COLOURING NUMBERS OF THE DIRECT PRODUCT OF TWO HYPERGRAPHS

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A "hypergraph" is defined by a set of points called the "vertices" and a family of subsets called the "edges" and generalizes the concept of "graph". In this paper, the transversal number and the chromatic number of a graph are extended to hypergraphs; the authors show that this generalization yields new results for many combinatorial problems such as :

(1) *The Zarankiewicz Problem :* What is the least integer k such that every subset of k points of an $m \times n$ rectangular lattice contains rs points situated on the same r columns and the same s rows.

(2) *The polarized partition problem :* What is the least number of colours required to colour the points of an $m \times n$ rectangular lattice so that no rs points situated in r columns and s rows can have the same colour?

Definitions

In the following, $H=(X, \xi)$ will denote a hypergraph with vertex set $X=\{x_1, x_2.., x_n\}$, and edge family $\xi=(E_i/i \in I)$. $n(H)=n$ is the *order* of H, $m(H)= | I |$ is the number of edges, and $r(H) = \max | E_i |$ is the *rank* of H.

A set $S \subset X$ is said to be *stable* if it contains no edge; the maximum cardinal of a stable set is denoted by $\beta(H)$ and is called the *stability number* of H.

A set $T \subset X$ is said to be a *transversal* if it meets each edge; the minimum cardinal of a transversal of H is denoted by $\tau(H)$ and is called the *transversal number* of H. Other numbers can be associated with hypergraph H; for instance, $\nu(H)$ denotes the maximum number of pairwise disjoint edges; $\rho(H)$ denotes the minimum number of edges which together cover X; $\delta(H)$, the *maximum degree,* is the maximum number of edges which meet at the same vertex. $\varkappa(H)$, the chromatic number, is the least integer k for which there exists a partition of X into k stable sets.

It is well known that the following inequalities hold :

(1) $\varkappa(H) \cdot \beta(H) \geqslant n(H)$

(2) $\varkappa(H) + \beta(H) \leqslant n(H) + 1$

(3) $\beta(H) = n(H)\tau(H)$

(4) $\tau(H) \geqslant \nu(H)$

(5) $\tau(H) \leqslant r(H)\nu(H)$

For a proof of the above Berge (1970) may be compared.

Given two hypergraphs $H = (X, \delta)$ and $H' = (Y, \tau)$, with $\delta=(E_i/i\in I)$, $\tau=(F_j/j\in J)$, their *direct product* is a hypergraph $H\times H'$ with vertex set $X\times Y$ and with edges $E_i\times F_j$ for $(i, j) \in I\times J$.

The aim of this paper is to find upper bounds and lower bounds for the numbers associated with hypergraph $H\times H'$. These results can easily be extended to the direct product of more than two hypergraphs.
First, it should be noticed that we have :

(6) $r(H\times H') = r(H) \times r(H')$

Moreover, some of the associated numbers of $H\times H'$ can be obtained from other coefficients by the duality principle, using the following results:

Proposition 1: $(H\times H')^* = H^* \times H'^*$

By definition of the dual, $(H\times H')^*$ has vertex set $\{(e_i, f_j)/i\in I, j\in J\}$; the edge corresponding to a vertex (x_p, y_q) of $H\times H'$ must contain all the (e_i, f_j) such that $E_i\ni x_p$ and $F_j\ni y_q$, and therefore is the set $X_p\times Y_q$.
The proposition follows.

The Transversal Number

Let H and H' be two hypergraphs of order m and n, respectively.
From (3) we have

$$\beta(H\times H') = mn - \tau(H\times H'),$$

So, the problem of finding a lower bound for β is the same as the problem of finding an upper bound for τ. This problem often occurs in Combinatorics.

Example 1: What is the least number of points in a $m\times n$ rectangular unit lattice (integer points of the plane), such that each square of side r has a least one of these points as a corner? The answer is $\tau\left(D^r_m \times D^r_n \right)$, where D^r_n is a simple graph with vertices 1, 2, ..n, two vertices x, y being joined if $|x-y| = r$.
One can easily show that if $r = 1$ and mn is even, we have

$$\tau\left(D^1_m \times D^1_n \right) = [m/2]^* \, [n/2]^*.$$

Example 2: The Zarankiewicz problem. Let $1 \leqslant r \leqslant m$, $1 \leqslant s \leqslant n$. Zarankiewicz has asked for the least integer $k_{rs}(m, n)$ such that every subset of $k_{rs}(m, n)$ points of an $m\times n$ rectangular unit lattice should contain rs points situated in r columns and s rows. If K^r_m denotes the complete r-uniform hypergraph on m points, we have

$$\beta\left(K^r_m \times K^s_n \right) = k_{rs}(m, n)-1$$

An extensive literature exists on this problem (*see* Guy 1968, 1969). For the sake of simplicity, consider first the case $m=n$. It is known (Kôvary *et al.* 1954) that if $r \leqslant s$, then

$$\text{(i)} \quad \beta\left(K_n^r \times K_n^s \right) \leqslant c_{r,s}\, n^{2-1/r}$$

where $c_{r,s}$ is a constant. Furthermore, if $r=s=2$, (i) is sharp, that is if $n \to \infty$, we have

$$\frac{\beta\left(K_n^2 \times K_n^2 \right)}{n} \to 1$$

It follows easily from (Brown 1966) that if $s \geqslant 3$, then

$$\text{(ii)} \quad c'_s\, n^{5/3} \leqslant \beta\left(K_n^3 \times K_n^s \right) \leqslant c''_s\, n^{5/3}$$

Unfortunately, the lower bounds for the general case are far from the upper bound given in (i).

Another simple case is when is n much greater than m. Thus if $n \geqslant (s-1)\binom{m}{r}$ Culik (1956) has determined the exact value:

$$\beta\left(K_m^r \times K_n^s \right) = (r-1)\, n + (s-1)\binom{m}{r}.$$

For example, $\beta\left(K_4^2 \times K_6^2 \right) = 6 + 6 = 12$, and a maximum stable and set with 12 vertices is given by the ones in the following array :

$$n = 4 \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$
$$m = 6$$

Proposition 2: Let H and H′ be two hypergraphs. Then

$$\tau(H \times H') \leqslant \tau(H)\, \tau(H')$$

Let $T \subset X$ and $T' \subset Y$ be two minimum transversals respectively for H and H'. Since $T \times T'$ is a transversal for $H \times H'$, we have

$$\tau(H \times H') \leqslant |T \times T'| = \tau(H)\, \tau(H')$$

Q.E.D.

Instead of showing that the inequality of Proposition 2 is the best possible, we shall show that a very large class of hypergraphs H satisfy

$$\tau(H \times H') = \tau(H)\, \tau(H') \qquad \text{for all } H'.$$

First, we shall prove the two lemmas. In fact, these lemmas have been proved independently by L. Lovasz (1973) and the authors and can be used for a different purpose.

— Let s be a positive integer.

— Let $\varphi(x)$ be an integer function on X ; for $A \subset X$, let $\varphi(A) = \sum_{x \in A} \varphi(x)$.

If $\varphi(E_i) \geqslant s$ for all $i \in I$, the function φ is said to be an *s-covering* for H. The minimum of $\varphi(x)$ over all s-coverings φ will be denoted by $\tau_s(H)$. Clearly, $\tau(H) = \tau_1(H)$.

Now, let H be a hypergraph with vertices $x_1, x_2, \ldots, x_n$, with $m(H)$ edges, and with maximum degree $\delta(H)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n non-negative real numbers. Let

$$\tau^*(H) = \min \left\{ \sum_{i=1}^{n} \alpha \Big/ \sum_{x_i \in E_j} \alpha_i \geqslant 1 \quad \text{for all } j \right\}.$$

Lemma 1; Let H be a hypergraph. Then

$$\max\left\{ \nu(H), \frac{m(H)}{\delta(H)} \right\} \leqslant \tau^*(H) \leqslant \frac{\tau_s(H)}{s} \leqslant \tau(H)$$

We have $\tau_s(H) \leqslant s\tau(H)$, because if T is a minimum transversal set and if $\varphi_T(x)$ is its characteristics function, then $s\varphi_T$ is an s-covering, and, consequently,

$$\tau_s(H) \leqslant s\varphi_T(X) = s\,\tau(H)$$

We have $\frac{1}{s}\tau_s(H) \geqslant \tau^*(H)$, because if φ is a minimum s-covering, then by putting $\alpha_i = \frac{1}{s}\varphi(x_i)$, we obtain

$$\tau^*(H) \leqslant \sum_{i=1}^{n} \alpha_i = \frac{1}{s}\tau_s(H)$$

Now, we shall show that $\tau^*(H) \geqslant \frac{m(H)}{\delta H}$. Consider n real numbers α_i such that $\sum_{x_i \in E_j} \alpha_i \geqslant 1$ for all j and such that $\sum \alpha_i = \tau^*(H)$.

Denote by $\delta_x(H)$ the degree of vertex x. We have

$$m(H) \leqslant \sum_{j=1}^{m} \sum_{x_i \in E_j} \alpha_i \leqslant \sum_{i=1}^{n} \alpha_i\, \delta_{x_i}(H) \leqslant \delta(H) \sum_{i=1}^{n} \alpha_i = \delta(H)\,\tau^*(H)$$

Also, if $(E'_1, E'_2, \ldots, E'_\nu)$ is a maximum matching of H, then

$$\tau^*(H) = \sum_{i=1}^{n} \alpha_i \geqslant \sum_{k=1}^{\nu} \sum_{x_i \in E_k'} \alpha_i \geqslant \nu(H)$$

The first inequality follows.

Lemma 2: $s^{-1}\tau_s(H)$ tends to a limit, and

$$\lim_{s \to \infty} \frac{\tau_s(H)}{s} = \tau^*(H)$$

A well known theorem of Fekete states that if a sequence (a_n) of positive numbers is such that $a_{m+n} \leqslant a_m + a_n$, then the sequence $\left(\frac{a_n}{n}\right)$ tends to a limit.

Let φ be a minimum p-covering and φ' be a minimum q-covering. Then $\varphi + \varphi'$ is a $(p+q)$—covering, and therefore

$$\tau_{p+q}(H) \leqslant \varphi(X) + \varphi'(X) = \tau_p(H) + \tau_q(H).$$

Hence, by Fekete's theorem, there exists a number ξ such that $\frac{\tau_s(H)}{s} \to \xi$.

By Lemma 1, $\xi \geqslant \tau^*(H)$.

Furthermore, the α_i whose sum is $\tau^*(H)$ are defined by a linear programming problem with integral coefficients, and therefore, the α_i are rational, and we can write :

$$\alpha_i = \frac{\alpha'_i}{s}, \ \alpha'_i \text{ and } s \text{ integers.}$$

Hence

$$s^{-1}\tau_s(H) \leqslant \frac{1}{s} \sum \alpha'_i = \sum \alpha_i = \tau^*(H)$$

This shows that $\xi = \tau^*(H)$.

Q.E.D.

Theorem 1: A necessary and sufficient condition for a hypergraph H to satisfy $\tau(H \times H') = \tau(H)\,\tau(H')$ for all H' is that $\tau(H) = \tau^*(H)$.

Necessity: Assume that $\tau(H) \neq \tau^*(H)$. Then by Lemma 1, $\tau(H) > \tau^*(H)$ and by Lemma 2, there axists an integer $s > 2$ such that $\frac{\tau_s(H)}{s} < \tau(H)$. We shall show that there exists a hypergraph H' such that $\tau(H \times H') < \tau(H)\,\tau(H')$.

Let $\varphi(x)$ be a minimal s-covering for H. Put $\varphi(X) = \tau_s(H) = t$, $Y = \{1, 2, \ldots, t\}$.

It is always possible to associate with each $x \in X$ a set $A(x) \subset Y$ so that :

(1) $|A(x)| = \varphi(x)$ for all $x \in X$

(2) $x \neq x'$ implies $A(x) \cap A(x') = \phi$

Let $H' = K_t^{t-s+1} = [Y, (F_j)]$. We shall show that the direct product $H \times H'$ admits

$$T_0 = \{(x, y)/x \in X, y \in A(x)\}$$

as a transversal.

Clearly, $E_i \times Y$ contains at least s different elements of T_0. Since no two of them have the same projection on Y, $E_i \times F_j$ contains at least one element of T_0, for all i, j. Thus, T_0 is a transversal of $H \times H'$. Moreover, $\tau(H') = s$.

Hence,

$$\tau(H \times H') \leqslant | T_0 | = \tau_s(H) < s\tau(H) = \tau(H)\ \tau(H').$$

Q.E.D.

Sufficiency: Let H be a hypergraph such that $\tau(H) = \tau^*(H)$. Then, by Lemma 1, we have $\tau_s(H) = s\tau(H)$ for every integer s. Let $T_0 \subset X \times Y$ be a minimum transversal of $H \times H'$. Let

$$\varphi_0(x) = | \{y/(x, y) \in T_0, y \in Y\} |$$

Since the projection on Y of $(E_i \times Y) \cap T_0$ is a transversal of H',

$$\varphi_0(E_i) = | (E_i \times Y) \cap T_0 | \geqslant \tau(H').$$

Thus, φ_0 is an s-covering for $s = \tau(H')$.

Hence,

$$\tau(H \times H') = | T_0 | = \varphi_0(X) \geqslant \tau_s(H) = s\ \tau(H) = \tau(H')\ \tau(H)$$

Therefore, the equality holds.

Q.E.D.

Corollary 1: If H satisfies $\nu(H) = \tau(H)$ (and in particular if H is balanced) then $\tau(H \times H') = \tau(H)\ \tau(H')$ for every H'.

This follows immediately from Lemma 1.

In particular, if H is balanced, i.e., if each odd cycle of H possesses an edge containing three vertices of the cycle, it is known (Berge 1970) that $\nu(H) = \tau(H)$, and consequently, the required equality holds.

Corollary 2: Let G be a graph, Then

$$\tau(G \times H') = \tau(G)\ \tau(H') \quad \textit{for every hypergraph } H' \textit{ if and only if } \tau(G) = \nu(G).$$

By a theorem of Lovász (1973), $\tau(G) = \tau^*(G)$ if and only if $\tau(G) = \nu(G)$. The proof follows.

Q.E.D.

Corollary 3: Let H be a hypergraph such that $m(H) = \tau(H)\ \delta(H)$. Then $\tau(H \times H') = \tau(H)\ \tau(H')$ for every hypergraph H'.

This follows immediately from lemma 1.

Corollary 4: Let H and H' be two hypergraphs. Then

$$\rho(H \times H') \leqslant \rho(H)\ \rho(H')$$

Furthermore, if H is balanced, then $\rho(H \times H') = \rho(H)\ \rho(H')$ for every H'.

Clearly, if H^* is the dual of H, then $\rho(H) = \tau(H^*)$.

If H is balanced, then H^* is also balanced.

Thus, the result follows immediately from Proposition 1, Proposition 2 and Corollary 1.

Corollary 5: Let H and H' be two hypergraphs. Then

$$\beta(H\times H')\geqslant\beta(H)\,n(H') + \beta(H')\,n(H) - \beta(H)\,\beta(H')$$

Equality holds for every H' if and only if $\tau(H = \tau^*(H)$.

We have

$$\beta(H\times H') = n(H\times H') - \tau(H\times H')\geqslant n(H)\,n(H')-\tau(H)\,\tau(H') = n(H)\,n(H')$$
$$-\{n(H) -\beta(H)\,(n(H') - \beta(H')\} = \beta(H)\,n(H') + \beta(H')\,n(H)-\beta(H)\,\beta(H')$$

The equality holds if it holds in Theorem 1.

Theorem 2: Let H and H' be two hypergraphs, Then

$$\tau(H\times H') \geqslant \tau(H) + \tau(H') -1$$

A hypergraph $H = (E_i \,/\, i \in I)$ satisfies $\tau(H\times H') = \tau(H) + \tau(H') - 1$ for every H' if and only if $\bigcap_{i\in I} E_i \neq \phi$.

Let H and H' be two hypergraphs on X and Y respectively.

Let T_0 be a minimum transversal of $H\times H'$, and for $x \in X$, let

$$\varphi_0(x) = |\,\{y \,/\, (x, y) \in T_0, y \in Y\}\,|$$

Clearly, φ_0 is an s-covering of H for $s = \tau(H')$. Let $T_1\subset X \times Y$ be obtained from T_0 be removing exactly $s-1$ vertices, and let

$$\varphi_1(x) = |\,\{y/(x, y) \in T_1, y \in y\}\,|$$

We have, for all edge E_i of H,

$$\varphi_1(E_i)\geqslant\phi_0(E_i)-(s-1)\geqslant 1$$

Hence, φ_1 is a covering of H, and therefore $\varphi_1(X)\geqslant\tau(H)$.

Hence

$$\tau(H\times H') = \varphi_0(X) = \varphi_1(X) + (s-1)\geqslant\tau(H) + \tau(H')-1.$$

Now, consider a hypergraph $H = (E_i/i\in I)$ such that $\bigcap E_i \neq\phi$.
Then $\tau(H) = 1$.

Let $x_0 \in \bigcap E_i$. Clearly, $H\times H'$ has a transversal $T_0\subset\{x_0\}\times Y$ such that

$$|\,T_0\,| = \tau(H') = \tau(H) + \tau(H') - 1$$

Hence, by Part 1 of the theorem, T_0 is a minimum transversal of $H\times H'$, and

$$\tau(H\times H') = |\,T_0\,| = \tau(H) + \tau(H') - 1$$

Since this equality holds for every H', the second part of the theorem is proved.

It remains to show that if $\tau(H)>1$, there exists a hypergraph H' such that $\tau(H\times H')> \tau(H)+\tau(H')-1$. Take any balanced hypergraph H' with $\tau(H')=s\geqslant 2$. By Corollary 1 to Theorem I, we have

$\tau(H \times H') = s\ \tau(H) > \tau(H) + (s-1) = \tau(H) + \tau(H') - 1$

The required inequality follows.

Q.E.D.

Remark: Proposition 2 shows that, for all p, q,

(1) $\max \{\tau(H \times H') / \tau(H = p, \tau(H') = q\} = pq$

However, Theorem II shows only that

(2) $\min \{\tau(H \times H') / \tau(H) = p, \tau(H') = q\} = p + q - 1$

holds for $p = 1$ (or $q = 1$). However, it is easy to show that (2) holds for all p, q.

Put $H = K^q{}_{p+q-1}$, $H' = K^p{}_{p+q-1}$ (the complete hypergraph on $p+q-1$ vertices with ranks respectively q and p). Clearly, $\tau(H) = p$, $\tau(H') = q$. If the vertex set of H is $\{x_1, .., x_{p+q-1}\}$ and the vertex set of H' is $\{y_1, .., y_{p+q-1}\}$, then $T_0 = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{p+q-1}, y_{p+q-1})\}$ is a transversal of $H \times H'$, because, otherwise there exists an edge E_i of H and an edge F_f of H' such that $(E_i \times F_j) \cap T_0 = \phi$, which contradicts that $|E_i| + |F_j| = p + q$.

Thus, (2) follows from Theorem 2.

In fact, we can have a better inequality by using the number τ^*. We have

Theorem 3: Let H and H' be two hypergraphs. Then

$\tau(H \times H') \geqslant \max \{\tau^*(H)\ \tau(H'), \tau(H)\ \tau^*(H')\}$.

Let T_0 be a minimum transversal of $H \times H'$, and let

$\varphi_0(x) = |\{y/(x, y) \in T_0, y \in Y\}|$

φ_0 is an s-covering of H for $s = \tau\ (H')$. Hence by Lemma 1,

$\tau(H \times H') = |T_0| = \varphi_0(x) \geqslant \tau_s(H) \geqslant s\ \tau^*(H) = \tau(H')\ \tau^*(H)$.

The required inequality follows.

Corollary 1: $\tau(H \times H') \geqslant \max \left\{ \frac{m\,(H)}{\delta\,(H)} \tau\ (H'), \frac{m\,(H')}{\delta\,(H')} \tau\,(H) \right\}$

This follows immediately from Lemma 1.

Corollary 2: $\tau(H \times H') \geqslant \max \{\nu(H)\ \tau(H'), \nu(H')\ \tau(H)\}$

This follows immediately from Lemma 1.

The Chromatic Number

We shall now consider the chromatic number of the direct product $H \times H'$.

Example: Polarized partition relations among cardinal numbers (Erdös & Rado 1956; and Chvātal 1969). What is the least number of colors required to color the points of a $m \times n$ rectangle unit lattice so that rs points situated in r columns and s rows cannot have the same color? Clearly, this number is

$\chi\,(K^r_m \times K^s_n)$.

For instance, $\chi(K_6^2 \times K_4^2) = 2$, and a bicoloring of the 6×4 rectangle unit lattice is shown in example 2, the Zarankiewicz Problem, in the section under "Transversal Number."

Also, we have

$$\chi(K_5^2 \times K_5^2) = 3$$

Otherwise, there exists a bicoloring of the 5×5 matrix (a_j^i) where the 0 denote the points of the first color and the 1 the points of the second color.

The first column $(a_1^1, a_1^2, a_1^3, a_1^4, a_1^5)$ having necessarily three entries of equal values, suppose $a_1^1 = a_1^2 = a_1^3 = 0$.

The first two rows have, in each column, one of the combinations 00, 11, 01, 10, and there exist two columns with the same combination (because $2^2<5$).

Since this repeated combination cannot be 00 nor 11, we may assume

$$a_2^1 = a_3^1 = 0$$

$$a_2^2 = a_3^2 = 1$$

None of a_2^3, a_3^3 can be zero; hence

$$a_2^3 = a_3^3 = 1$$

The submatrix $\begin{bmatrix} a_2^2 & a_3^2 \\ a_2^3 & a_3^3 \end{bmatrix}$ having only ones, the 0 and 1 in $\left[a_j^i \right]$ do not define a bicoloring of $K_5^2 \times K_5^2$

Q.E.D.

This type of reasoning has been extended by Chvatal (1969, 1970), who showed that

$$(A) \qquad c_1 n^{1/r} \leqslant \chi(K_n^r \times K_n^r) \leqslant c_2 n^{1/r}$$

In fact, the lower bound also follows from a result of Kövary *et al.* (1954), while the upper bound was obtained by so-called probabilistic methods. Moreover, replacing the probabilistic method by a finite geometrical construction, Chvatal showed that

$$(B) \qquad \chi(K_n^2 \times K_n^2)/n^2 \longrightarrow 1$$

Later, (B) and the upper bound in (A) were also obtained by Sterboul and the authors of this paper, independently. Finally, Sterboul (1973) showed that in some cases, the same kind of arguments gives the exact value of $\chi(K_m^2 \times K_n^2)$.

The problem of finding a lower bound for $\chi(H \times H')$ was also considered by Chvátal (1970), who gave the two following inequalities :

$$\chi(H \times H') \geqslant \min \{\chi(H), \chi(H')^{1/n(H)}\}$$

$$\chi(H \times H') \geqslant \min \{\chi(H), m(H)^{-1} \chi(H')\}$$

An obvious result is :

Proposition 3: $\chi(H \times H') \leqslant \min \{\chi(H), \chi(H')\}$

Assume that $\chi(H) \leqslant \chi(H')$, and let $g(x)$ be a coloring of H in $p = \chi(H)$ colors. Then $h(x, y) = g(x)$ is a coloring of $H \times H'$ in p colors. Hence $\chi(H \times H') \leqslant \chi(H)$ Q.E.D.

Equality is obtained in some degenerated cases, for example when $\chi(H) = 2$. However in general, Proposition 3 is far from being best possible.

Theorem 4: Max $\{\chi(H \times H') / \chi(H) = p, \chi(H') = q\} = \chi(K_p^2 \times K_q^2)$.

We have only to show that if H and H' are two hypergraphs with $\chi(H) = p$, $\chi(H') = q$ then

$$\chi(H \times H') \leqslant \chi(K_p^2 \times K_q^2)$$

Consider a coloring $c(x)$ of H with P symbols $a_1, a_2, \ldots, a_p$, and a coloring $c'(y)$ of H' with q symbols $b_1, b_2, \ldots, b_q$. Consider a complete graph K_p^2 with vertex set $\{a_1, a_2, \ldots, a_p\}$ and a complete graph K_q^2 with vertex set $[b_1, b_2, \ldots, b_q]$. Let $g(a_i, b_j)$ be a coloring of $K_p^2 \times K_q^2$ in $t = \chi(K_p^2 \times K_q^2)$ colors. Now, put

$$h(x, y) = g(c(x), c'(y))$$

To show that $h(x, y)$ is a coloring of $H \times H'$, consider an edge $E \times F$ of $H \times H'$. E contains two vertices x_1 and x_2 with $c(x_1) \neq c(x_2)$, and F contains two vertices y_1 and y_2 with $c'(y_1) \neq c'(y_2)$.

Since $\{c(x_1), c(x_2)\} \times \{c'(y_1), c'(y_2)\}$ is an edge of $K_p^2 \times K_q^2$, it contains two points, say $\{c(x_3), c'(y_3)\}$ and $\{c(x_4)\ c'(y_4)\}$, with

$$g\{c(x_3), c'(y_3)\} \neq g\{c(x_4), c'(y_4)\}$$

Hence, $E \times F$ contains two vertices (x_3, y_3) and (x_4, y_4) with $h(x_3, y_3) \neq h(x_4, y_4)$. This shows that h is a coloring of $H \times H'$.

Hence,

$$\chi(H \times H') \leqslant t = \chi(K_p^2 \times K_q^2)$$

Q.E.D.

The problem of finding a good estimate for

$$f(p, q) = min\{\chi(H \times H') \,/\, \chi(H) = p, \chi(H') = q\}$$

seems to be difficult. In particular, we can ask *if it is true that, when p and q tend to infinity, f (p, q) also tends to infinity.*

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