MUTUALLY PARTIALLY ORTHOGONAL LATIN RECTANGLES AND MUTUALLY PARTIALLY ORTHOGONAL LATIN CUBES OF SECOND ORDER

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Let $r(s, v)$ be a latin rectangle of order $s \times v$ $(v \leqslant s)$ (cf. Ryser 1951) in s symbols to be denoted by 1, 2,... s. Without loss of generality the first column of $r(s, v)$ will be taken in natural order. Two latin rectangles $r_1(s, v)$ and $r_2(s, v)$ are said to be partially orthogonal, if on super-imposition on each other identical pairs of symbols occur exactly once and non-identical ordered pairs of symbols occur atmost once. Further a set of latin rectangles of order $s \times v$ will be called a set of mutually partially orthogonal latin rectangles (MPOLR), if every pair of the set, is partially orthogonal. The total number of MPOLR of order $s \times v$ $(v \leqslant s)$ is atmost $(s—1)$. Further the number of MPOLR of order $s \times v$ $(v \leqslant t)$ is at least $(t—1)$, if $s = 2t$ and t is an odd prime.

Let us consider the second-order latin cubes of side s with first square to be in some natural order. Two second order latin oubes of side s are said to be partially orthogonal if on superimposition on each other all identical pairs of symbols occur exactly once and non-identical ordered pairs of symbols, occur atmost once. A set of second order latin cubes of side s, will be called a set of mutually partially orthogonal latin cubes (MPOLC) if every pair of the latin cubes of the set is partially orthogonal. The total number of second-order MPOLC of side s is atmost $(s—1)^2$. The existence of t mutually orthogonal latin sequares (MOLS) of order s, implies, the existence of t^2 second-order MPOLC of side s.

A second order latin cube of side s can be used as an orthogonal design for the s^2 treatments eliminating heterogeneity in three directions.

INTRODUCTION

We present the concepts of mutually partially orthogonal latin rectangles (MPOLR) of order $s \times v$ and second-order mutually partially orthogonal latin cubes (MPOLC) of side s. The new method of construction of second-order latin cubes of side s, discusssed in this paper, will help in studying their combinatorial properties, in some more detail. Initially, Kishen (1949) gave a method of construction of a second-order latin cube of side s, when s is a prime or a prime power. Later on, Saxena (1960) constructed the second-order latin cubes of side s, for any positive integral value of s. An use of the second-order latin cubes of side s as orthogonal designs for the s^2 treatments, eliminating heterogeneity in three directions, is discussed in the Section under "Use of Second order Latin cubes of side s" of this paper.

The concepts introduced in this paper, will lead to the construction of some useful confounded factorial experiments and this aspect will be dealt in a subsequent paper. For the definitions of terms and notations used in this paper, we refer to Raghavarao (1971).

Mutually Partially Orthogonal Latin Rectangles

Ryser (1951) defined a latin rectangle as follows:

Definition 1: An $s \times v$ rectangle in s symbols denoted by 1, 2, .., s, is called a latin rectangle of order $s \times v$, if every symbol occurs exactly once in each column and atmost once in each row.

We shall represent an $s \times v$ latin rectangle by $r(s, v)$. Without loss of generality, we shall take the first column of a latin rectangle of order $s \times v$ to be in natural order. Usual orthogonality of two latin rectangles $r_1(s, v)$ and $r_2(s, v)$, is not defined. We define a pair of partially orthogonal latin rectangles as follows:

Definition 2: Two latin rectangles $r_1(s, v)$ and $r_2(s, v)$ are said to be partially orthogonal, if on super-imposition on each other identical pairs of symbols occur exactly once and non-identical ordered pairs occur atmost once.

Definition 3: A set of latin rectangles of order $s \times v$, will be called a set of mutually partially orthogonal latin rectangles (MPOLR), if every pair of latin rectangles of the set, is partially orthogonal.

The following theorem can easily be proved :

Theorem 1: The total number of MPOLR of order $s \times v$ ($v \leqslant s$) is atmost (s—1).

The (s—1) MPOLR of order $s \times v$ can be obtained, when s is a prime or a prime power, by deleting the last (s—v) columns of the (s—1) mutually orthogonal latin squares (MOLS) of order s [*see* Raghavarao (1971), Chapter I]. It is quite interesting to note that (t—1) MPOLR of order $s \times m (m \leqslant t)$ exist when t is an odd prime and $s=2t$. Now, we shall construct these (t—1) MPOLR of order $s \times t$.

Let $M=(0, 1, 2, .., s-1)$ be a module of s elements. Let x be a primitive root of GF(t). Either x or $x+t$ will be odd. Let this odd integer be denoted by a. It can be seen that $a^0, a^1, .., a^{t-2}$ (mod s) will give all the odd elements of M except t and $a^{t-1} = 1$[cf. Griffin (1954)]. The (t—1) initial ordered sets

$$(0, a^{i-1}, a^i, .., a^{t+i-3}),\ i = 1, 2, .., (t-1) \pmod s \qquad .. \qquad .. \qquad (1)$$

when generated in the sense of Bose (1939), give the (t—1) MPOLR of order $s \times t$. The (t—1) MPOLR of order $s \times m$ ($m \leqslant t$) can be obtained by deleting the last (t—m) columns of the above obtained set of (t—1) MPOLR.

We state the following theorems proofs for which can be constructed easily :

Theorem 2: The number of MPOLR of order $s \times v$ ($v \leqslant t$) is at least (t—1), if $s = 2t$ and t is an odd prime.

Theorem 3: The (s—1) MPOLR of order $s \times 2$ can always be constructed.

The concept of partially orthogonal latin rectangles is likely to open many new combinatorial problems. One obvious problem is the maximum number of MPOLR of order $s \times v$ ($v \leqslant s$) that can be constructed when s is a composite number for a particular value of v.

Second-Order Mutually Partially Orthogonal Latin Cubes of Side s

Kishen (1949) introduced the concept of second-order latin cubes of side s.

Definition 4: A second order latin cube of side s is an arrangement of the s^2 symbols in s squares of s rows and s columns such that each symbol occurs exactly once in each square, in the ith rows of all the squares ($i=1, 2, \ldots s$) and in the jth columns of all the squares ($j=1, 2, \ldots s$).

The arrangement of 9 symbols represented by 1, 2, 3, ... 9 as given in Eq, (2) is a second order latin cube of side 3.

$$\begin{array}{ccc} \text{Square 1} & \text{Square 2} & \text{Square 3} \end{array}$$

$$\begin{bmatrix} 1 & 4 & 7 & 9 & 3 & 6 & 5 & 8 & 2 \\ 2 & 5 & 8 & 7 & 1 & 4 & 6 & 9 & 3 \\ 3 & 6 & 9 & 8 & 2 & 5 & 4 & 7 & 1 \end{bmatrix} \quad .. \quad (2)$$

We, now, present the new method of construction of a second-order latin cube of side s.

We define the symbolic Hadamard product of two vectors of treatment combinations as following :

Definition 5: The symbolic Hadamard product of two ordered sets of treatment combinations $(a_1, a_2, \ldots, a_s)$ and $(b_1, b_2, \ldots, b_s)$ is defined as $(a_1\ b_1, \ldots, a_z, b_z)$.

For example the symbolic Hadamard product of (00, 12, 21) and (02, 30, 11) will be (0002, 1230, 2111).

Illustration 1: We, now, construct a second-order latin cube of side 3. Let us take a latin square of order 3 as given in (3), below :

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad .. \quad .. \quad .. \quad (3)$$

The symbolic Hadamard product of (0, 1, 2) with the three rows of the latin square given in (3) are

$$\begin{array}{lll} (00, & 11, & 22), \\ (01, & 12, & 20), \\ (02, & 10, & 21). \end{array} \quad .. \quad .. \quad .. \quad (4)$$

Further taking the symbolic Hadamard product of the three rows given in (4), with the three rows of the latin square given in (3), we get the following nine rows of the 27 treatment combinations of the 3^3 symmetrical factorial experiment :

Treatment	*Combinations*		*Row No.*
000	111	222	1
010	121	202	2
020	101	212	3
001	112	220	4
011	122	200	5
021	102	210	6
002	110	221	7
012	120	201	8
022	100	211	9

.. .. (5)

We number the 9 rows of (5) as 1, 2, ..., 9. Let the 27 treatment combinations of the 3^3 symmetrical factorial experiment, be arranged in the three squares of the 27 treatment combinations of the 3^3 experiment as following :

Square 1			Square 2			Square 3		
000	001	002	100	101	102	200	201	202
010	011	012	110	111	112	210	211	212
020	021	022	120	121	122	220	221	222

.. (6)

Replacing the treatment combinations of the squares given in (6) by the number of the rows in which they occur in (5), we get the second-order latin cube C_1 of side 3 given in (2).

This procedure is general and is valid to construct a second order latin cube of side s where s is any positive integer.

Without loss of generality, the s^2 symbols of the first square of the second-order latin cube of side s, can be written as

$$\begin{array}{cccccc} 1 & s+1 & . & . & . & s^2-s+1 \\ 2 & s+1 & . & . & . & s^2-s+1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ s & 2s & . & . & . & s^2 \end{array} \quad .. \quad (7)$$

The usual orthogonal latin cubes of the second-order, cannot be defined in three dimensions (cf. Saxena 1960). We define two second-order partially orthogonal latin cubes (POLC) of side s as following :

Definition 6: Two second-order latin cubes of side s, are said to be partially orthogonal, if on superimposition on each other all identical pairs occur exactly once and non-identical ordered pairs occur atmost once.

Illustration 2: Two second-order latin cubes C_1, C_2 of side 3 given in (2) and (8) are partially orthogonal.

$$\mathbf{C_2} = \begin{bmatrix} 1 & 4 & 7 & 6 & 9 & 3 & 8 & 2 & 5 \\ 2 & 5 & 8 & 4 & 7 & 1 & 9 & 3 & 6 \\ 3 & 6 & 9 & 5 & 8 & 2 & 7 & 1 & 4 \end{bmatrix} \quad .. \quad (8)$$

Definition 7: A set of second order latin cubes of side s, will be called a set of mutually partially orthogonal latin cubes (MPOLC), if every pair of the latin cubes of the set is partially orthogonal.

Illustration 3: Cubes C_1, C_2, C_3, C_4 given in (2), (8) and (9) are 4 second order MPOLC of side 3.

$$C_3 = \begin{bmatrix} 1 & 4 & 7 & 5 & 8 & 2 & 9 & 3 & 6 \\ 2 & 5 & 8 & 6 & 9 & 3 & 7 & 1 & 4 \\ 3 & 6 & 9 & 4 & 7 & 1 & 8 & 2 & 5 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1 & 4 & 7 & 8 & 2 & 5 & 6 & 9 & 3 \\ 2 & 5 & 8 & 9 & 3 & 6 & 4 & 7 & 1 \\ 3 & 6 & 9 & 7 & 1 & 4 & 5 & 8 & 2 \end{bmatrix}. \quad \ldots (9)$$

We prove

Theorem 4: Total number of second-order MPOLC of side s is atmost $(s-1)^2$.

Proof: Consider a set of second-order MPOLC of side s. Let us consider the symbols occurring in the (i,j)th position of the kth square ($i=1, 2, \ldots s$; $k=2, 3, \ldots, s$) of each of the latin cubes of the set. The symbols $(2s-1)$ in number occurring in the ith row and jth column of the Ist square as given in (7) cannot occur in the i, j)th position of each of the latin cubes of the set, otherwise the condition of their being second order latin cubes of side s, will be violated. Further, the symbols occurring in the (i, j)th position of the kth square of the latin cubes of the set, will be all distinct because of the partial orthogonality condition. This implies that the (i, j)th position of the kth square ($i, j = 1, 2, \ldots, s$; $k=2, 3, \ldots, s$) of each of these latin cubes can be filled in atmost in $(s-1)^2$ ways by the remaining $(s-1)^2$ symbols. Hence atmost $(s-1)^2$ second-order MPOLC of side s can be constructed. This completes the proof of the Theorem 4.

The $(s-1)^2$ second-order MPOLC of side s, can be constructed when s is a prime or a prime power. We illustrate this fact by constructing 4 second-order MPOLC of side 3. The two mutually orthogonal latin squares (MOLS) are

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

and

$$\begin{matrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{matrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Taking the symbolic Hadamard product of (0, 1, 2) with rows of the two latin squares given in (10) and (11), we get

$$\begin{array}{ccc} 00 & 11 & 22 \\ 01 & 12 & 20 \\ 02 & 10 & 21 \end{array} \qquad \ldots \qquad (12)$$

and

$$\begin{array}{ccc} 00 & 12 & 21 \\ 01 & 10 & 22 \\ 02 & 11 & 20 \end{array} \qquad \ldots \qquad (13)$$

Further taking the symbolic Hadamard product of the rows of (12) with the rows of (10) and (11); and the rows of (13) with the rows of (10) and (11) and numbering each of these nine sets of treatment combinations of the 3^3 symmetrical factorial experiment by 1, 2, ..9 and replacing the treatment combinations of the 3^3 experiment given by (6) by the number of the rows in which they occur, we get the 4 second-order MPOLC of side 3, given by C_1, C_2, C_3, C_4, of (2), (8) and (9).

It may be noted that the procedure followed in constructing a set of 4 second order MPOLC of side 3 discussed above is general and can be extended to construct a complete set of $(s-1)^2$ MPOLC of side s, when s is a prime or a prime power.

It may, further be noted that the procedure followed, in constructing a second order latin cube of side 3, given in illustration (1), is similar to the one explained by Saxena (1960) except that he had restricted to s being a prime or a prime power whereas we have removed this restriction and have generalized it for any s as one latin square of order s, will be always available. Further the $(s-1)^2$ second order latin cubes of side s, obtained by Saxena (1960), form the complete set of $(s-1)^2$ second order MPOLC of side s, after suitably remaining the s^2 symbols so that the first horizontal square of each of the second-order latin cubes of side s, is in the order as given in (7), The procedure of constructing the $(s-1)^2$ second order MPOLC of side s with the help of $(s-1)$ MOLS of order s, seems to be simpler than obtaining them through the flats of the Eucledian geometry as done by Saxena (1960), Further, an obvious advantage of the new method of construction of a second-order latin cube of side s, is that we can construct t^2 second-order MPOLC of side s, if t MOLS of order s exist. We state this result in the form of the following theorem :

Theorem 5 : The existence of t MOLS of order s implies, the existence of t^2 second-order MPOLC of side s.

Use of Second Order Latin Cubes of Side s

The second order latin cubes of side s, can be used as orthogonal designs involving s^2 treatments, for eliminating heterogeneity in three directions. Let the s^2 treatments be represented by the s^2 symbols of a second order latin cube of side s. Let a second order latin cube of side s, be selected at random from the set of all possible second order latin cubes of side s. Let

$$Y_{ijkl} = m + t_i + h_j + r_k + c_l + e_{ijkl}, \quad i=1,2,\ldots,s^2; j, \ k,l=1,2,\ldots,s \qquad \ldots (14)$$

be the observational set up with e_{ijkl} 's to be normally and independently distributed with means 0 and variances σ^2. Let m be the general mean, t_i be the ith treatment effect, h_j be the jth horizontal square effect, r_k be the kth row square effect and c_l be the lth column square effect. All these effects are assumed to be fixed-effects satisfying the relations $\sum_i t_i = 0$, $\sum_j h_j = 0$, $\sum_k r_k = 0$ and $\sum_l c_l = 0$. The set of equations (14) with these assumptions is called a fixed-effects model or model I or intrablock model without interactions.

Let T_i, HS_j, RS_k and CS_l be the totals of the yields of the plots to which the ith treatment is allotted, of the plots occurring in the jth horizontal square, of the plots occurring in the kth row square and of the plots occurring in the lth column square, respectively. Let $Y....$ be the total of the yields of all the plots. Then the anova to test the significance of the various effects, is given in Table I. The usual test procedure is followed to test the significances of the various effects.

TABLE I

Anova Table

Source	D.F.	S.S.	M.S.
Horizontal squares	$(s-1)$	$(\sum_j HS^2_j/s^2)-(Y^2..../s^3)$	MS_{HS}
Row squares	$(s-1)$	$(\sum_h RS^2_k/s^2)-(Y^2..../s^3)$	MS_{RS}
Column squares	$(s-1)$	$(\sum_l CS^2_l/s^2)-(Y^2..../s^3)$	MS_{CS}
Treatments	(s^2-1)	$(\sum_v T^2_i/s)-(Y^2..../s^3)$	MS_T
Error	$(s-1)(s^2-3)$	By substraction	MS_E
Total	(s^3-1)	$\sum_i\sum_j\sum_k\sum_l Y^2_{ijkl}-(Y^2../s^3)$	

The design discussed, in this para, will be called a second-order latin cubic design. In a second order latin cubic design, the number of plots required will be s^3 for the s^2 treatments whereas in a latin square design, the number of plots required for the same number of treatments, will be s^4. Further in the latin cubic design, the heterogeneity is removed in one more direction.

The problem of determining the number of distinct second order latin cubes of sides $s \leqslant 10$ and selecting a second order latin cube of side $s \leqslant 10$ at random will be discussed in the subsequent work.

CONCLUDING REMARKS

It may be noted that $(s-2)$ MPOLR of order $s \times v$ $(v<s)$ cannot always be extended to a complete set of $(s-1)$ MPOLR of order $s \times v$. The results pertaining to the second order mutually partially orthogonal latin cubes (MPOLC) of side s, can be extended to higher order hypercubes of side s, easily.

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