STRONGLY REGULAR GRAPHS CONTAINING STRONGLY REGULAR SUBGRAPHS

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In this paper we review some results on strongly regular graphs which contain two disjoint strongly regular subgraphs.

INTRODUCTION AND PRELIMINARY RESULTS

Let G be a finite, undirected graph on a set $V(1, 2, \ldots, v)$ of v vertices with no edges from x to x and utmost one edge from x to y when $x \neq y \in V$. The graph is called regular if the number of edges passing through each vertex is a constant n_1, called the valence of G. If there is an edge from x to y we say that x and y are adjacent or 1-associates. Otherwise if $x \neq y$ they are non-adjacent or 2-associates. A regular graph is called strongly regular if for any pair (x, y) of vertices which are i-associates, the number $p^i_{jk}(x, y)$ of vertices which are simultaneously j-associates of x and k-associates of y is a constant p^i_{jk} which is independent of the pair (x, y). If n_2 is the number of 2-associates of any vertex in a strongly regular graph, then it is well known (cf. Bose 1963) that

$$n_2 = v-1-n_1.$$

$$p^1_{12} = p^1_{21},\ p^2_{12} = p^2_{21}.$$

$$p^1_{11} + p^1_{12} + 1 = n_1 = p^2_{11} + p^2_{12}. \qquad \ldots \ldots (1)$$

$$p^2_{21} + p^2_{22} + 1 = n_2 = p^1_{21} + p^1_{22}.$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ji};\ i, j, k = 1, 2.$$

In view of the above we denote a strongly regular graph with the above parameters by $G\ (v, n_1, p^1_{11}, p^2_{11})$.

For any graph G on v vertices the square symmetric matrix $\mathbf{A} = \mathbf{A(G)} = (\mathrm{a}_{mn})$ is called the adjacency matrix of G if $\mathrm{a}_{mm} = 0$, and $\mathrm{a}_{mn} = 1$ if and only if m and n are 1-associates; $m \neq n = 1, 2 \ldots, v$.

A graph G is said to be connected if given any two vertices m and n $(m \neq n)$ there is a sequence $i_0 = m, i_1, \ldots, i_p = n$ of vertices such that i_a and i_{a+1} are adjacent, $a = 0, 1, \ldots p-1$.

Any (0, 1) symmetric matrix **A** with o along the main diagonal can be regarded as the adjacency matrix of a unique graph G. We say that **A** is regular, strongly regular or connected according as the graph G is regular, strongly regular or connected.

The following results can be easily proved :

Lemma 1: A regular adjacency matrix of valence n_1 is strongly regular, if and only if there exist non-negative integers e and f such that

$$A^2 = (e-f)\,A + f\mathbf{J} + (n_1-f)\,\mathbf{I}$$

where **I** is the identity matrix and **J** the square matrix with all elements 1. The values of p^1_{11} are then given by $p^1_{11} = e,\ p^2_{11} = f.$

The following result is proved in Shrikhande and Bhagwandas (1965).

Lemma 2 : A regular connected adjacency matrix **A** of valence n_1 is strongly regular if and only if **A** has two other distinct characteristic roots θ_1 and θ_2 besides n_1 which is a characteristic root of multiplicity 1, and then

$$p^1_{11} = n_1 + \theta_1\,\theta_2 + \theta_1 + \theta_2$$

$$p^2_{11} = n_1 + \theta_1\,\theta_2.$$

Mesner (1967) has defined the negative latin square graph $NL_g\,(n)$ for non-negative integers g and n with $g+1 \leq n \leq g(g+3)$. The parameters are

$$v = n^2,\ n_1 = g(n+1)$$

$$p^1_{11} = (g+1)\,(g+2) - n - 2,\ p^2_{11} = g(g+1), \qquad \text{.. (2)}$$

where the remaining parameters can be easily calculated from the set of parameters (1). We note that $p^1_{11} = 0$ if and only if $n = g(g+3)$. For $g = 1$ and 2 we get respectively NL_1 (4) and NL_2 (10) graphs which are known to exist. It is not known whether NL_g (g^2+3g) graphs exist for $g \geq 3$.

A balanced incomplete block design (bibd) is a system of v treatments (symbols) arranged in b blocks of size $k < v$ such that any two treatments occur together in precisely λ blocks. Then each treatment occurs in exactly r blocks. It is then known that $vr = bk$; $\lambda\,(v-1) = r\,(k-1)$ and $b \geq v$. If $b = v$ the design is called a symmetric bibd and then any two blocks intersect in exactly λ treatments. The parameters of a bibd are denoted by (v, b, r, k, λ).

The incidence matrix of a bibd D with parameters (v, b, r, k, λ) is a $v \times b$ matrix $\mathbf{N} = (n_{ij})$ where $n_{ij} = 1$ or 0 according as treatment numbered i occurs or does not occur in block numbered j. The incidence matrix of any incomplete design on v treatments i.e., a design in which each block contains $k < v$ distinct treatments is defined in the same manner.

A quasi-symmetric balanced incomplete block design (bibd) is a design with parameters (v, b, r, k, λ), with $v > b$ and such that any block of the design intersects in either μ_1 or μ_2 treatments. The following result is given Shrikin hande (1973).

Lemma 3: Let there exist a quasi-symmetric bibd with parameters (v, b, r, k, λ) such that any two blocks interesect in either μ_1 or μ_2 treatments. Let G be the graph whose vertices are the blocks of this design. Define two blocks adjacent (non-adjacent) according as they intersect in μ_1 (respectively μ_2) treatments. Then G is strongly regular with parameters $(b, n_1, p^1_{11}, p^2_{11})$ where

$$n_1 = \frac{k(r-1)+\mu_2(1-b)}{\mu_1-\mu_2},$$

$$p^1_{11} = n_1 + \theta_1\theta_2 + \theta_1 + \theta_2,$$

$$p^2_{11} = n_1 + \theta_1\theta_2,$$

$$\theta_1 = \frac{(r-\lambda)-(k-\mu_2)}{\mu_1-\mu_2},$$

$$\theta_2 = \frac{-(k-\mu_2)}{\mu_1-\mu_2}.$$

A 3-design $D = (b_3; v, k, 3)$ is a set V of v elements called treatments and a collection of k-subsets of V called blocks such that every 3-set of V is contained in exactly b_3 blocks. It then follows that the number of blocks b_i containing any i-set of V is a constant. If we define b_0 as the total number of blocks, then the following relations hold:

$$b_0\binom{k}{i} = b_i\binom{v}{i}\,;\; i = 0, 1, 2, 3.$$

$$b_i\binom{k-i}{3-i} = b_3\binom{v-i}{3-i};\; i = 0, 1, 2, 3.$$

$$b_i(k-i) = b_{i+1}(v-i);\; i = 0, 1, 2.$$

Putting $b_0 = b$, $b_1 = r$, $b_2 = \lambda$ we see that a 3-design is a bibd with parameters (v, b, r, k, λ).

By considering the blocks of D containing a given treatment x and omitting x from these blocks we get the derived designs $D_d(x)$ which is a bibd with

$$v' = v-1,\; b' = r,\; r' = \lambda,\; k' = k-1,\; \lambda' = b_3.$$

The remaining blocks then form the residual design $D_r(x)$ which is again a bibd with

$$v'' = v-1,\; b'' = b-r,\; r'' = r-\lambda,\; k'' = k,\; \lambda'' = \lambda-b_3.$$

A 3-design D is called symmetric if $D_d(x)$ for any (and hence all) x is a symmetric bibd i.e., if the number of treatments $v-1$ and the number of blocks b' in $D_d(x)$ are equal. It can easily be shown that a 3-design D is symmetric if and only if any block of D is disjoint with some of the other blocks of D and intersects the remaining blocks of D in $g = 1 + b_3$ treatments. A possible infinite family of symmetric 3-designs corresponds to the parameters $(b_3;\, v, k, 3)$ of a design D where

$$D: \begin{array}{l} v = g(g^2+3g+1), \quad b = (g^2+2g-1)(g^2+3g+1), \\ r = (g+1)(g^2+2g-1), \; k = g(g+1), \\ \lambda = g^2+g-1, \; b_3 = g-1, \end{array} \qquad \ldots (3)$$

where g is a positive integer. Than $D_d(x)$ has parameters

$$v' = b' = (g+1)(g^2+2g-1), \; r' = k' = g^2+g-1 \qquad \ldots (4)$$
$$\lambda' = g-1$$

and $D_r(x)$ has parameters

$$D_r: \begin{array}{l} v'' = (g+1)(g^2+2g-1), \; b'' = (g^2+2g-1)(g^2+2g) \\ r'' = g^2(g+2), \; k'' = g(g+1), \; \lambda'' = g^2. \end{array} \qquad \ldots (5)$$

We note that any 2 blocks of D or $D_r(x)$ intersect in 0 or g treatments.

It may be mentioned that the 3-designs corresponding to parameter (3) are known to exist for $g = 1$ or 2. Nothing is known for higher values of g.

Since D and $D_r(x)$ given by parameters (3) and (5) are quasi-symmetric bibd's with any two blocks inter-secting in $\mu_1 = 0$ or $\mu_2 = g$ treatments, we can use Lemma 3 to obtain strongly regular graphs $G(D)$ and $G(D_r)$ which have the following parameters

$$G(D): \begin{array}{l} v = (g^2+2g-1)(g^2+3g+1), \\ n_1 = g^2(g+2), \\ p_{11}^1 = 0, \\ p_{11}^2 = g^2. \end{array} \qquad \ldots (6)$$

$$G(D_r): \begin{array}{l} v = g(g+2)(g^2+2g-1), \\ n_1 = g(g^2+g-1), \\ p_{11}^1 = 0, \\ p_{11}^2 = g(g-1). \end{array} \qquad \ldots (7)$$

In these graphs two vartices are 1-associates if the corresponding blocks are disjoint in D and D_r respectively and 2-associates if they intersect in g treatments.

If $\mathbf{A}$ and $\mathbf{A}_r$ are the adjacency matrices of graphs $G(D)$ and $G(D_r)$ given respectively by (6) and (7) then without loss of generality we can write

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A}_r \end{pmatrix} \qquad \ldots (8)$$

where 0 is a square matrix of order $(g+1)(g^2+2g-1)$ corresponding to the blocks of D which contain a given treatment x and $\mathbf{A}_r$ is the adjacency matrix of the graph $G(D_r) = G(D_r(x))$ corresponding to the treatment x,

If G is a graph whose adjacency matrix can be put in the form (8) where $\mathbf{A}$ and $\mathbf{A}_r$ are the adjacency matrices of graphs with parameters (6) and (7) respectively, then we call G a pseudo-$G(D)$ graph where D is a symmetric 3-design with parameters (3).

Bose (1963) has defined a partial geometry (r, k, t) as a system of points and lines satisfying the following four axioms. Using the usual geometric language, the axioms are as follows :

A_1 : Any two points are incident with at most one line.

A_2 : Each point is incident with r lines.

A_3 : Each line is incident with k points.

A_4 : If a point P is not incident with the line l, there pass through P exactly t lines intersecting l where $1 \leq t \leq \min(r, k)$.

It can then be shown that the number of points v and the number of lines b in a partial geometry (r, k, t) are given by

$$v = k[(r-1)(k-1)+t]/t,$$

$$b = r[(r-1)(k-1)+t]/t.$$

The graph G of a partial geometry is defined as the graph whose vertices are the points of the geometry, and in which two vertices of G are adjacent or non-adjacent according as the corresponding points are incident or non-incident with a common line. G is then called a geometric graph (r, k, t) and is strongly regular with parameters.

$$v = k[(r-1)(k-1)+t]/t, \ n_1 = r(k-1),$$

$$p_{11}^1 = k-2+(r-1)(t-1), \ \ p_{11}^2 = rt. \tag{9}$$

A strongly regular graph with parameters (9) is called a pseudo-geometric-graph (r, k, t).

Let G be a pseudo-geometric graph $(r, k, 1,)$. Then G is said to have the property (P) with respect to the vertex θ_0, if the $r(k-1)$ vertices adjacent to θ_0 can be partitioned into r disjoint sets $S_1, S_2, \ldots, S_r$ such that any two vertices belonging to the same S_i are adjacent. Then $K_i = S_i \cup \theta_0$ is a clique of size k. Since $t = 1$ any vertex θ_{iu} of S_i is non-adjacent to any vertex $\theta_{i'u'}$ of $S_{i'}$, if $i \neq i'$.

We shall say that G has the additional property (P^*) with respect to θ_0, if any two different vertices β_j and $\beta_{j'}$, both non-adjacent to θ_0 and both adjacent to θ_{iu} and $\theta_{i'u'}$ are themselves non-adjacent, where θ_{iu} and $\theta_{i'u'}$ are any two vertices belonging to S_i and $S_{i'}$ respectively, $i \neq i'$.

Given a strongly regular graph $G(v, n_1, p_{11}^1, p_{11}^2)$ on v objects a partially balanced incomplete block design (pbibd) based on this graph G is an arrangement of v treatments (objects) in b blocks of size $k < v$ such that any treatment occurs in r blocks and any two treatments which are i-associates in G occur together in λ_i blocks; $i = 1, 2$; $\lambda_1 \neq \lambda_2$. The parameters $b, r, k, \lambda_1, \lambda_2$ are not determined by the parameters of G.

If $v = mn$ objects are partitioned into m disjoint sets, each containing n objects, then we get a group divisible (GD) association scheme (strongly regular graph) by taking two objects to be 1-associates if and only if they belong to the same set. The parameters of this (GD) graph are then

$$v = mn,\ n_1 = n-1,\ p_{11}^1 = n-2,\ p_{11}^2 = 0. \tag{10}$$

A pbibd based on a (GD) association scheme is said to be a (GD) design. Two treatments which belong to the same set occur together in λ_1 blocks, and two treatments not belonging to the same set occur together in λ_2 blocks. The parameters of the design are $v, b, r, k;\ m, n;\ \lambda_1, \lambda_2$.

The combinatorial properties of GD designs are given by Bose and Connor (1952). They can be divided into three types, singular, semi-regular, or regular according as (i) $r = \lambda_1$, (ii) $r > \lambda_1$, $rk - \lambda_2 v = 0$. (iii) $r > \lambda_1$, $rk - \lambda_2 v > o$.

It can be shown that for a semi-regular group divisible ($SRGD$) design each block contains exactly k/m treatments from each set. Conversely, if in a GD design each block contains precisely k/m treatments from each set the design is a $SRGD$ design.

Graphs Connected With $NL_g(g^2 + 3g)$ Graphs

Consider an $NL_g(g^2 + 3g)$ graph with parameters

$$\begin{aligned} v &= g^2(g+3)^2, \quad n_1 = g(g^2+3g+1), \\ p_{11}^1 &= 0, \qquad\qquad p_{11}^2 = g(g+1). \end{aligned} \tag{11}$$

Without loss of generality its adjacency matrix P can be written as

$$\mathbf{P} = \begin{pmatrix} o & j^T & o^T \\ j & 0 & \mathbf{D} \\ o & \mathbf{D}^T & \mathbf{A} \end{pmatrix} \tag{12}$$

where j is a column vector of all ones of order n_1 and o is a column vector of all zeros of order n_2.

The following theorem has been proved by Mesner (1967).

Theorem 1 : If P given by (12) is the adjacency matrix of an $NL_g\ (g^2 + 3g)$ graph given by (11), then $\mathbf{D}$ *is the incidence matrix of a symmetric 3-design with the set of parameters (3),* $\mathbf{A}$ *is the adjacency matrix of the graph G (D) given by (6) corresponding to the blocks of D and* $\mathbf{D}^T$ *is the incidence matrix of a pbibd based on the graph* $\mathbf{A}$ *and has parameters*

$$\bar{v} = n_2\ \bar{b} = n_1,\ \bar{r} = p_{11}^2,\ \bar{k} = p_{12}^1,\ \bar{\lambda}_1 = o,\ \bar{\lambda} = g.$$

Conversely, if **D** is the incidence matrix of asymmetric 3-design with the set of parameters (3), than P given by (12) is the adjacency matrix of an $NL_g(g^2+3g)$ graph.

The proof essentially depends upon Lemma 1 and the expression for $\mathbf{P}^2$ when **P** is regarded as a partitioned matrix (12).

Since **A** is the adjacency matrix of the graph $G(D)$ given by parameters (6) corresponding to a symmetric 3-design given by parameters (3), the following theorem is obvious.

Theorem 2 : The matrix **A** *of the previous theorem can be partitioned as*

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A}_r \end{pmatrix}$$

where 0 *is a square matrix of order* $(g+1)(g^2+2g-1)$, $\mathbf{A}_r$ *is the adjacency matrix of a graph* $G(D_r)$ *given by (7) and B is the incidence matrix of a quasisymmetric design* D_r *given by (5).*

Recalling the definition of a pseudo-$G(D)$ graph we have the following generalisation (Shrikhande 1971) of the previous theorem :

Theorem 3: The adjacency matrix **A** *of a pseudo-* $G(D)$ *graph can be partitioned as in the above theorem. Conversely, if* **B** *is the incidence matrix of a quasi-symmetric design* D_r *given by (5) and* $\mathbf{A}_r$ *is the adjacency matrix of the blocks of* D_r*, where two blocks are adjacency (non-adjacent) if they interest in* $\mu_1 = o$ $(\mu_2 = g)$ *treatments, then* **A** *given in Theorem 2 is the adjacency matrix of a pseudo-G* (D) *graph.*

Let D be a quasi-symmetric bibd with parameters (v, b, r, k, λ) in which any two blocks intersect in either μ_1 or μ_2 treatments. Let **N** be the incidence matrix of D and call two blocks of D adjacent (non-adjacent) according as they intersect in μ_1 (respectively μ_2) treatments. Let **A** be the adjacency matrix of the blocks of D. The following theorem is due to Shrikhande, M.S. (1973).

Theorem 4: With the above notation

$$\mathbf{P} = \begin{pmatrix} 0 & \mathbf{N} \\ \mathbf{N}^T & \mathbf{A} \end{pmatrix}$$

is the adjacency matrix of a strongly regular graph G if and only if there exists an integer g with

$$1 \leq g \leq \frac{-1+\sqrt{1+4k}}{2}$$

and such that

$$v = \frac{k^2+kg-g^2-g}{k-g^2}$$

$$r = \frac{k(k+g)}{g+1}$$

$$\lambda = \frac{k(k-g^2)}{g+1}$$

$$\mu_1 = k-g^2-g$$

$$\mu_2 = k-g^2.$$

Then G is $\left(b+v,\, n_1,\, p^1_{11},\, p^2_{11}\right)$ *strongly regular graph with*

$$n_1 = r,$$

$$p^1_{11} = g+\lambda-r,$$

$$p^2_{11} = \lambda.$$

GRAPHS CONNECTED WITH PSEUDO-GEOMETRIC GRAPHS $(q^2+1, q+1, 1)$

Let G be a pseudo-geometric $(q^2+1, q+1, 1)$ graph with parameters $(v, n_1, p^1_{11}, p^2_{11})$ *where*

$$\begin{aligned} v &= (q+1)(q^3+1), & n_1 &= q(q^2+1), & n^2 &= q^4, \\ p^1_{11} &= q-1, & p^2_{11} &= q^2+1, & & \\ p^1_{12} &= q^3, & p^2_{12} &= (q^2+1)(q-1), & & \qquad (13) \\ p^1_{22} &= q^3(q-1), & p^2_{22} &= q(q-1)(q^2+1), & & \end{aligned}$$

with the property (P) and (P^*) defined in the first Section with respect to a vertex θ_0. Without loss of generality taking θ_0 to be the initial vertex the adjacency matrix of G can be put in the following form

$$\mathbf{Q} = \begin{pmatrix} o & j^T & o^T \\ j & \mathbf{L} & \mathbf{D} \\ o & \mathbf{D}^T & \mathbf{A} \end{pmatrix} \qquad (14)$$

where j is a column vector of all ones of order n_1 and o is a column vector of all zeros of order n_2 and $\mathbf{L}$ is the diagonal matrix $(J_q-I_q, J_q-I_q, .., J_q-I_q)$ of order $q(q^2+1)$. The following theorem is due to Bose and Shrikhande (1972).

Theorem 4: If (14) *is the adjacency matrix of a pseudo-geometric* $(q^2+1, q+1, 1)$ *graph with the property* (P) *and* (P^*) *with respect to the initial vertex* θ_0, *then* $\mathbf{D}$ *is the incidence matrix of a SRGD design with parameters*

$$\begin{aligned} v' &= q(q^2+1), b' = q^4, r' = q^3, k' = q^2+1, \\ m' &= q^2+1, n' = q, \lambda'_1 = o, \lambda'_2 = q^2, \end{aligned} \qquad (15)$$

which has the property (I_1) that any two blocks intersect in either $\mu_1 = 1$ or $\mu_2 = q+1$ treatments. **A** is then the adjacency matrix of the blocks of this *SRGD* design in which two blocks are adjacent (non-adjacent) according as they intersect in μ_1 (respectively μ_2) treatments. Further, **A** is the adjacency matrix of a NL_{q-1} (q^2) graph and $\mathbf{D}^T$ is the incidence matrix of a pbibd based on the NL_{q-1} (q^2) graph.

Conversely, if **D** is the incidence matrix of a *SRGD* design with property (I_1) and parameters (15) then (14) is the adjacency matrix of a pseudo-geometric (q^2+1, $q+1$, 1) graph having the property (P) and (P^*) with respect to the initial vertex in Q.

The existence of a partial geometry (q^2+1, $q+1$, 1) is known (cf. Bose & Shrikhande (1972) when q is a prime power. This implies the existence of a SRGD design with property (I_1) given by (15) and the existence of a NL_{q-1} (q^2) graph for the same values of q.

Concluding Remarks

The examples considered in the last two sections open the possibility of characterising the class of all strongly regular graphs G in which the graphs induced by the 1-associates and 2-associates of a vertex in G are themselves strongly regular. In Section under Graphs connected with NL_g (g^2+3g), the graph induced by a 1-associates is the empty graph while in the last Section it is a (GD) graph.

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