CHARACTERIZATION OF THREE-PERFECT GRAPHICAL DEGREE SEQUENCES

by VASANTI N. BHAT, *Department of Mathematics University of Bombay, Bombay*

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A sequence $d_1 \geq d_2 \geq \ldots \geq d_p$ of non-negative integers, $p \geq 3$, where $\sum_{i=1}^{p} d_i$ is even and $d_1 \leq p-1$ is defined to be 3-perfect if $d_i = d_{i+1} = d_{i+2}$ for some i, $1 \leq i \leq p-2$ and all other d_j's are mutually distinct and distinct from d_i. Such a sequence is called graphical if there exists a graph with $d_1, \ldots, d_p$ as its degree sequence. In this paper graphical 3-perfect degree sequences have been completely characterized.

INTRODUCTION

We consider only finite undirected graphs without loops and multiple edges. The degrees $d_1, \ldots d_p$ of vertices of a graph G of order p form a sequence of non-negative integers whose sum is even and $p-1 \geq d_i$ for $1 \leq i \leq p$. In general a sequence $d_1 \geq d_2 \geq \ldots \geq d_p$ of nonnegative integers whose sum is even and where $p-1 \geq d_1$ is not a *graphical sequence*, that is, there may not exist a graph G of order p with this sequence as its degree sequence. If there does exist one, we call such a sequence *graphical*. We quote two well known results giving necessary and sufficient conditions for a sequence of non-negative integers to be graphical. The first one is due to Havel (1955) and Hakimi (1962) who proved it independently. The second is by Erdös and Gallai (1960).

Theorem (Havel 1955; and Hakimi 1962) : A sequence $d_1 \geq d_2 \geq \ldots \geq d_p$ of non-negative integers whose sum is even and where $p-1 \geq d_1$ is graphical if and only if the sequence $d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, d_{d_1+3}, \cdots d_p$ is graphical.

Theorem (Erdös and Gallia 1960) : A sequence $d_i \geq d_2 \geq \ldots \geq d_p$ of nonnegative integers whose sum is even and where $p-1 \geq d_1$ is graphical if and only if

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{p} \min \left\{ r, d_i \right\}$$

for every integer r, $1 \leq r \leq p-1$.

Behzad and Chartrand (1967) defined a graph with two or more vertices to be *perfect* if for each pair of distinct vertices u and v, deg $u \neq$ deg v, that is, no two vertices have the same degree. It immediately follows that no graph is perfect. They further defined a graph with at least two vertices to be *quasiperfect* if there are precisely two vertices with the same degree. They gave the following complete characterization of quasiperfect graphs.

Theorem (*Behzad & Chartrand 1967*) : *For every integer* $p \geq 2$, *there are exactly two nonisomorphic quasiperfect graphs with p vertices. Furthermore, these two graphs are complementary and the equal degrees are* $\left[\frac{p}{2}\right]$ *in the connected graph and* $\left[\frac{p-1}{2}\right]$ *in the disconnected graph.*

In this paper we pursue these ideas further :

Definition 1. A graph G of order $p \geq 3$ is said to be *3- perfect* if there are precisely three vertices with the same degree and among all other vertices no two have the same degree.

Definition 2. A sequence $d_1 \geq d_2 \geq \ldots \geq d_p$ of nonnegative integers, $p \geq 3$, whose sum is even and where $p-1 \geq d_1$ is said to be *3-perfect* if $d_i = d_{i+1} = d_{i+2}$ for some i, $1 \leq i \leq p-2$ and all other d_j's are mutually distinct and distinct from d_i.

We call d_i *special degree.* Obviously the degree sequence of a 3-perfect graph is 3-perfect and if a 3-peerfect degree sequence is graphical then a realization is a a 3-perfect graph.

Let G be a graph of order p. The possible values for degree of a vertex of G are $p-1$, $p-2$, .., 1, 0. If there is a vertex of degree $p-1$ then no vertex of G can have degree 0 and if G has an isolated vertex then G cannot have a vertex of degree $p-1$.

Definition 3. A 3-perfect graph G of order p is said to be of *type I* if G has a vertex of degree $p-1$. Similarly, a 3-perfect degree sequence $d_1 \geq d_2 \ldots \geq d_p$ is said to be of Type I if $d_1 = p-1$.

If G is 3-perfect, Type I, then besides 0 there is exactly one more integer m, $1 \leq m \leq p-2$, which is not degree of any vertex of G. We call m *missing degree.* We use this terminology for 3-perfect degree sequences too.

We observe that G is 3-perfect if and only if its complementary graph $\bar{G}$ is 3-perfect. If G is of Type I, special degree i, missing degree m, then in $\bar{G}$ there is a vertex of degree 0, $p-1-i$ is special degree and $p-1-m$ is missing degree.

Definition 4. A 3-perfect graph G of order p is said to be of *Type II* if G has a vertex of degree 0. Similarly, a 3-perfect degree sequence $d_1 \geq d_2 \geq \ldots \geq d_p$ is said to be of *Type II* if $d_p = 0$.

G is of Type I if and only if $\bar{G}$ is of Type II.

Definition 5. A 3-perfect graph G of order p is said to be of *Type III* if G has no vertex of degree $p-1$ and no vertex of degree 0. Similarly a 3-perfect degree sequence $d_1 \geq d_2 \geq \ldots \geq d_p$ is said to be of Type III if $d_1 \neq p-1$, and $d_p \neq 0$.

We note that G is 3-perfect, Type III, special degree i if and only if $\bar{G}$ is 3-perfect, Type III, special degree $p-1-i$. In case of Type III there is no question of missing degree other than $p-1$ and 0.

Let G be a 3-perfect graph of order p, Type III, special degree i. Then its degree sequence is

$$p-2 \geq \ldots \geq i \geq i \geq i \geq \ldots \geq 1.$$

Since the sum of the degrees has to be even it follows that $p \equiv 1, 2 \pmod 4$.

We give below all possible 3-perfect graphical sequences when $3 \leq p \leq 7$. These are calculated by considering all possibilities and using Theorem HH[1].

TABLE I

All possible 3-perfect graphical sequences with $p \leqslant 7$.

p	Type I	Type II	Type III
3	2, 2, 2	0, 0, 0	
4	3, 1, 1, 1	2, 2, 2, 0	
5	4, 3, 3, 3, 1	3, 1, 1, 1, 0	3, 2, 1, 1, 1
			3, 2, 2, 2, 1
			3, 3, 3, 2, 1
6	5, 4, 3, 2, 2, 2	3, 3, 3, 2, 1, 0	4. 3, 2, 1, 1, 1
	5, 4, 2, 2, 2, 1	4, 3, 3, 3, 1, 0	4, 3, 2, 2, 2, 1
	5, 4, 3, 3, 3, 2	3, 2, 2, 2, 1, 0	4, 3, 3, 3, 2, 1
	5, 4, 4, 4, 3, 2	3, 2, 1, 1, 1, 0	4, 4, 4, 3, 2, 1
7	6, 5, 5, 5, 4, 3, 2	4, 3, 2, 1, 1, 1, 0	
	6, 4, 4, 4, 3, 2, 1	5, 4, 3, 2, 2, 2, 0	
	6, 5, 4, 4, 4, 3, 2	4, 3, 2, 2, 2, 1, 0	
	6, 4, 3, 3, 3, 2, 1	5, 4, 3, 3, 3, 2, 0	
	6, 5, 4, 3, 3, 3, 2	4, 3, 3, 3, 2, 1, 0	
	6, 4, 3, 2, 2, 2, 1	5, 4, 4, 4, 3, 2, 0	
	6, 5, 4, 3, 2, 2, 2	4, 4, 4, 3, 2, 1, 0	

Some of these 3-perfect sequences have more than one realizations. For example, the 3-perfect degree sequence 4, 3, 2, 2, 2, 1 which is of Type III has the following four nonisomorphic realizations :

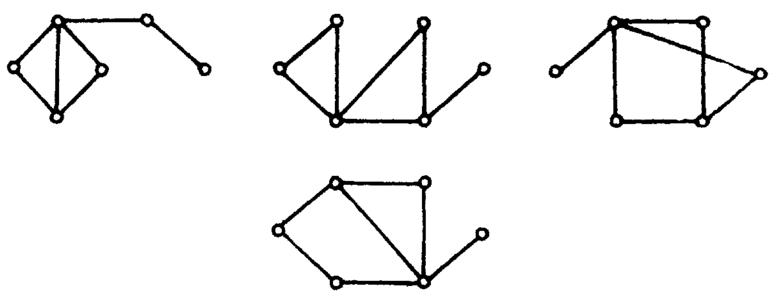

Fig. 1. Nonisomorphic realizations of a 3-perfect degree sequence 4, 3, 2, 2, 2, 1 (Type III)

[1]HH refers to Havel (1955) and Hakimi (1962).

Fig. 2. Nonisomorphic realizations of a 3-perfect degree sequence 3, 2, 2, 2, 1, 0 (Type II)

The degree sequence 3, 2, 2, 2, 1, 0 which is 3-perfect of Type II has two realizations as shown above.

Definition 6. In a 3-perfect degree sequence of order p, Type I or Type II, missing degree m is said to be *compatible* with special degree i if this 3-perfect degree sequence is graphical.

Corresponding to a special degree i there can be several compatible missing degrees m. To illustrate this and for subsequent use we give below a Table showing all possible special degrees and corresponding compatible missing degrees when $p=7$.

TABLE II

Type I		Type II	
Special degree	Missing degrees	Special degree	Missing degrees
5	1	1	5
4	1, 5	2	5, 1
3	1, 5	3	5, 1
2	1, 5	4	5, 1

PRELIMINARY RESULTS

We state without proof the following :

Proposition 1. Suppose G is a 3-perfect graph of order $p \geq 3$. Then we have the following results :

(*a*) Let $p \equiv 0 \pmod 4$. If G is of Type I then missing degree is even whereas if G is of Type II then missing degree is odd.

(*b*) Let $p \equiv 1 \pmod 4$. Whether G is of Type I or Type II missing degree is even.

(*c*) Let $p \equiv 2 \pmod 4$. If G is of Type I then missing degree is odd whereas if G is of Type II then missing degree is even.

(*d*) Let $p \equiv 3 \pmod 4$. Whether G is of Type I or Type II missing degree is odd.

For an easy reference we give below a Table showing the parity of missing degree as in Proposition 1.

Theorem 1 : Let $p \equiv 0 \pmod 4 \geq 3$. If G is a 3-perfect graph of order $p=4t$, Type I, then $4t-k$, $1 \leq k \leq t$ cannot be special degree. Also, l, $1 \leq l \leq t-1$ cannot be special degree.

TABLE III

	Type I	Type II
$p \equiv 0 \pmod 4$	even	odd
$p \equiv 1 \pmod 4$	even	even
$p \equiv 2 \pmod 4$	odd	even
$p \equiv 3 \pmod 4$	odd	odd

Proof: The proof depends on Theorem EG[2] of Erdös and Gallai (1960). Throughout the proof r has the same meaning as in Theorem EG as stated in the Introduction.

Evidently $p-1$ cannot be special degree for any 3-perfect graph G of order p and Type I. Also, we may assume $t \geq 2$ since the case $p=4$ is completely known.

If possible, let $4t-k$, $2 \leq k \leq t$ be special degree for a 3-perfect graph G of order $4t$, Type I. Let m be missing degree, $1 \leq m \leq p-2$.

Case A: Suppose $m=4t-j$, $2 \leq j < k \leq t$.

With $r=2k-j+1$ in Theorem EG, since $p \geq 2r$, we must have

$$pr - \frac{(2k-j)(2k-j+1)}{2} - 2k+j \leq r(r-1) + \frac{r(r+1)}{2} + (p-2r)r.$$

This inequality reduces to $1 \leq 0$ which is impossible.

Case B: Suppose $m=4t-(k+j)>k+2$, $j \geq 1$.

With $r=k+2$ in Theorem EG, since $p \geq 2r$, we must have

$$pr - \frac{k(k+1)}{2} - 2k \leq r(r-1) + \frac{r(r+1)}{2} + (p-2r)r.$$

This inequality reduces to $3 \leq 0$ which is impossible.

Case C: Suppose $m=k+2$.

With $r=k+2$ in Theorem EG, since $p \geq 2r-1$ we must have

$$pr - \frac{k(k+1)}{2} - 2k \leq r(r-1) + \frac{(k+1)(k+2)}{2} + (p-2r+1)r.$$

This inequality reduces to $3 \leq 0$ which is impossible.

Case D: Suppose $m=k+1$.

With $r=k+1$ in Theorem EG, since $p \geq 2r-1$ we must have

$$pr - \frac{k(k+1)}{2} - k \leq r(r-1) + \frac{k(k+1)}{2} + (p-2r+1)r.$$

This inequality reduces to $1 \leq 0$ which is impossible.

[2]EG refers to Erdös and Gallai (1960).

Case E : Suppose $m \leq k$.

From Proposition 1, we know that m has to be even. Hence $m \neq 1$. Take $r = 2k-m+2$. We then have $r \geq k+2 > m$ and $p \geq 2r-1$. Therefore we must have

$$pr - \frac{(2k-m)(2k-m+1)}{2} - 2k \leq r(r-1) + \frac{r(r+1)}{2} - m + (p-2r+1)r.$$

This inequality reduces to $1 \leq 0$ which is impossible.

We have now completely proved that for a 3-perfect graph G of order $4t$, Type I, $4t-k$, $2 \leq k \leq t$, cannot be special degree.

Next we prove the later part of the theorem. If possible, let l, $1 \leq l \leq t-1$ be special degree for a 3-perfect graph G of order $4t$, Type I. Let m be missing degree $1 \leq m \leq p-2$.

Case A_1 : Suppose $1 \leq m < l$.

With $r = 2l-m+1$ in Theorem EG, since $p \geq 2r+1$ we must have

$$pr - \frac{r(r+1)}{2} \leq r(r-1) + \frac{r(r+1)}{2} + 2l-m+(p-2r-1)r.$$

This inequality reduces to $1 \leq 0$ which is impossible.

Case B_1 : Suppose $m = l+1$.

With $r = l+1$ in Theorem EG, since $p \geq 2r+1$, we must have

$$pr - \frac{r(r+1)}{2} \leq r(r-1) + \frac{l(l+1)}{2} + 2l+(p-2r-1)r.$$

This inequality reduces to $2 \leq 0$ which is impossible.

Case C_1 : Suppose $m = l+2$.

With $r = l+2$ in Theorem EG, since $p \geq 2r+1$, we must have

$$pr - \frac{r(r+1)}{2} \leq r(r-1) + \frac{(l+1)(l+2)}{2} + 2l+(p-2r-1)r.$$

This inequality reduces to $4 \leq 0$ which is impossible.

Case D_1 : Suppose $l+2 < m \leq 4t-(l+3)$.

With $r = l+2$ in Theorem EG, since $p \geq 2r+2$, we must have

$$pr - \frac{r(r+1)}{2} \leq r(r-1) + \frac{(l+2)(l+3)}{2} + 2l + (p-2r-2)r.$$

This inequality reduces to $4 \leq 0$ which is impossible.

Case E_1 : Suppose $m \geq 4t-(l+2)$.

Let $m = 4t-s$, $2 \leq s \leq l+2$. Take $r = 2l-s+3$. We then have $s \leq r+1$, $l \leq r$ and $p \geq 2r+2$. Therefore we must have

$$pr - \frac{(r+1)(r+2)}{2} + s \leq r(r-1) + \frac{r(r+1)}{2} + 2l + (p-2r-2)r.$$

This inequality reduces to $2 \leq 0$ which is impossible.

The theorem is now completely proved.

We reformulate Theorem (1) in terms of degree sequences as

Corollary 1 : A 3-perfect degree sequence of order $4t$, Type I and special degree $4t-k$, $1\leq k\leq t$, or special degree l, $1\leq l\leq t-1$ cannot be graphical whatever may be missing degree.

Consequently, we have

Corollary 2 : A 3-perfect degree sequence of order $4t$, Type II and special degree l, $0\leq l\leq t-1$ or special degree $4t-k$, $2\leq k\leq t$, cannot be graphical whatever may be missing degree.

We now prove that certain 3-perfect degree sequences of order $4t$ are graphical.

Theorem 2: Let $p\equiv 0 \pmod 4)\geq 3$. The 3-perfect degree sequence of order $p=4t$, Type 1, special degree $4t-k$, $t+1\leq k\leq 3t$. and missing degree $4t-2$ is graphical.

Proof: A degree sequence under consideration is as follows :

$$4t-1,\ 4t-3,\ ..,\ 4t-(k-1),\ 4t-k,\ 4t-k,\ 4t-k,\ ..,\ 1.$$

We verify that the Erdös Gallai (1960) criterion goes through for this degree sequence.

If $r=1$, we must have $4t-1\leq 4t-1$ which is true.

Let $2\leq r\leq k-2$. We have the following possibilities :

(*i*) $r<4t-k$,

(*ii*) $4t-k\leq r\leq 4t-(r+2)$, and

(*iii*) $r>4t-(r+2)$.

In case of (*i*), since $p\geq 2r$, we must have

$$pr-\frac{(r+1)\ (r+2)}{2}+2\leq r(r-1)+\frac{r(r+1)}{2}+(p-2r)r.$$

This inequality reduces to $1\leq r$ which is true.

In case of (*ii*) since $p\geq 2r+2$, we must have

$$pr-\frac{(r+1)\ (r+2)}{2}+2\leq r(r-1)+\frac{r(r+1)}{2}+2(4t-k)+(p-2r-2)r.$$

This inequality reduces to

$$0\leq 8t-r-2k-1.$$

We have $2r\leq 4t-2$ and hence $r\leq 2t-1$. Also $k\leq 3t$. This gives $8t-r-2k-1\geq 0$.

In case of (*iii*) we must have

$$pr-\frac{(r+1)\ (r+2)}{2}+2\leq r(r-1)+\frac{\{4t-(r+2)\}\{4t-(r+2)+1\}}{2}+2(4t-k).$$

This inequality reduces to

$$2r^2+2r+\frac{p^2}{2}-2pr+\frac{p}{2}-2k\geq 0.$$

We have $p=4t$, $k\leq 3t$, $r\geq 2t-1$ and hence

$$r^2+r+4t^2-4tr+t-k\geq r^2+r+4t^2-4tr+t-3t=\{(r-(2t-1)\}\ (r-2t)\geq 0.$$

Let $r=k-1$. We have the following possibilities :

(*i*) $r<4t-k$.

(*ii*) $r\geq 4t-k$.

In case of (*i*), since $p\geq 2r$, we must have

$$pr-\frac{k(k+1)}{2}+2\leq r(r-1)+\frac{r(r+1)}{2}+(p-2r)r.$$

This inequality reduces to $1\leq r$ which is true.

In case (*ii*), we must have

$$pr-\frac{k(k+1)}{2}+2\leq r(r-1)+\frac{(4t-k)(4t-k+1)}{2}+4t-k.$$

This inequality reduces to

$$0\leq 2r^2+\frac{p^2}{2}-2pr+\frac{p}{2}-2.$$

We have, since $r\geq 2t$,

$$r^2+4t^2-4tr+t-1=(r-2t)^2+(t-1)\geq 0.$$

Let $r=k$. We have the following possibilities :

(*i*) $r<4t-k$,

(*ii*) $r\geq 4t-k$.

In case (*i*), since $p\geq 2r$, we must have

$$pr-\frac{k(k+1)}{2}-k+2\leq r(r-1)+\frac{r(r+1)}{2}+(p-2r)r.$$

This inequality reduces to $2\leq r$ which is true.

In case (*ii*), we must have

$$pr-\frac{k(k+1)}{2}-k+2\leq r(r-1)+\frac{(4t-k)(4t-k+1)}{2}.$$

This inequality reduces to

$$0\leq 2r^2+\frac{p^2}{2}-2pr+\frac{p}{2}-2.$$

But

$$r^2+4t^2-4tr+t-1=(r-2t)^2+(t-1)\geq 0.$$

Let $r\geq k+1$. We have the following possibilities :

(*i*) $r<4t-r$,

(*ii*) $r\geq 4t-r$.

In case (i), since $p>2r$, we must have

$$pr-\frac{(r-1)r}{2}-2k+2\leq r(r-1)+\frac{r(r+1)}{2}+(p-2r)r.$$

This inequality reduces to

$$0\leq 2k-r-2.$$

Since $t+1<k$ and $r\leq 2t$, it is true that $0\leq 2k-r-2$.

In case (*ii*) we must have

$$pr - \frac{(r-1)r}{2} - 2k + 2 \leq r(r-1) + \frac{(4t-r)(4t-r+1)}{2}.$$

This inequality reduces to

$$0 \leq 2r^2 - 2r + \frac{p^2}{2} - 2pr + \frac{p}{2} + 2k - 2.$$

We have to show that

$$0 \leq r^2 - r + 4t^2 - 4tr + t + k - 1 = (r-2t)^2 + t + k - r - 1.$$

Let $k = t+1+i$, $0 \leq i \leq 2t-1$ and $r = 2t+j$, $0 \leq j \leq 2t$. Then the above inequality reduces to $0 \leq j^2 - j + i$ which is true.

Corollary 3: Let $p \equiv 0 \pmod 4$. The 3-perfect degree sequence of order $p = 4t$, Type II, special degree i, $t \leq i \leq 3t-1$, and missing degree 1 is graphical. We define

$$S(4t, \mathrm{I}) = \{i,\ 0 \leq i \leq 4t-1 \mid i \text{ is special degree in some graphical sequence of order } 4t, \text{ Type I}\}.$$

We define $S(4t, \mathrm{II})$ and in general $S(p, \mathrm{I})$, $S(p, \mathrm{II})$, $S(p, \mathrm{III})$ in a similar fashion. Let α, β be integers we define

$$[\alpha, \beta] = \{i \mid i \text{ is an integer and } \alpha \leq i \leq \beta\}.$$

From Theorem 2 and Corollary 3 we have

$$S(4t, \mathrm{I}) = [t, 3t-1] = S(4t, \mathrm{II}).$$

We observe that $|S(4t, \mathrm{I})| = |S(4t, \mathrm{II})| = 2t$ and $S(4t, \mathrm{I}) = S(4t, \mathrm{II})$ forms the middle half of the range 0 to $4t-1$ of possible degrees of vertices in a graph of order $4t$. Theorem 2 and Corollary 3 lead us to the following :

Definition 7: A missing degree m which is compatible with every special degree in $S(p, \mathrm{I})$ [respectively $S(p, \mathrm{II})$] is called *first universal missing degree* (abbreviated f.u.m.d.) for $S(p, \mathrm{I})$ [respectively for $S(p, \mathrm{II})$].

For example, $4t-2$ is f.u.m.d. for $S(4t, \mathrm{I})$ while 1 is that for $S(4t, \mathrm{II})$.

Theorem 3: Let $p \equiv (\mathrm{mod}\ 4) \geq 3$, If G_p is a 3-perfect graph of order $p = 4t+1$, Type III, then $4t-k$, $1 \leq k < t-1$ cannot be special degree. Also, l, $1 \leq l \leq t-1$ cannot be special degree.

Proof: G_p exist if and only if $\bar{G}_p$ exists. Hence it is s ufficient to provethat $4t-k$, $1 \leq k \leq t-1$ cannot be special degree. If not, the degree sequence of G_p is

$$4t-1, \ldots, 4t-k, \quad 4t-k, \quad 4t-k, \ldots, 1.$$

With $r = 2k+2$ in Theorem EG, since $p \geq 2r$, we must have

$$(p-1)r - \frac{2k(2k+1)}{2} - 2k \leq r(r-1) + \frac{r(r+1)}{2} + (p-2r)r$$

$$\text{i.e.,} \quad 1 \leq 0$$

which is impossible.

Theorem 4: *Let* $p \equiv 2$ (*mod* $4 \geq 3$. *If* G_p *is a 3-perfect graph of order* $p = 4t+2$, *Type III, then* $4t-8$, $0 \leq k \leq t-2$ *cannot be a special degree. Also* l, $1 \leq l \leq t-1$ *cannot be special degree.*

Proof: G_p exists if and only if $\bar{G}_p$ exists. Hence it is sufficient to prove that $4t-k$, $0 \leq k \leq t-2$ cannot be a special degree for G_p. If not, the degree sequence of G_p is

$$4t, \ldots, 4t-k, 4t-k, 4t-k, \ldots, 1.$$

With $r=2k+4$ in Theorem EG, since $p \geq 2r$, we must have

$$(p-2)r-\frac{(2k+1)(2k+2)}{2}-2k \leq r(r-1)+\frac{r(r+1)}{2}+(p-2r)r$$

i. e., $1 \leq 0$

which is impossible.

Stepwise Development

We prove that a 3-perfect graphical sequence of order p, whatever Type it may be, can always be constructed from one of order $p-1$. We may assume without loss of generality that $t \geq 2$.

Lemma 1: Let $p \geq 3$. A 3-perfect graph G_p of order $p \equiv 0 \pmod 4$ and Type I, special degree i, missing degree m (respectively Type II, special degree i', missing degree m') exists if and only if a 3-perfect graph G_{p-1} of order $p-1 \equiv 3 \pmod 4$ and Type II, special degree $i-1$, missing degree $m-1$ (respectively Type I, special degree i', missing degree m') exists.

Proof: Let G_p be a 3-perfect graph of order $p \equiv 0 \pmod 4$, Type I, special degree i, missing degree m. There is a unique vertex of degree $p-1$ and since missing degree in G_p has to be even, G_p has a vertex of degree 1. The graph G_{p-1} obtained by deleting the vertex of degree $p-1$ evidently has the required properties.

Conversely, starting with $\bar{G}_{p-1}$, order $p-1 \equiv 3 \pmod 4$, Type II, special degree $i-1$, missing degree $m-1$, by taking a new vertex and joining it to all the vertices of G_{p-1} we get G_p as required.

We know that G_p, Type I, special degree i, missing degree m exists if and only if the complementary graph G_p, Type II, special degree $i' = p-1-i$, missing degree $m' = p-1-m$ exists. But then G_{p-1}, Type I, special degree $(p-1)-1-(i-1) = p-1-i = i'$, missing degree $(p-1)-1-(m-1) = p-1-m = m'$ exists and conversely.

Following corollary is immediate :

Corollary 4: $S(4t-1, \text{I}) = S(4t, \text{II}) = [t, 3t-1]$ and $S(4t-1, \text{II}) = [t-1, 3t-2]$. Further, 1 is f.u.m.d. for $S(4t-1, \text{I})$ while $4t-3$ is f.u.m.d. for for $S(4t-1, \text{II})$.

Lemma 2: Let $p \geq 3$. A 3-perfect graph G_p of order $p \equiv 1 \pmod 4$ and Type I, special degree i, missing degree m (respectively Type II, special degree i', missing degree m') exists if and only if a 3-perfect graph G_{p-1} of order $p-1 \equiv 0 \pmod 4$ and Type II, special degree $i-1$, missing degree $m-1$ (respectively Type I, special degree i', missing degree m') exists.

Proof is similar to that of Lemma 1. We have the following :

Corollary 5: $S(4t+1, \text{I}) = [t+1, 3t]$ and $S(4t+1, \text{II}) = S(4r, \text{I} = [t, 3t-1]$. Further, 2 is f.u.m.d. for $S(4t+1, \text{I})$ while $4t-2$ is f.u.m.d. for $S(4t+1, \text{II})$.

Lemma 3: Let $p \equiv 1 \pmod 4 \geq 3$. A 3-perfect graph G_p of order $p=4t+1$, Type III, special degree i exists for every $t \leq i \leq 3t$.

Proof: From Theorem 2 we know that $S(4t, \text{I}) = [t, 3t-1]$ and $4t-2$ is f.u.m.d. To start with we consider a 3-perfect graph G_{p-1} of order $p-1 = 4t$, Type I, special degree s, $2t-1 \leq s \leq 3t-1$ and missing degree $4t-2$. Hence $s = 4t-(t+j)$, $1 \leq j \leq t+1$ and the degree sequence of G_{p-1} is

$$4t-1,\ 4t-3,\ \ldots,\ 4t-(t+j),\ 4t-(t+j),\ 4t-(t+j),\ \ldots,\ 1.$$

Let the corresponding vertices be $v_1, v_2, \ldots, v_{4t}$ where v_1 has degree $4t-1$, v_2 has degree $4t-3$, and so on.

Take a new vertex and join it to $v_2, \ldots, v_{2t+1}$. Since $1 \leq j \leq t+1$ we see that except when $j=t+1$, this new vertex is joined to all the three vertices with special degree $4t-(t+j)$ and since degree of v_{2t+1} is $2t$ we get a required G_p of order $p=4t+1$, type III, special degree $s+1$. If $j=t+1$, this new vertex is joined to exactly two of the vertices with special degree $4t-(t+t+1) = 2t-1$ and again G_p so obtained is of order $p=4t+1$, Type III, special degree $2t$.

We have so far constructed G_p of order $p=4t+1$, Type III, special degree i, $2t \leq i \leq 3t$. But then G_p is of Type III, special degree i, $t \leq i \leq 2t$. The proof is now complete.

Theorem (3) and Lemma (3) give the following :

Corollary 6: $S(4t+1, \text{III}) = [t, 3t]$.

Lemma 4: Let $p \geq 3$. A 3-perfect graph G_p of order $p \equiv 2 \pmod 4$ and Type I, special degree i, missing degree $m \neq 1$ (respectively Type II, special degree i', missing degree $m' \neq p-2$) exists if and only if a 3-perfect graph G_{p-1} of order $p-1 \equiv 1 \pmod 4$ and Type II, special degree $i-1$, missing degree $m-1$ (respectively Type I, special degree i', missing degree $m' \neq p-2$) exists.

Proof is similar to that of Lemma (1).

Lemma 5: Let $p \geq 3$. A 3-perfect graph G_p of order $p \equiv 2 \pmod 4$ and Type I, special degree i, missing degree 1 exists if and only if a 3-perfect graph G_{p-1} of order $p-1$ and Type III, special degree $i-1$ exists.

Proof: Suppose G_p exists. By deleting the (unique) vertex of degree $p-1$ we get a required G_{p-1}.

Conversely, suppose that G_{p-1} of Type III, special degree $i-1$ exists. We take a new vertex and join it to all vertices of G_{p-1} to obtain G_p.

The following corollary is immediate.

Corollary 7: $S(4t+2, \text{I}) = [t+1, 3t+1]$ and 1 is f.u.m.d. for $S(4t+2, \text{I})$. However, for $i=3t+1$. 1 is the only possible compatible missing degree whereas for all $i \in S(4t+2, \text{I})$. $i \neq 3t+1$, $4t-1$ is a compatible missing degree.

This leads us to the following :

Definition 8: A missing degree m which is compatible with every special degree in $S(p, \text{I})$ (respectively $S(p, \text{II})$) except with exactly one element of $S(p, \text{I})$ (respectively of $S(p, \text{II})$) is called *second universal missing degree* (abbreviated s.u.m.d.) for $S(p, \text{I})$ (respectively for $S(p, \text{II})$).

For example $4t-1$ is s.u.m.d. for $S(4t+2, \text{I})$ and the exceptional value of i is $3t+1$. We call such i as the *exceptional degree for s.u.m.d. for* $S(p, \text{I})$ The exceptional degree for s.u.m.d. for $S(p, \text{II})$ is similarly defined.

From Corollary 7 by complementation we get,

Corollary 8: $S(4t+2, \text{II}) = [t, 3t]$ and $4t$ is f.u.m.d. for $S(4t+2, \text{II})$. However, for $i = t$, $4t$ is the only possible compatible missing degree whereas for all $i \in S\ (4t+2, \text{II})$ $i \neq t$, 2 is s.u.m.d.

Lemma 6: Let $p \equiv 2 \pmod 4 \geq 3$. A 3-perfect graph G_p of order $p=4t+2$, Type III, special degree i exists for every $t \leq i \leq 3t+1$.

Proof: From Corollary 5 we know that $S\ (4t+1, \text{I}) = [t+1, 3t]$ and 2 is f.u.m.d. for $S(4t+1, \text{I})$. Let G_{p-1} be a 3-perfect graph of order $p-1$, Type I, special degree s, missing degree 2. We take a new vertex and join it to exactly two vertices of G_{p-1} with degree s to get G_p of order p, Type III, special degree $s+1$, Hence we have proved the existence of G_p, order p, Type III, special degree i, $t+2 \leq i \leq 3t+1$. Since G_p is also of Type III, the result follows. For, G_p with special degree $3t+1$ gives $\overline{G}_p$ with special degree t and that with special degree $3t$ gives one with special degree $t+1$.

From Theorem 4 and Lemma 6 we have,

Corollary 9: $S(4t+2, \text{III}) = [t, 3t+1]$.

Lemma 7: Let $p \geq 3$. A 3-perfect graph G_p of order $p \equiv 3 \pmod 4$ and Type I, special degree i, missing degree $m \neq 1$ (respectively Type II, special degree i', missing degree $m' \neq p-2$) exists if and only if a 3-perfect graph G_{p-1} or order $p-1 \equiv 2 \pmod 4$ and Type II, special degree $i-1$, missing degree $m-1$ (respectively Type I, special degree i', missing degree $m' \neq p-2$) exists.

Proof is similar to that of Lemma 1.

Lemma 8: Let $p \geq 3$. A 3-perfect graph G_p of order $p \equiv 3 \pmod 4$ and Type I, special degree i, missing degree 1 exists if and only if a 3-perfect graph G_{p-1} or order $p-1$ and Type III, special degree $i-1$ exists.

Proof is similar to that of Lemma 5.

Corollary 10: $S(4t+3, \text{I}) = [t+1, 3t+2]$ and 1 is f.u.m.d. for $S(4t+3, \text{I})$. However 1 is the only possibly compatible missing degree when $i = 3t+2$. Further, $4t+1$ is s.u.m.d., $i=3t+2$ being the exceptional degree for s.u.m.d.

From Lemma 10 by complementation we get,

Corollary 11: $S(4t+3, \text{II}) = [t, 3t+1]$ and $4t+1$ is f.u.m.d. for $S(4t+3, \text{II})$. However, $4t+1$ is the only possible compatible missing degree when $i=t$. Further, 1 is s.u.m.d., $i=t$ being the exceptional degree for s.u.m.d.

Since $t \geq 2$ is arbitrary, from Lemma 1 and Corollary 11 we have

Corollary 12: 2 is s.u.m.d. for $S(4t, \text{I})$, the exceptional degree for s.u.m.d. being $i=t$.

From Lemma 1 and Corollary 10 we have

Corollary 13: $4t-3$ is s.u.m.d. for $S(4t, \text{II})$, the exceptional degree for s.u.m.d. being $i=3t-1$.

Lemma 2 and Corollary 12 give us

Corollary 14: $4t-2$ is s.u.m.d. for $S(4t+1, \text{I})$ exceptional degree for s.u.m.d. being $i=3t$.

Lemma (2) and Corollary (13) give us

Corollary 15: 2 is s.u.m.d. for $S(4t+1, \text{II})$ exceptional degree for s.u.m.d. being $i=t$.

One may define *third universal missing degree* for $S(4t+2, \text{I})$ (respectively $S(4t+1, \text{II})$) and so on. Third universal missing degree will have its corresponding two exceptional degrees and so on. In general, when $p=4t$, in Type I $l=2t-1$ will have all the even integers between 2 and $4t-2$ as compatible missing degrees and one can think of α^{th} universal missing degree, $1\leq\alpha\leq 2t-1$. Naturally, α^{th} universal missing degree has corresponding $\alpha-1$ exceptional degrees. For $p=4t$, Type II, $i=2t$ will have all the odd integers between 1 and $4t-3$ as compatible missing degrees and one can think of α^{th} universal missing degree, $1\leq\alpha\leq 2t-1$. Corresponding parameters for $p=4t+1$, Type I, are $i=2t+1$, $1\leq\alpha\leq 2t-1$; for type II are $i=2t-1$, $1\leq\alpha\leq 2t-1$. For $p=4t+2$, they are, for type I, $i=2t$, $1\leq\alpha\leq 2t$; for type II, $i=2t+1$, $1\leq\alpha\leq 2t$. When $p=4t+3$, they are for Type I, $i=2t+2$, $1\leq\alpha\leq 2t+1$; for type II, $i=2t$, $1\leq\alpha\leq 2t+1$, These considerations will enable us to write down directly (that is, not by starting with $p=3$ and going stepwise) for an arbitrary p all possible compatible missing degrees for a given special degree i.

We give below the results of this section in tabular form.

TABLE IV

p	$S(p, -)$	f.u.m.d.	s.u.m.d.	Exceptional degree for s.u.m.d.
$4t$	$S(4t, \text{I}) = [t, 3t-1]$	$4t-2$	2	t
	$S(4t, \text{II}) = [t, 3t-1]$	1	$4t-3$	$3t-1$
	$S(4t+1, \text{I}) = [t+1, 3t]$	2	$4t-2$	$3t$
$4t+1$	$S(4t+1, \text{II}) = [t, 3t-1]$	$4t-2$	2	t
	$S(4t+1, \text{III}) = [t, 3t]$			
	$S(4t+2, \text{I}) = [t+1, 3t+1]$	1	$4t-1$	$3t+1$
$4t+2$	$S(4t+2, \text{II}) = [t, 3t]$	$4t$	2	t
	$S(4t+2, \text{III}) = [t, 3t+1]$			
$4t+3$	$S(4t+3, \text{I}) = [t+1, 3t+2]$	1	$4t+1$	$3t+2$
	$S(4t+3, \text{II}) = [t, 3t+1]$	$4t+1$	1	t

CONCLUDING REMARKS

We have completely characterized 3-perfect graphical degree sequences. We have proved that a 3-perfect graphical degree sequence of order p can be constructed from a suitable 3-perfect graphical degree sequence of order $p-1$ and conversely 3-perfect graphical degree sequence of order $p-1$ gives rise to those of order p. But 3-perfect graphs have been only partially characterized. We have not proved in general that given any arbitrary 3-perfect graph G_p of order p there always exists a vertex v of G_p such that $G_p - v$ is a 3-perfect graph of order $p-1$ and conversely, given a 3-perfect graph G_{p-1} of order $p-1$ we can always add a new vertex and join it to suitable vertices of G_{p-1} to get a 3-perfect graph G_p of order p. We conjecture that this is true. We observe that Lemma 3 and Lemma 6 prove only the existence. We do not know if every graph of order $4t+1$ (respectively $4t+2$), Type III, 'comes from' a suitable graph of order $4t$ (respectively $4t+1$) or possibly from a graph of some lower order. We have not touched upon the problem of nonisomorphic realizations. For example, in Lemma 6 we can join the new vertex to any two of the three vertices with the special degree. Different choices might lead to non-isomorphic graphs. Also, a 3-perfect degree sequence of order p could perhaps be obtained, from several such of order $p-1$.

Appendix

We give for p, $8 \leq p \leq 15$, all the possible compatible missing degrees corresponding to all possible special degrees. Though it is enough to give these values only for Type I, to help visualization we given these values for Type II also. We use the following abbreviations :

s.d. = special degree, c.m.d. = compatible missing degrees.

Table V

$p = 8$

Type I		Type II	
s.d.	c.m.d.	s.d.	c.m.d.
5	6, 2	2	1, 5
4	6, 2	3	1, 5
3	6, 4, 2	4	1, 3, 5
2	6	5	1

Table VI

$p = 9$

Type I		Type II	
s.d.	c.m.d.	s.d.	c.m.d.
6	2	2	6
5	2, 4, 6	3	6, 4, 2
4	2, 6	4	6, 2
3	2, 6	5	6, 2

TABLE VII

$p = 10$

Type I		Type II	
s.d.	c.m.d.	s.d.	c.m.d.
7	1	2	8
6	1, 3, 7	3	8, 6, 2
5	1, 3, 7	4	8, 6, 2
4	1, 3, 5, 7	5	8, 6, 4, 2
3	1, 7	6	8, 2

TABLE VIII

$p = 11$

Type I		Type II	
s.d.	c.m.d.	s.d.	c.m.d.
8	1	2	9
7	1, 3, 9	3	9, 7, 1
6	1, 3, 5, 7, 9	4	9, 7, 5, 3, 1
5	1, 3, 7, 9	5	9, 7, 3, 1
4	1, 3, 7, 9	6	9, 7, 3, 1
3	1, 9	7	9, 1

TABLE IX

$p = 12$

Type I		Type II	
s.d.	c.m.d.	s.d.	c.m.d.
8	10, 2	3	1, 9
7	10, 8, 4, 2	4	1, 3, 7, 9
6	10, 8, 4, 2	5	1, 3, 7, 9
5	10, 8, 6, 4, 2	6	1, 3, 5, 7, 9
4	10, 8, 2	7	1, 3, 9
3	10	8	1

TABLE X

$p = 13$

Type I						Type II					
s.d.	c.m.d.					s.d.	c.m.d.				
9	2,					3	10				
8	2,				10	4	10,				2
7	2,	4,	6,	8,	10	5	10,	8,	6,	4,	2
6	2,	4,		8,	10	6	10,	8,		4,	2
5	2,	4,		8,	10	7	10,	8,		4,	2
4	2,				10	8	10,				2

TABLE XI

$p = 14$

Type I							Type II						
s.d.	c.m.d.						s.d.	c.m.d.					
10	1						3	12					
9	1,	3,				11	4	12,	10,				2
8	1,	3,	5,		9,	11	5	12,	10,	8,		4,	2
7	1,	3,	5,		9,	11	6	12,	10,	8,		4,	2
6	1,	3,	5,	7,	9,	11	7	12,	10,	8,	6,	4,	2
5	1,	3,			9,	11	8	12,	10,			4,	2
4	1,					11	9	12,					2

TABLE XII

$p = 15$

Type I								Type II							
s.d.	c.m.d.							s.d.	c.m.d.						
11	1							3	13						
10	1,	3,					13	4	13,	11,					1
9	1,	3,	5,			11,	13	5	13,	11,	9,			3,	1
8	1,	3,	5,	7,	9,	11,	13	6	13,	11,	9,	7,	5,	3,	1
7	1,	3,	5,		9,	11,	13	7	13,	11,	9,		5,	3,	1
6	1,	3,	5,		9,	11,	13	8	13,	11,	9,		5,	3,	1
5	1,	3,				11,	13	9	13,	11,				3,	1
4	1,						13	10	13,						1

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