximation or expansion of the t-matrix. However, the UPA approach, which is far simpler to compute and useful for spin-dependent potentials as well (*see* Fuda 1968) has not been used so far for the three-body calculations using local, hard-core potentials. The results of such a calculation are reported herein.

We first report a UPA calculation for the binding energy of a three spinless particle system interacting in pairs according to Herzfeld potential. A good agreement of our result with the exact result of Kim and Tubis (1970) points to the effectiveness of the UPA for hard-core potentials. Subsequently a calculation for the triton binding energy is reported using a spin-dependent local potential with hard-core and exponential shape. The results are compared with those of a variational calculation done much earlier by Ohmura (1959).

UPE for Hard-core Potentials

The Unitary Pole Expansion of the off-shell elements of the S-wave part of the two-body t-matrix, according to Harms (1970) and Mathur *et al.* (1972) is :

$$2\pi^2 t_o\ (k,k';s)=\frac{\pi}{2}\sum_{n,m}^{N}\psi_{n,l=o}\ (k,-B)\triangle_{n,m}(s)\ \psi_{n,l=o}\ (k',-B), \tag{1}$$

where

$$[\triangle^{-1}(s)]_{n,m}=\lambda_n\ \delta_{n,\ m}+\int_0^{\infty}\frac{\psi_{n,l=o}\ (k,-B)\ \psi_{m,l=o}\ (k,-B)}{s-k^2}k^2dk \tag{2}$$

and $\psi_n\ (\vec{\kappa},-B)=<\vec{\kappa}\mid\psi_n\ (-B)>$ are the form factors obtained by the solution of the homogeneous Lippman-Schwinger equation

$$Vg_o\ (-B)\mid\psi_n\ (-B)>=\lambda_n^{-1}\mid\psi_n(-B)> \tag{3}$$

Also $g_o(s)=(s-H_o)^{-1}$ and $B\equiv\kappa^2$ $(n=M=1)$ denotes the binding energy of the two-particle system characteristic of the given two-body potential $V(r)$, and λ_n are the eigen values of the kernel of the Lippman-Schwinger equation for $s=-B$.

In the case of Herzfeld potential viz.,

$$\begin{aligned} V(r) &= \infty,\ r\leqslant a \\ &= -V_o,\ a\leqslant r\leqslant b \\ &= O,\ r>b \end{aligned} \tag{4}$$

the expression for the form factor $\psi_{n,l=0}(k,-B)$ is given by (*see* Lagu 1972):

$$k\psi_{n,l=o}\ (k,-B)=\sqrt{\frac{2}{\pi}}\ C_n\Big[K_n\sin ka-\lambda_n V_o\Big\{\frac{\sin Ka}{2(\kappa_n+k)}+\frac{\sin\ (K_n a-(K_n+K)b}{2\ (\kappa_n+k)}$$

$$+\tfrac{1}{2}(b-a)\cos\tfrac{1}{2}\Big(a\ (K_n+k)-b\ (K_n-k)\Big)\frac{\sin\gamma}{\gamma}\Big\}\Big] \tag{5}$$

where

$$\gamma=\tfrac{1}{2}\ (K_n-k)\ (b-a) \tag{6}$$

and

$$C_n^2=\frac{2}{\lambda_n V_o}\Big[(b-a)-\frac{\sin\{2\ K_n\ (b-a)\}}{2K_n}\Big]^{-1} \tag{7}$$

Here K_ns are the roots of the equation :

$$K\cot\{K(b-a)\}+\kappa=0 \tag{8}$$

while the eigen values λ_ns are related to K_n by

$$K_n^2=\lambda_n V_o-\kappa^2 \tag{9}$$

For the case of the exponential potential with hard core, viz.

$$V(r)=\infty,\ r\leqslant a$$
$$=-V_0 e^{-\alpha(r-a)},\ r>a \tag{10}$$

the expression for the form factor has been given in our earlier communication (Mathur *et al.* 1972, Eqns. 26 and 27). In either case, to write down the UPE of the t-matrix one is only required to substitute for the form factors into Eqns. (1) and (2).

We find that in the case of Herzfeld potential the UPE of the off-shell elements of the t-matrix is as convergent (*see* Lagu 1972) as it is in the case of the hard core exponential potential. In both cases a one-term approximation (UPA) is found to be very good. On the other hand the convergence of the unitary pole expansion was not so good in the case of soft-core potentials as shown by Harms (1970). One may thus infer that the UPA is a more effective approximation for hard core potentials than for soft core ones.

Binding Energy of a 3-Boson System Using Herzfeld Potential

If one used n terms of the unitary pole expansion of the two-body t-matrix in the three-body equations, then the latter would, according to Harms (1970), reduce to a set of n-coupled one-dimensional integral equations. However, since a one-term approximation (UPA) has been found to be good enough in the case of Herzfeld potential, one may use the same for the input t-matrix in the three-body equations thereby reducing the problem to that of solving a single one-dimensional integral equation. It seems logical to believe that the convergence of the UPE in the three-body calculations would be as good as the convergence of the unitary pole expansion of the two body t-matrix, for it is the off-shell elements of the two body t-matrix that are the input parameters in the three-body equations.

Now, for a three (spin-less) particle system the three-body equations reduce in the UPA, to the following one-dimensional integral equation (Lagu 1972)

$$f_1(p)=\int_0^\infty Z_{11}(p,p';E)\ \Delta_{11}(E-\tfrac{3}{4}p^2)\, f_1(p')\, p'^2 dp' \tag{11}$$

where

$$Z_{11}(p,p';E)\equiv\int_{-1}^{+1} d\zeta \frac{\psi_{1,l=0}\left(\left|\frac{\vec{p}}{2}+\vec{p}'\right|,-B\right)\psi_{1,l=0}\left(\left|\vec{p}+\frac{\vec{p}'}{2}\right|,-B\right)}{E-p^2-p'^2-\vec{p}\cdot\vec{p}'} \tag{12}$$

and

$$\zeta=(\vec{p}\cdot\vec{p}')/pp',$$

and where the form factors $\psi_{1,l=0}(k,-B)$ are defined by Eqn. (5). The function $f(p)$ is the 'spectator' function i.e., the wave function of one of the particles with respect to the other two. Also $|E| = -E$ is the binding energy of the three-body system which occurs as a parameter in the kernel of the above homogeneous integral equation. The latter may be solved by using a Gauss Legendre quadrature for integration and finding the value of E by trial and error method so that the integral equation holds i.e., the maximum eigen value of the kernel matrix becomes

equal to unity within a desired accuracy (*see* Brady *et al.* 1969). The 'spectator' function from our calculation is given in Fig. 1. The binding energy of the three-particle system is thus found to be 7.96 MeV which may be compared to the value 8.13 MeV obtained through an exact calculation by Kim and Tubis (1970) using the same two-body potential and involving the solution of two-dimensional integral equations. This agreement lends further support to the effectiveness of the UPA in the case of hard-core potentials.

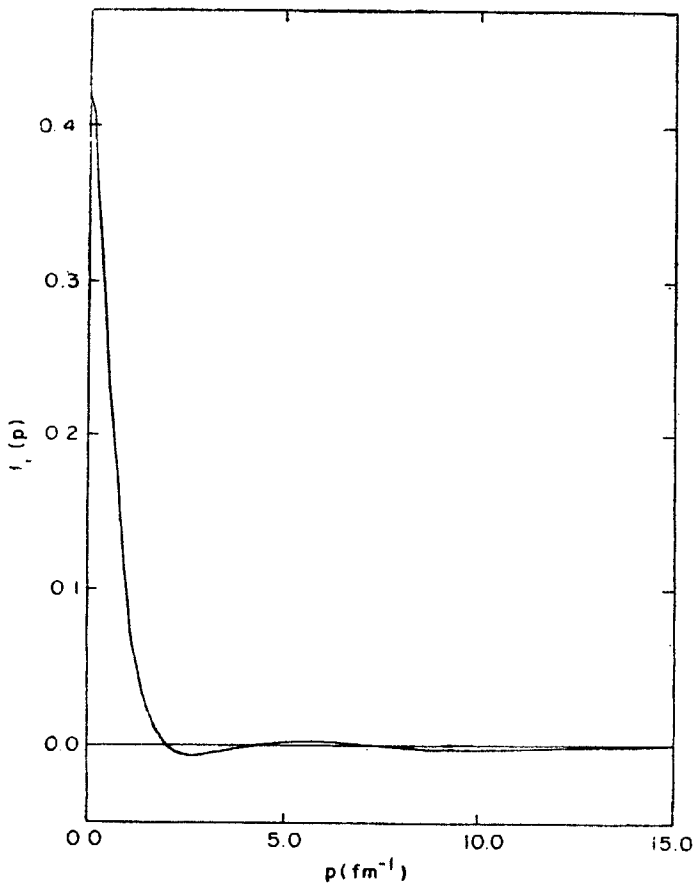

FIG. 1. The 'spectator' function $f_1(p)$ of the three-particle wave function when the two-body potential used has a hard-core and square well shape.

TRITON BINDING ENERGY WITH SPIN-DEPENDENT HARD-CORE POTENTIAL

In order to obtain the binding energy of the realistic three nucleon system it is necessary to take into account the difference between the potentials in the ^3S and ^1S state as a kind of spin-dependence in the two body-potential. So we consider the two-body potential to be :

$$V(r)=V_s(r)\,P_\sigma^- + V_t(r)\,P_\sigma^+ \tag{13}$$

where P_σ^+ and P_σ^- are the projection operators for the spin-triplet and spin-singlet states and V_t and V_s have the exponential form (10) with different range and depth parameters in the two cases as given by Ohmura (1959). If we assume the two-body interaction to be in the S-state then two particles are either in the spin-triplet ($S=1$) and isospin singlet ($T=0$) state or *vice-versa*. Consequently, $<S=1, T=0 \mid t(s) \mid S=1, T=0> \equiv t^{10}(s)$ is the transition operator defined by the Lippman-Schwinger equation for the spin-triplet potential while $<S=0, T=1 \mid t(s) \mid S=0, T=1> \equiv t^{01}(s)$ is the transition operator defined by the Lippman-Schwinger equation for the spin-singlet potential. The UPE for the t-matrices $t^{10}(s)$ and $t^{01}(s)$ can be written using Eqns. (1) and (2). In the one-term approximation we have simply :

$$<\vec{k} \mid t^{10}(s) \mid \vec{k'}> \approx \psi^T{}_{1,l=0}(k,-B_T)\,\triangle^T{}_{11}(s)\,\psi^T{}_{1,l=0}(k',-B_T) \quad (14)$$

$$<\vec{k} \mid t^{10}(s) \mid \vec{k'}> \approx \psi^S{}_{1,l=0}(k,-B_S)\,\triangle^S{}_{11}(s)\,\psi^S{}_{1,l=0}(k',-B_S) \quad (15)$$

where $\psi^T{}_{1,l=0}$ and $\psi^s{}_{1,l=0}$ are given by Eqns. (26) and (27) of our earlier communication (Mathur *et al.* 1972) on putting $n=1$ and using correspondingly the range and depth parameters of the triplet and singlet potentials. B_T is the binding energy of the two-particle system for the spin-triplet potential while B_s is that for the spin-singlet potential. We take $B_S=0$.

Now, the equations for the bound state of three nucleons ($s=\frac{1}{2}$, $t=\frac{1}{2}$) can be written down following Sitenko and Kharchenko (1963) who formulated the equations by taking into account the overall antisymmetrization of the three-body wave function according to Verde (1957). In the case when the N-N interaction is in the S-state only, the three body equations reduce to two coupled equations viz.,

$$V(\vec{k},\vec{p}) = -\tfrac{1}{4}\frac{1}{k^2-E+\frac{3}{4}p^2}\int\Big\{<\vec{k} \mid t^{10}(E-\tfrac{3}{4}p^2) \mid -\tfrac{1}{2}\vec{p}-\vec{p'}> + <\vec{k} \mid t^{10}(E-\tfrac{3}{4}p^2) \mid \tfrac{1}{2}\vec{p}+\vec{p'}>\Big\} \times \Big\{ V\Big(\vec{p}+\tfrac{1}{2}\vec{p'},\vec{p'}\Big) + 3u\Big(\vec{p}+\tfrac{1}{2}\vec{p'},\vec{p'}\Big)\Big\} d^3p' \quad (16)$$

$$u\Big(\vec{k},\vec{p'}\Big) = -\tfrac{1}{4}\frac{1}{k^2-E+\frac{3}{4}p^2}\int\Big\{<\vec{k} \mid t^{01}\Big(E-\tfrac{3}{4}p^2\Big) \mid -\tfrac{1}{2}\vec{p}-\vec{p'}> + <\vec{k} \mid t^{01}\Big(E-\tfrac{3}{4}p^2\Big) \mid \tfrac{1}{2}\vec{p}+\vec{p'}>\Big\} \times \Big\{3v\Big(\vec{p}+\tfrac{1}{2}\vec{p'},\vec{p'}\Big) + u\Big(\vec{p}+\tfrac{1}{2}\vec{p'},p'\Big)\Big\} d^3p' \quad (17)$$

where u and v are the spatial functions in terms of which one constructs the spatial wave functions of various symmetries. Substituting for the functions v and u the forms

$$v(\vec{k},\vec{p}) = \psi^T{}_{1,l=0}(k,-B_T)\,F(p)/\,-E+k^2+\tfrac{3}{4}p^2) \quad (18)$$

$$u(\vec{k},\vec{p}) = \overset{S}{\psi}_{1,l=0}(k,-B_s)\, G(p)/(-E+k^2+\tfrac{3}{4}p^2) \tag{19}$$

one gets the following coupled integral equations* in $F(p)$ and $G(p)$:

$$F(p)=\tfrac{1}{4}\,\overset{T}{\triangle}_{11}\left(E-\tfrac{3}{4}p^2\right)\int_0^{\infty} p'^2\,dp'\left\{3I_{TT}\left(p,p';E\right)F(p')+I_{TS}\left(p,p';E\right)G(p')\right\} \tag{20}$$

and

$$G(p)=\tfrac{1}{4}\,\overset{S}{\triangle}_{11}\left(E-\tfrac{3}{4}p^2\right)\int_0^{\infty} p'^2\,dp'\left\{I_{ST}\left(p,p';E\right)F(p')+3I_{SS}\left(p,p';E\right)G(p')\right\} \tag{21}$$

where

$$I_{XX'}(p,p';E)=\int_{-1}^{+1} d\zeta\, \frac{\psi^{x}_{1,l=0}\left(\left|\frac{\vec{p}}{2}+\vec{p}'\right|,-B_x\right)\overset{X'}{\psi}_{1,l=0}\left(\left|\vec{p}+\frac{\vec{p}'}{2}\right|,-B_{X'}\right)}{-p^2-p'^2-\vec{p}.\vec{p}'+E} \tag{22}$$

$$\zeta=(p.p')/pp', \qquad \text{and } X, X'=S,T$$

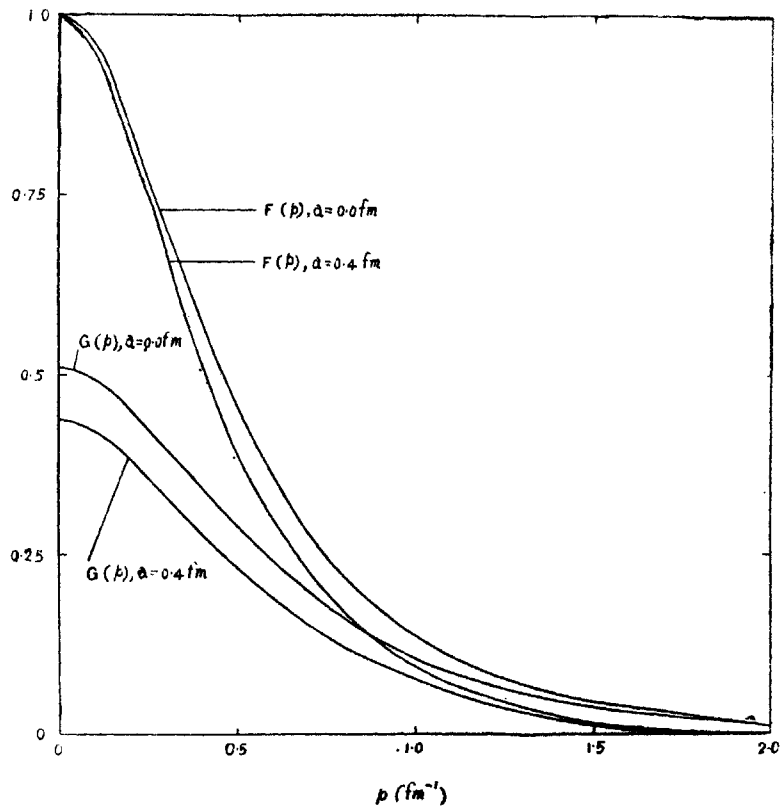

FIG. 2. The 'spectator' functions of the three-particle wave function when a local and spin-dependent two-body potential is used which has a hard core and exponential shape.

*Similar equations were obtained by Bhakar (1963) following Mitra's method and using separable potentials.

Solutions of these coupled integral equations were obtained by the method of Brady *et al.* (1969). Fig. 2 shows the 'spectator' functions $G(p)$ and $F(p)$ for hard core radius $a = 0$—0.4 fm. The values of the triton binding energy in the two cases are listed in Table I in comparison with those obtained by Ohmura (1959) using the variation method for the same potentials. One observes that the introduction of hard-core causes comparatively less reduction in the triton binding energy in a UPA calculation than it does in a variational calculation.

TABLE I

a(fm.)	Binding energy of ^{3}H (MeV)	
	UPA	Variational calculation
0.0	10.36	10.26
0.4	9.33	7.52

The three-body wave function, determined by the 'spectator' functions $G(p)$ and $F(p)$ is being used for the calculation of the electromagnetic form factors of the trinucleons.

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