REMARKS ON A RECENT PAPER OF DE BRUIJN*

by PRABIR BHATTACHARYA, *St. Stephen's College, University of Delhi, Delhi-110007*

(Communicated by Prof. P. K. Menon, F.N.A.)

(Received 9 May 1974)

Let χ, θ be two permutation representations of a finite group G on two finite sets D and R respectively. Then χ, θ induce an equivalence in R^D. The elements of R^D are given weights. The Store-Enumerator of the mapping patterns is calculated. This gives some results proved earlier by de Bruijn as particular cases.

INTRODUCTION

A basic result in the theory of permutation groups is Burnside's Lemma :

"*Let G be finite group and π a permutation representation of G on a finite set X. Then the number of orbits of G is equal to :*

$$\frac{1}{|G|} \sum_{g \in G} \psi(g) \qquad \qquad .. \quad (1)$$

where ψ is the character of the permutation representation.

It is interesting to observe that Burnside's Lemma plays a key role in the study of various enumeration problems concerning structures involving graphs, finite state machines etc. Many such problems can be solved by a celebrated combinatorial theorem proved by Polya in 1937 which depends on Burnside's Lemma for its derivation. However, Polya's theory of enumeration was anticipated much earlier by Redfield in 1927 which was overlooked at that time. Later on, there had been many outstanding contributions to the subject by de Bruijn, Harary, Head and others.

In a recent paper de Bruijn (1972), has considered the following combinatorial situation:

G is a finite group and D, R are two finite sets. R^D denotes the set of all mappings from D into R. χ and θ are two permutation representations of G on the sets D, R respectively. In R^D an equivalence relation is defined as :

$$f_1 \sim f_2 \text{ if there exists } g \in G \text{ such that } f_1\,\chi(g) = \theta(g) f_2 \qquad .. \quad (2)$$

where $f_1, f_2 \in R^D$

Each equivalence class F is called a '*mapping pattern*'.

De Bruijn (1972) enumerates the number of all mapping patterns and also the number of patterns consisting of injective mappings. These results are expressed in the theorems 1 & 2 of the same paper.

*'Enumeration of Mapping Patterns' by N. G. de Bruijn. *J. Combl. Th.*, Vol. **12**, No. **1**, Jan. 1972, 14–20.

In this note, we give 'weights' to the elements of R^D in such a manner that equivalent elements have equal weights. We then obtain the Store Enumerator of the mapping patterns. It is interesting to find that the Store Enumerator gives as special cases the two theorems of de Bruijn (1972). Also, we show how the weighted form of Burnside's Lemma and a key Lemma of de Bruijn (1959), can be derived as special cases. The main idea of this note is somewhat implicit in de Bruijn's (1972) paper though, however, he does not explicitly prove the theorem which we intend to prove here. Also, D. A. Klarner in his review of de Bruijn (1972) in the Mathematical Reviews, suggests similar possibilities.

Remarks

Let Q be a commutative ring containing the rationals. Let W be a mapping of R^D into Q such that :

$$f_1 \sim f_2 \Rightarrow W(f_1) = W(f_2) \quad \ldots (3)$$

where $f_1, f_2 \in R^D$.

The image $W(f)$ is called the *'weight'* of the function $f \in R^D$. There is a quite general method of defining W so as to satisfy (3). This is one of the important contributions of de Bruijn and is described in de Bruijn (1964), page 163.

Now, we define the weight of a mapping pattern F as $W(f)$ for any f in F. (Because of (3), we can take any f in F).

The sum $\sum_F W(F)$, summed over all the mapping patterns F of R^D, is called the Store Enumerator of the mapping patterns.

Theorem: The store enumerator $\sum_F W(F)$ is given by :

$$\sum_F W(F) = \frac{1}{|G|} \sum_{g \in G} \sum_{f \in \mathfrak{V}(g)} W(f) \quad \ldots (4)$$

$$\text{where } \mathfrak{V}(g) = \{f \in R^D : f\chi(g) = \theta(g) f\} \quad \ldots (5)$$

Proof : Consider the permutation representation π of G on R^D induced by χ and θ :

$$\pi(g) f = \theta(g) f \chi(g)^{-1} \quad \ldots (6)$$

for every $f \in R^D$.

This determines in a natural way an equivalence relation in R^D :

$f_1 \sim f_2$, if there exists $g \in G$ such that $f_1 \chi(g) = \theta(g) f_2$ where $f_1, f_2 \in R^D$

Let F be a typical pattern under the above equivalence. We observe that $\pi/_F$ is a transitive permutation representation, i.e., there is only one orbit.

$$\therefore \text{ By Burnside's Lemma, } 1 = \frac{1}{|G|} \sum_{g \in G} |F \cap \mathfrak{V}(g)|$$

$$\therefore\ W(F) = \frac{1}{|G|} \sum_{g \in G} W(F) \cdot |F \cap \mathfrak{V}(g)|$$

$$=\frac{1}{|G|}\sum_{g\in G}\sum_{f\in F\cap \mho(g)} W(f),$$ as all functions in $F\cap\mho(g)$ have the same weight $W(F)$.

Summing over all the mapping patterns, we obtain

$$\sum_{F} W(F)=\frac{1}{|G|}\sum_{F}\sum_{g\in G}\sum_{f\in F\cap\mho(g)} W(f)=$$

$$\frac{1}{|G|}\sum_{g\in G}\Big(\sum_{F}\sum_{f\in F\cap\mho(g)} W(f)\Big)$$

or $$\sum_{F} W(F)=\frac{1}{|G|}\sum_{g\in G}\sum_{f\in \mho(g)} W(f).$$

Examples: We first show that the theorems 1 and 2 of the Bruijn's (1972) paper can be obtained as special cases of our theorem.

(*i*) Let W be chosen such that $W(f)=1$ for all f in R^D. Clearly this definition of W satisfies the condition that equivalent functions have equal weights. This kind of definition of weight function is due to de Bruijn (1959). Then the Store Enumerator for this choice of weight W is nothing but the number of mapping patterns. The right hand side of Eq. (4) is the same as Eq. (1) of de Bruijn's (1972) paper. This, then following the paper, leads to the theorem 1 of de Bruijn that the number of mapping patterns is equal to :

$$T\left(\chi,\theta\,;\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\ldots\ldots;e^{z_1+z_2+z_3+\ldots},e^{2(z_2+z_4+z_6+\ldots)}\right),$$

evaluated at $z_1=z_2=\ldots\ldots=0$

with the same notations as given in de Bruijn (1972).

(*ii*) Choose W such that $W(f)=1$ if f is 1—1 and 0 if f is otherwise. Again, the Store Enumerator represents the number of injective mapping patterns. The right hand side of Eq. (4) leads to the result expressed by de Bruijn in Theorem 2 of his 1972 paper, that the number of injective mapping patterns is equal to :

$$T\left(\chi,\theta\,;\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\ldots\ldots;1+z_1,1+2z_2,\ldots\ldots\right),$$

evaluated at $z_1=z_2,\ldots\ldots=0.$

(*iii*) We can also get weighted form of Burnside's Lemma as a special case of Eq. (4). Take G to be a permutation group acting on D. Let the representation χ be taken as the natural action of G on D. Also, take θ to be the identity permutation representation. Then the definition of equivalence becomes :

$f_1\sim f_2$ if $\exists\, g\in G$ such that $f_1\,g=f_2$

Let $W: R\rightarrow Q$. The image $w(r)$ of r is called the weight of $r\in R$. Define the function W by

$$W(f) = \prod_{d \in D} w(f(d)) \qquad .. \quad (7)$$

Thus, in this special case, the Store Enumerator $\sum_{F} W(F)$ is the same as equation (5.17), page 158 of de Bruijn (1964). This is called the 'Weighted form of Burnside's Lemma' (*see* Harary 1969). From this, one can prove the famous theorem of Polya :

$$\sum_{F} W(F) = P_G \left(\sum_{r \in R} w(r), \sum_{r \in R} w(r)^2, \ldots \right).$$

with the same notations as in de Bruijn (1959).

(*iv*) Let $G^* = G \times H$ where G, H are two permutation groups on D & R respectively. Define $\chi(g, h) = g$, $\theta(g, h) = h$, $\forall \, g \in G, h \in H$. This construction was given by de Bruijn (1972). In this special case, the definition of equivalence becomes :

$f_1 \sim f_2$ if $f_1 g = h f_2$ for some $g \in G$, $h \in H$

Thus we have the same situation as described by de Bruijn in his generalisation of Polya's theorem. In this special case, we observe that the Eq. (4) gives the key Lemma of de Bruijn (1964). We remark that Theorem 1 of de Bruijn's same paper can be generalised without much difficulty using the theorem of our paper, to express the Store Enumerator in terms of the cycle indices of the two permutation groups χ (G) and θ (*G*). Such a possible generalisation has been suggested by de Bruijn (1972).

Acknowledgement

This work is partially supported by a grant from the University Grants Commission, New Delhi, India.

References

de Bruijn, N. G. (1959). Generalisation of Polya's Fund. thm. in enumerative combinatorial analyses. *Nedert. Akad. Wetensch, Proc. Ser A* **62**, *Indiag Math.* **21**, 59–67.

de Bruijn, N. G. (1964). Polya's theory of counting. *Applied Combinatorial Mathematics. ed.* E. F. Beckenbach, Wiley, New York, Chap. 5, pp. 144–184.

de Bruijn, N. G. (1972). Enumeration of Mapping Patterns. *J. Combl. Th.*, **12**, (1), 14–20.

Harary, F. (1969). *Graph Theory*. Addison Wesley.