NOTE ON BICIRCULANT MATRICES

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The note presents a new relative of the back—circulant matrix, called the bicirculant matrix, and discusses some of its properties. The elements of a bicirculant matrix circulate in pairs and are symmetric about both the main diagonals. The eigenvectors of the matrix are the discrete Walsh vectors. This matrix appears important in the regular representation of abstract dyadic groups.

INTRODUCTION

Hadamard, skew-Hadamard, circulant and back-circulant matrices have been discussed extensively by Marshall Hall Jr. (1967) and Jennifer Wallis (1970, 1971). A Hadamard matrix $\mathbf{H} = (h_{ij})$ is a matrix of order n, all of whose elements are 1 and—1 and which satisfieds $\mathbf{HH^T} = n\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. A Hadamard matrix $\mathbf{H} = \mathbf{U} + \mathbf{I}$ is called skew-Hadamard if $\mathbf{U^T} = -\mathbf{U}$. A set of elements $D = \{x_1, x_2, \ldots, x_k\}$ will be said to generate a circulant $(1, -1)$ matrix $\mathbf{A} = (a_{ij})$ if $a_{ij} = a_{1,\, j-i+1} = 1$ when $j-i+1 \in D$ (all numbers modulo k) and -1 otherwise. A back-circulant matrix $\mathbf{A} = (a_{ij})$ of order k has $a_{1i} = a_{1+j,\, i-j}$ where $1+j$ and $i-j$ are reduced modulo k.

In this note we study some properties of a relative of the back-circulant matrix called the bicirculant matrix (cf. Kak 1972).

BICIRCULANT MATRICES

Definition : A bicirculant matrix $\mathbf{B} = (b_{i,\, j})$ of order $n = 2^k$ is defined by :

$$b_{i,\, j} = b_{j,\, i} = b_{n2^{-m}-i+1,\; n2^{-m}-j+1}\; ;\; i, j \leqslant n2^{-m}$$

$$b_{i,\, 2^l+j} = b_{j,\, 2^l+i}\; ;\; i, j \leqslant 2^l$$

Example : A fourth order bicirculant matrix $\mathbf{B}_4$ is given below :

$$\mathbf{B}_4 = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_2 & b_1 & b_4 & b_3 \\ b_3 & b_4 & b_1 & b_2 \\ b_4 & b_3 & b_2 & b_1 \end{bmatrix}$$

We notice that the elements circulate in pairs while undergoing interchanges amongst themselves.

It is obvious that if $\mathbf{B}_{ij}$, $(i, j = 1, \ldots N)$ satisfy the conditions on $b_{i,j}$ in the above definition, and $\mathbf{B}_{ij}$ themselves be bicirculant matrices of order M, then $\mathbf{B} = (\mathbf{B}_{ij})$ is also a bicirculant matrix of order NM.

Also if **B** and **C** are bicirculant matrices of the same orders, so is their product **BC**. Generalizing, any $\mathbf{B}^n \mathbf{C}^m$ would also be a bicirculant matrix.

Since the elements of a bicirculant matrix circulate in pairs, and are symmetric about both the main diagonals the product matrix **BC** should commute. Or **BC** = **CB.**

One Hadamard matrix, **H,** of order 2^n can be shown to be built up from Hadamard sub-matrices of lower orders. These sub-matrices have the same sign alternations as would be obtained in Hadamard matrix of that order. This is generally expressed as the result that the Kronecker product of two Hadamard matrices of orders M and N is another Hadamard matrix of order MN. The matrix **B** can also be expressed as being built of submatrices $\mathbf{B}_i$'s. Now since any 2nd order bicirculant matrix post-multiplied by a Hadamard matrix can, on simple manipulations, be shown to give a matrix with orthogonal columns, the product matrix **BH** shall also have orthogonal columns. Using similar reasoning **HA** has orthogonal rows.

To sum up, **B** is diagonalized with the Hadamard matrix as the modal matrix. Mathematically :

$$\mathbf{HBH} = N\ \mathbf{D}$$

where **D** is the diagonalized (spectral) matrix and N the order of **H** and **B.** The Hadamard matrix being used is assumed symmetric. Now since the columns of the Hadamard matrix are the Walsh vectors (cf. Welch 1970) and therefore the eigenvectors of a bicirculant are also the discrete Walsh vectors.

A bicirculant matrix of order 2^n will lead to 2^n independent (1, 0) matrices if $b_i = \delta_{ij}$ with $j = 1, 2, 3, \ldots, 2^n$ in turn. Thus for a fourth order matrix the four independent matrices are :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Since all these matrices are also bicirculant matrices, they have Walsh vectors. as their eigenvectors.

These matrices can be easily shown to be the regular representation of an abstract dyadic group (cf. Kak 1972).

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