LINE-CLIQUE COVER NUMBER OF A GRAPH*

by S. A. Choudum, K. R. Parthasarathy *and* G. Ravindra,
Mathematics Department, Indian Institute of Technology, Madras-600036

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In this paper, a new invariant for finite graphs is studied, namely, the line-clique cover number $\theta_1(G)$= the minimum number of cliques that cover all the lines of the graph G. A number of criteria equivalent to the condition $\theta_1(G) = \beta_0(G)$ are established, through the join $G+G'$, of two graphs. θ_1 perfect graphs are then defined imitating Berge's definition of α-perfect graphs. These graphs are characterized as the ones without P_4 or C_4 as induced subgraph. As corollaries it is proved that, a Hamiltonian θ_1-perfect graph is pancyclic and a θ_1 perfect graph has a vertex adjacent to all the other vertices. The value of θ_1 for a class of r- partile graphs is obtained.

Introduction

The minimal number of cliques (maximal complete subgraphs) that cover all the vertices of a graph is well known in graph theory as the partition number θ introduced by Berge (see e.g., Berge 1969) and has been celebrated in Berge's conjecture on perfect graphs. We call this number the point-clique-cover number of a graph and denote it by θ_0 to distinguish it from another invariant of a graph: the line-clique-cover number θ_1 defined as the minimal number of cliques that cover all the lines of a graph. This is defined only for graphs without isolated points. The importance of θ_1 came up naturally in our investigations on the intersection numbe1 ω of a graph. An earlier reference to this concept, however, occurs in a problem posed by Culik (1963). The relation between θ_1 and ω is investigated in another paper (cf. Choudum & Parthasarathy 1975). In this paper we study this concept in relation to θ_0 and the independence number β_0 (see Harary 1969 for definition), which is denoted by α and called the internal stability number by Berge (1962). In analogy with Berge's definition of α-perfect graphs (which we call θ_0-perfect), we define θ_1-perfect graphs and characterise such graphs. While we leave open the problem of finding an algorithm for the determination of θ_1 for a geneıal graph, we find a formula for θ_1 for a special type of complete r-partite graph. The graphs considered here are finite, undirected and without loops or multiple edges. For notation and terminology not explicitly defined here see Harary (1969). By $u\ A\ v$ we denote that u is adjacent to v and $u \not A v$ denotes u is not adjacent to v.

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θ_0, θ_1 AND β_0

We first introduce a few definitions : A point/line of G is *unicliqual* if it belongs to only one clique in G, otherwise it is non-unicliqual. A clique of G is point-free/line-free (*p-free*/*l-free* for short) if it contains at least one unicliqual point/line. It is obvious that a θ_0-cover/a θ_1-cover should contain all the p-free/l-free cliques of G. We denote a θ_1-cover for G by $\{\theta_1(G)\}$ and the line-clique-cover number by $\theta_1(G)$. Clearly,

$$\beta_0(G) \leqslant \theta_0(G) \leqslant \theta_1(G) \tag{1}$$

Our next result is the following :

Theorem 1: For any connected graph G the following are equivalent :

(*i*) $\theta_1(G) = \beta_0(G)$.

(*ii*) $\theta_1(G+G) = (\theta_1(G))^2$.

(*iii*) $\theta_1(G) = \theta_0(G)$.

(*iv*) *The members of any θ_1-cover for G are p-free cliques.*

(*v*) *If M is the set of non-unicliqual points in G, then* $\theta_1(G) = \theta_0(G-M)$.

(*vi*) $\theta_1(G) = \theta_0(G-S)$ where $S \subseteq M$.

(*vii*) $\theta_1(G) = \theta_0(G-v)$, $v \in M$.

Proof: (*i*) $\Rightarrow$ (*ii*) : By definition, $G+G$ contains two separate copies of G, say G and G' with every point of the former joined by a line to every point of the latter. It is generally true that $\theta_1(G+G') \leqslant (\theta_1(G))^2$: For, if Q and Q' are any two cliques in G and G', then $Q+Q'$ is a clique in $G+G'$; so that the collection $\{Q_i+Q_j' : Q_i \in \{\theta_1(G)\}, Q_j' \in (\theta_1(G'))$ containing $\{\theta_1(G)\}^2$ elements is a line-clique cover for $G+G'$. The inequality follows by the minimality of $\theta_1(G+G')$.

For the reverse inequality in (*ii*) we need the assumption (*i*) Let S and S' be corresponding maximal independent sets in G and G', with $|S| = |S'| = \beta_0(G)$. The set $L = \{(v_i, v_j') : v_i \in S, v_j' \in S'\}$ of β_0^2 lines of $G+G'$ has the property that no two of its lines can belong to the same clique in $G+G'$ (since neitgher $(v_i, v_j) \in S \times S$ nor $(v_i', v_j') \in S' + S'$ is an edge in $G+G'$). Thus each line of L needs to be covered by a different clique in a θ_1-cover for $G+G'$. Hence $\theta_1(G+G') \geqslant |L| = \beta_0^2 = \theta_1^2$ by (*i*).

(*ii*) $\Rightarrow$ (*iii*): Suppose (*iii*) is false, that is, $\theta_0(G) < \theta_1(G)$. Now, let $\{K_1, K_2, \ldots K_{\theta_1}\}$ be a θ_1–cover for G and $\{Q_1, Q_2, \ldots, Q_{\theta_0}\}$ be a θ_0-cover for G. Then it is easy to see that the set of cliques, $\{Q_i+Q_j' : i, j = 1, 2, \ldots, \theta_0\} \cup \{K_i + K_i' : i = 1, 2, \ldots, \theta_1\}$ cover all the lines of $G+G'$. Thus $\theta_1(G+G') \leqslant \theta_1(G) + (\theta_0(G))^2 < (\theta_1(G))^2$ for $\theta_1(G) > 1$, contradicting (*ii*). If $\theta_1(G) = 1$, then (*iii*) holds trivially.

(*iii*) $\Rightarrow$ (*iv*): Suppose (*iv*) is not true. Then there exists a clique $K \in \{\theta_1(G)\}$ which is not p-free. If $v \in V(K)$, v is in some other clique of G also. There is a line e incident with v, which is not in K. This e is covered by some member, other than K, of $\{\theta_1(G)\}$. Thus every $v \in V(K)$ is in some other member of $\{\theta_1(G)\}$. Therefore $\{\theta_1(G)\} - \{K\}$ is a θ_0-cover for G. That is, $\theta_0(G) \leqslant \theta_1(G)-1$, contradicting (*iii*).

(*iv*) $\Rightarrow$ (*i*) : Each member of $\{\theta_1(G)\}$ contributes 1 to β_0, under (*iv*). Hence $\theta_1(G) \leqslant \beta_0(G)$. Now, (*i*) follows by Theorem 1.

$(iv) \Rightarrow (v)$: By (iv) each clique of $\{\theta_1(G)\}$ contains one member of a maximal independent set and the same set is also a maximal independent set for $G-M$. Therefore $\beta_0(G-M)=\beta_0(G)$. Since (iv) implies (i) we have,

$$\theta_1(G)=\beta_0(G)=\beta_0(G-M)\leqslant\theta_0(G-M)\leqslant\theta_0(G).$$

Since $\theta_0(G)\leqslant\theta_1(G)$ equality obtains throughout in the above. Therefore $\theta_1(G)=\theta_0(G-M)$.

$(v) \Rightarrow (vi)$: Since $S \subseteq M$ and $G-M$ is an induced subgraph of $G-S$, $\theta_0(G-M) \leqslant\theta_0(G-S)$. Hence $\theta_1(G)=\theta_0(G-M) \leqslant \theta_0(G-S) \leqslant \theta_0(G)$. As before, equality holds throughout and $\theta_1(G)$-$\theta_0(G-S)$.

$(vi) \Rightarrow (vii)$: Follows easily, as above, since $G-S$ is an induced subgraph of $G-v$.

$(vii) \Rightarrow (iii)$: By hypothesis, $\theta_1(G)=\theta_0(G-v)$, $v\in M$. Hence $\theta_1(G)=\theta_0(G-v)\leqslant\theta_0(G)$ which together with Theorem 1 implies $\theta_1(G)=\theta_0(G)$.

Corollary 1: Since $\theta_0(G)=\chi(\bar{G})$, where $\bar{G}$ is the complement of G, we have $\theta_1(G)=\chi(G)$ if $\theta_1(G)=\beta_0(G)$.

Corollary 2: $\theta_1(G_1+G_2)=\beta_0(G_1+G_2)$ if one of the graphs G_1, G_2 is complete and $\theta_1=\beta_0$ for the other.

Proof:

Sufficiency: Without loss of generality let G_1 be complete and $\theta_1(G_2)=\beta_0(G_2)$. Then G_1 is complete implies that $\theta_1(G_1+G_2)=\theta_1(G_2)$ and $\beta_0(G_1+G_2)=\beta_0(G_2)$. Since $\theta_1(G_2)=\beta_0(G_{\cdot 2})$ the result follows.

Necessity: Since $\beta_0(G_1+G_2)=\max\{(\beta_0(G_1), \beta_0(G_2)\}$, let without loss of generality, $\beta_0(G_1+G_2)=\beta_0(G_2)$. Then $\theta_1(G_1+G_2)\geqslant\theta_1(G_2)\geqslant\beta_0(G_2)=\beta_0(G_1+G_2)$. By hypothesis, we get the equality throughout, and hence $\theta_1(G_2)=\beta_0(G_2)$. Next $\beta_0(G_1)\beta_0(G_2)\leqslant \theta_1(G_1+G_2)=\beta_0(G_2)=\beta_0(G_2)$. So $\beta_0(G_1)=1$ and G_1 is complete.

θ_1-Perfect Graphs

Definition: A graph G is θ_1-perfect if every induced subgraph H of G without isolated points has the property, $\theta_1(H)=\beta_0(H)$.

We characterise θ_1-perfect graphs by the

Theorem 2: A graph G is θ_1-percfect if G has neither P_4 nor C_4 as an induced subgraph.

Proof:

Necessity: If P_4 or C_4 is an induced subgraph, G is not θ_1 perfect, because

$$\theta_1(P_4)=3>2=\beta_0(P_4) \text{ and}$$

$$\theta_1(C_4)=4>2=\beta_0(C_4).$$

Sufficiency: Suppose G is not θ_1-perfect and therefore has an induced subgraph H such that $\theta_1(H)\neq\beta_0(H)$. Then we will show that G has P_4 or C_4 as an induced subgraph.

By (iv) of Theorem 1, at least one member K of $\{\theta_1(H)\}$ is not a p-free clique. This implies that each vertex of K is in some other clique of $\{\theta_1(H)\}$. Also, since K is in $\{\theta_1(H)\}$ at least one edge (u, v) of K is not in any other member of $\{\theta_1(H)\}$. Let

K_u, K_v be members of $\{\theta_1(H)\}$ containing u and v respectively. Then there exist points $u' \in K_u$, $v' \in K_v$ which are distinct (otherwise $K_u \equiv K_v$ and $u, v \in K_u$) and are such that $u' \not A v$ and $v' \not A u$. Depending on whether u' and v' are adjacent or not, $\langle (u', u, v, v') \rangle$ is an induced subgraph of H which is a C_4 or P_4 (respectively). Since H is an induced subgraph of G, this implies that G has P_4 or C_4 as an induced subgraph. This establishes sufficiency.

Corollary 3: θ_1-perfect graphs are perfect graphs in the sense of Berge.

Proof: G is θ_1-perfect implies G has no induced cycle of length > 3 and is hence triangulated. But triangulated graphs are perfect (Berge 1969).

Corollary 4: Hamiltonian θ_1-perfect graphs are pancyclic (that is, they have cycles of length m, $3 \leqslant m \leqslant p$).

Proof: If G is a p-point θ_1-perfect graph and if it is also Hamiltonian there is a point v_i on a Hamiltonian cycle such that $v_{i-1}\ A\ v_{i+1}$ (otherwise there is an induced P_4 or C_4). That is, there is a $(p-1)$-point θ_1-perfect Hamiltonian induced subgraph in G. Repeating the argument we see that G has cycles of all lengths.

Corollary 5: If G is θ_1-perfect, then $\bigcap_{K \in \mathrm{K}(G)} V(K) \neq \phi$. where K$(G)$ is the set of all cliques of G.

Proof: It is enough to show that there is a point which is adjacent to all other points. Let x be a point with maximum degree and let $x_1, \ldots, x_k$ be the points adjacent to x. Suppose there is a point y not adjacent to x. Since G does not have P_4 as an induced subgraph, distance between x and y is 2. Let without loss of generality, y be adjacent to x_1. Since $d(x_1) \leqslant d(x)$, it follows that x_1 is not adjacent to some $x_i (2 \leqslant i \leqslant k)$. Now the subgraph induced by the four vertices y, x_1, x, x_i, is either P_4 or C_4, a contradiction.

Corollary 6: If G is θ_1-perfect then the clique graph of G is a complete graph.

Determination of θ_1 for a Class of r-Partite Graphs

Theorem 3: Let $K_{p(r)}$ denote the complete r-partite graph consisting of r vertex subsets each with p points. Then $\theta_1(K_{p(r)}) = p^2$ if every prime factor of p is greater than or equal to r.

Proof: Denote $K_{p(r)}$ by G and its vertex subsets by $V_1, V_2, \ldots, V_r$. Then G has p points and $\binom{r}{2} p^2$ lines. Since every clique in G contains r points we need at least $\binom{r}{2} p^2 / \frac{r(r-1)}{2}$ cliques to cover all the lines of G. Hence

$$\theta_1 \geqslant p^2 \tag{2}$$

We actually exhibit a set of p^2 cliques in G as follows : Let the vertices in V_i be denoted by $v_{i,1}, v_{i,2} \ldots, v_{i,p}$ and the p^2 cliques by $Q_{t,k} = \{v_{1,t}, v_{2,t+k}, v_{3,t+2k}, \ldots, v_{r,t+(r-1)k}\}$. $k = 0, 1, 2, \ldots, p$-1 and $t = 1, 2, \ldots, p$ where the numbers $t + (i-1)\ k$ are taken modulo p.

To complete the proof we can follow one of two procedures: We can show that the p^2 cliques are line disjoint in which case since each of these has $r(r-1)/2$

lines and the total number of lines in G is only $\binom{r}{2}p^2$, it follows that these cliques cover all the lines of G so that $\theta_1 \leqslant p^2$. That $\theta_1 = p^2$ now follows from eq. (2). Or, we can show that the p^2 cliques cover all the lines of G. Here we follow the former procedure :

Suppose an edge (x, y) is common to the two cliques Q_{t_1,k_1} and Q_{t_2,k_2}. Let $x \in V_i$ and $y \in V_j$. Then $(x,y) \in Q_{t_1,k_1} \cap Q_{t_2,k_2}$ if $x = v_{i,t_1+(i-1)k_1} = v_{i,t_2+(i-1)k_2}$ and $y = v_{j,t_1+(j-1)k_1} = v_{j,t_2+(j-1)k_2}$.

So $t_1+(i-1)k_1 \equiv t_2+(i-1)k_2 \pmod p$ and $t_1+(j-1)k_2 \equiv t_2+(j-1)k_2 \pmod p$.

So $(i-j)k_1 \equiv (i-j)k_2 \pmod p$.

So $(i-j)(k_1-k_2) \equiv 0 \pmod p$.

Since $0 \leqslant k_1, k_2 \leqslant p-1$, p does not divide (k_1-k_2), But if p is not prime, one factor of p may divide $(i-j)$ and anothei factor may divide (k_1-k_2). Thus, one sufficient condition that the above equation has no solution is that no prime factor of p is less than r. Under this condition the p^2 cliques are line disjoint and this completes the proof.

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