ON BERGE'S CONJECTURE CONCERNING PERFECT GRAPHS

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(Communicated by Prof. P. K. Menon, F.N.A.)

(*Received 9 May 1974; after revision 14 November 1974*)

Berge's conjecture that a graph is θ_0-perfect if and only if it has no C_{2n+1} or $\bar{C}_{2n+1}, n \geqslant 2$ as induced subgraphs is a famous unsettled problem of graph theory. Its importance is both theoretical (because of its bearing on the four-colour conjecture) and practical (because of its relation to perfect channels in communication theory). In this paper we show that the conjecture is equivalent to the following.

A θ_0-critical graph different from $\bar{C}_{2n+1}$ contains neither $C_{2t+1} + x$ nor $\bar{C}_{2s+1} + x$ as induced subgraphs, $n, s, t \geqslant 2$.

Introduction

Graphs considered here are finite undirected graphs, without loops and multiple edges. Generally the terminology in Harary (1969) is followed. A clique of a graph G is a maximal complete subgraph. The *point clique cover number*, $\theta_0(G)$ is defined as the minimum number of cliques covering all the points of G. A *point clique cover* $\{\theta_0(G)\}$ is a family of minimum number of cliques that cover all the points of G. A set $S \subseteq V(G)$ is called *independent set* in G if for $u, v \in S$, u is not adjacent to v. The largest number of points in such a set is called *independence number* β_0 (G).

A graph G is θ_0-perfect if $\theta_0(H) = \beta_0(H)$ for every induced subgraph H of G. G is called θ_0 imperfect if it is not θ_0-perfect.

Critical Graphs

A graph G is θ_0-critical if it is θ_0-imperfect and $G-v$ is perfect for every $v \in V(G)$. A graph G is *edge* θ_0-critical if $\beta_0(G) \neq \theta_0(G)$ and $G-x$ is θ_0-perfcct for every $x \in E(G)$. Clearly an edge θ_0-critical graph is θ_0-critical. But the converse is not true as $\bar{C}_7$ is θ_0-critical but not edge θ_0-critical. A point $v \in V(G)$ is a *unicliqual point* in G if it is not in two cliques. Otherwise it is called a non-unicliqual point. A clique containing a unicliqual point is called a *free clique*. A clique on three points with a unicliqual point is called a *free triangle.*

Statement of Berge's Conjecture

BC: A graph G is θ_0-perfect iff G has no induced subgraphs C_{2n+1} and $\bar{C}_{2n+1}$, $n \geqslant 2$.

H. Sachs (1970) has given six equivalent formulations of *BC*. E. Olaru (1969) has proved a result which is a bit weaker than *BC* viz.,

*Research supported by financial assistance of the CSIR, India.

Theorem 1: Every edge θ_0-critical graph is C_{2n+1}, $n \geqslant 2$.

In this paper we prove the following :

Theorem 2: The Berge conjecture is equivalent to the following conjecture.

(A) *A θ_0-critical graph, different from $\bar{C}_{2n+1}$ contains neither $C_{2s+1} + x$ nor $\bar{C}_{2t+1} + x$, as induced subgraphs, $n, s, t \geqslant 2$.*

Proof of equivalence of the conjecture : In proving this, in addition to Theorem 1, we use the following:

Theorem 3 (H. Sachs) : Every θ_0-imperfect graph contains a θ_0-critical graph as an induced subgraph.

We first establish the following two lemmas :

Lemma 1: If G is θ_0-*critical* then G has no unicliqual point.

Proof: To prove this we assume the following basic lemma due to Berge (1969) viz., if a connected graph G contains a complete subgraph K by whose removal, components $C_1, .., C_r$ $(r \geqslant 2)$ are created and if $< V(C_r) \cup V(K)>$ is θ_0-perfect for every r, then G is θ_0-perfect. Suppose G has an unicliqual point v. Let K_v be the clique containing the point v. Clearly $K_v - v$ is a complete subgraph by whose removal G will be disconnected into r components, $r \geqslant 2$, and $<v>$ will be one of these components. Let $G - K_v = C_1 \cup C_2 \cup .. \cup C_{r-1}$. Since $< V(C_i) \cup V(K_v - v) >$, $i = 1, 2 .. r-1$ and $< V(K_v - v) \cup \{v\} > = K_v$ are proper induced subgraphs of G, these are θ_0-perfect. Hence by above lemma G is θ_0-perfect. This violates the fact that G is θ_0-critical.

Lemma 2: If a point $v \in V(G)$ is a unicliqual point in G then it is unicliqual in any induced subgraph containing it. Proof is trivial.

Define $M(G) = \{v \in V(G) : v$ is a nonunicliqual point in $G\}$

$S(G) = \{v \in V(G) : v$ is a unicliqual point in $G\}$

Property P: A graph G is said to have property P if every edge of $<M(G)>$ is contained in a free triangle.

To prove $(A) \Rightarrow BC$.

Assume that result (A) is true. We prove BC. as a consequence of the following :

Theorem 4: If (A) holds true and if a graph G satisfies the property P and has no induced subgraphs C_{2n+1} and $\bar{C}_{2n+1}$ $(n \geqslant 2)$ then G is θ_0-perfect.

Proof: The result is obviously true for graphs with 1 or 2 points. Suppose the result is true for every graph having $p-1$ points. Let G be a graph with p points and suppose G has the property p and has no subgraphs C_{2n+1} and $\bar{C}_{2n+1}$, $n \geqslant 2$.

Case 1: There exists a unicliqual point v such that $G-v$ satisfies the property P. By induction hypothesis $G-v$ is θ_0-*perfect*. We show that G is θ_0-perfect. Suppose G is θ_0-imperfect. By theorem 3, G contains a θ_0-critical graph H as an induced subgraph. Therefore $V(H) \subseteq M(G)$ by lemma 1 and 2. But $<M(G)>$, being an induced subgraph of $G-v$, is θ_0-*perfect*. Hence H is also θ_0-*perfect*. This contradicts that H is θ_0-*critical*.

Case 2: There does not exist a unicliqual point $v \in V(G)$ such that $G-v$ satisfies the property P. It is easy to observe that for each $e \in E(<M(G)>)$ there exists a unique unicliqual point v adjacent to the terminals of e and conversely for each $v \in S(G)$, there exists a unique edge $e \in E (< M(G) >)$ such that v

is adjacent to the terminals of e. Thus there is a $1-1$ correspondence between $S(G)$ and $E(<M(G)>)$.

If for some $v \in S(G)$, $G-v$ is θ_0-perfect then G is θ_0-perfect as shown in the former case. Otherwise $G-v_i$ (for every $v_i \in S(G)$) contains a θ_0-critical graph H as an induced subgraph such that $V(H) \subseteq M(G-v)=M(G)$ (from lemma 1 and 2). But for every $u \in M(G)$, $G-u$ satisfies the property P. This implies by induction hypothesis that $<M(G)> - u$ is θ_0-perfect for every $u \in M(G)$. Thus $H = <M(G)>$. Since by our assumption H contains neither $C_{2n+1} + x$ nor $\bar{C}_{2n+1} + x$, $G-v_i-x_i$ does not contain C_{2n+1} and $\bar{C}_{2n+1}$ and satisfies the property P for every $v_i \in S(G)$ and corresponding $x_i \in E(H)<M(G)>$. Hence by induction hypothesis $G-v_i-x_i$ is θ_0-perfect for every i. This implies $H - x_i$ is θ_0-perfect for every $x_i \in E(H)$. Thus H is edge θ_0-critical. By E. Olarus theorem 1, $H=C_{2n+1}$, $n \geqslant 2$. This violates that G has no C_{2n+1}, $n \geqslant 2$ as an induced subgraph. So our assumption that $G-v_i$ is not θ_0-perfect for every $v_i \in S(G)$ is wrong i.e., $G-v_k$ is θ_0-perfect for some $v_k \in S(G)$. This implies that G is θ_0-perfect as in the previous case.

Theorem 5: If (A) holds true and if a graph G has no C_{2n+1} *and* $\bar{C}_{2n+1}$, $n \geqslant 2$ *as induced subgraphs then G is* θ_0*-perfect.*

Proof: Let $e_1 \ldots e_r$ be the edges of $<M(G)>$ which are not contained in a free triangle. Join a point v_i $V(G)$ to the terminals of the edge $e_i = (u_i, u_j)$ such that $<v_i, u_i, u_j>$ is a free triangle in the resultant graph G'. Now G' satisfies the property P. We also observe that G' has no C_{2n+1} and $\bar{C}_{2n-1}$ as induced subgraphs. From the above theorem G' is θ_0-perfect. Since G is an induced subgraph of G', G is also θ_0-perfect. Proof of BC immediately follows from the above theorem.

BC. $\Rightarrow$ *(A):* BC implies that every θ_0-critical graph is either C_{2n+1} or $\bar{C}_{2n+1}$, $n \geqslant 2$ (shown by Sachs 1970). Therefore (A) follows trivially.

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