GRAPH AND ITS COMPLEMENT

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For a given graph theoretic parameter f and positive integer p, the problem of Nordhaus-Gaddum class is to determine upper and lower bounds (preferably sharp bounds) for

$$f(G) + f(\bar{G}) \text{ and } f(G) \,.\, f(\bar{G}),$$

where G is a graph of order p; $\bar{G}$, the complement of G.
Recently, Alavi and Mitchem (1971) showed that, if $k(G)$ is the point connectivity of a graph of order p, then

$$0 \leq k(G) \,.\, k(\bar{G}) \leq M(p).$$

where

$$M(p) = \begin{cases} \left[\dfrac{(p-1^2)}{4} \right] & \text{if } p \neq 3 \bmod 4 \\ \dfrac{p^2 - 2p - 3}{4} & \text{if } p \equiv 3 \bmod 4 \end{cases}$$

In section titled "*Point Connectivity of a graph and its Complement*", it is shown that the bound $M(p)$ is the best possible when $p \neq 4$. This answers, in affirmative, a question of Chartrand and Mitchem (1971).
In the next section, it is proved that if $c_1(G)$ is the line core number of a graph G of order P (≥ 3), then

$$0 \leq c_1(G) + c_1 \bar{G} \leq \begin{cases} \dfrac{p^2+2p-3}{4} & \text{if } p \text{ is odd} \\ \dfrac{p^2+4p-8}{4} & \text{if } p \text{ is even} \end{cases}$$

and the bounds are the best possible. Similar best possible bounds are also obtained for the point core number of a graph . In the last section, some unsolved problems in the Nordhaus-Gaddum class are discussed.

INTRODUCTION

We consider only finite graphs without loops and multiple lines. For notation and terminology we follow Harary (1969). A *cut set* of a connected graph G is a set of points whose removal from G results in a disconnected or trivial graph. The *point connectivity* of G, denoted by $k(G)$ is the smallest number of points in a cut set of G.

Alavi and Mitchem (1971) proved, among other things, that for any graph G of order p,

$$k(G)\ .\ k(\bar{G}) \leqslant M(p)$$

where

$$M(p) = \begin{cases} \frac{(p-1)^2}{4} & \text{if } p\ 3 \not\equiv \text{mod } 4 \\ \frac{p^2-2p-3}{4} & \text{if } p\ 3 \equiv \text{mod } 4 \end{cases} \qquad \cdots \qquad \cdots \qquad (*)$$

In section 1, it is shown that $M(p)$ is the best possible bound when $p \neq 4$. This solves a problem, in affirmative, of Chartrand and Mitchem (1971). For the definitions of line core number, point core number of graph refer section titled, "*The line core number (point core number) of a graph and is complement*" of this paper.

Point Connectivity of a Graph and its Complement

In this section we prove

Theorem 1: For any graph G of order p (≥ 2), $k(G)$. $k(\bar{G}) \leqslant M(p)$, where $M(p)$ is defined as in equation (*), and the bound is the best possible except when $p = 4$; and if $p = 4$, then $k(G)\ .\ k(\bar{G}) \leqslant 1$.

The proof of this theorem is constructive in nature and is divided into three lemmas.

Lemma 1: Let $p = 2n$ and $G_1(2n) = G_1$ be the graph of order p defined as follows : G_1 has points $A \cup B$ where $A = \{1, 2, .., n\}$ and $B = \{1', 2', .., n'\}$. G_1 restricted to A or to B is the complete graph and the only other lines of G_1 are $\{(i, i')\ ;\ 1 \leqslant i \leqslant n\}$. Then (*i*) $k(G_1) = n$ and (*ii*) $k(\bar{G}_1) = n-1$ provided $n \neq 2$. Thus $k(G_1)$. $k(\bar{G}_1) = n(n-1) = M(p)$.

Proof: The case $n = 1$ is trivial. So let $n \geq 3$. To prove (*i*), let C be a cut set of G_1, then i' belongs to C whenever i does not belong to C, so that $|C| \geq n$. Hence $k(G_1) \geq n$ and the equality follows since G_1 is a regular graph of degree n.

To show that $k(\bar{G}_1) = n-1$, first observe that $\bar{G}_1$ is a regular graph of degree $n-1$. Secondly $\bar{G}_1$ is $(n-1)$—connected since the removal of any $n-2$ or less vertices leaves at least 3 vertices in one of A, B and at least two vertices in the other.

Lemma 2: Let $p = 4n+1$ ($n \geq 1$) and G_2 be the graph of order p defined as follows : G_2 has points $A \cup B \cup \{u_0\}$, where $A = \{1, 2, .., 2n\}$, $B = \{1', 2', .., (2n)'\}$ and u_0 is a point not in $A \cup B$. G_2 restricted to A is $K_{2n} - I$, where I is the 1-factor consisting of the lines $\{(1, 2,), (3, 4), .., (2n-1, 2n)\}$; G_2 restricted to B is the complete graph and the only other lines of G_2 are $\{(u_0, i), (i, i') : 1 \leqslant i \leqslant 2n\}$. Then (*i*) $k(G_2) = 2n$ and (*ii*) $k(\bar{G}_2) = 2n$.

Thus $k(G_2)\ .\ k(\bar{G}_2) = 4n^2 = M(p)$.

Proof: To prove (1), let, if possible, C be a cut set of G_2 with $k = |C| \leqslant 2n-1$. Then there is i_0, $1 \leqslant i_0 \leqslant 2n$, such that neither i_0 nor i_0' is in C, which means that (i_0, i_0') is a line of $G_2 - C$. If now u_0 is not in C, then since u_0 is joined to every point of $A - C$ it follows that $G_2 - C$ is a connected graph. Thus $u_0 \in C$.

Now since $k \leqslant 2n-1$, we have $|A-C| \geqq 2$. If $|A-C| = 2$, then $B \cap C = \phi$ and this coupled with the fact that every point of $A-C$ is joined to a point of B implies that G_2-C is connected. Thus $|A-C| \geqq 3$. Then, by definition of G_2, the subgraph of G_2 induced by $A-C$ is connected and since (i_0, i_0') is a line of G_2-C it follows that G_2-C is also connected. This contradicts the fact that C is a cut set of G_2. Thus $k(G_2) \geqq 2n$ and the equality follows since G_2 is a regular graph of degree $2n$.

To show that $k(\bar{G}_2) = 2n$, first note that if $n=1$, then $\bar{G}_2$ is the 5-cycle. So assume that $n \geqq 2$, and let, if possible, C be a cut set of $\bar{G}_2$ with $k = |C| \leqslant 2n-1$. If u_0 is in C, then since $\bar{G}_1(4n)$ of Lemma 1 is a spanning subgraph of $\bar{G}_2-u_0$ it follows that $\bar{G}_2-u_0$ is $(2n-1)$-connected and hence $\bar{G}_2-C$ is not disconnected. Thus u_0 is not in C and we have $|B-C| \geqq 1$ and $|A-C| \geqq 1$. If $|B-C| = 1$, then $k = 2n-1$ and $A \cap C = \phi$. Assume without loss of generality that $1' \in B-C$. Then $(1', i)$, $2 \leqslant i \leqslant 2n$, are lines of $\bar{G}_2-C$ and this coupled with the fact that $(2, 1)$, $(u_0, 1')$ are lines of $\bar{G}_2-C$ gives us that $\bar{G}_2-C$ is a connected graph. This shows that $|B-C| \geqq 2$. In this case every point of $A-C$ is joined to at least one point of $B-C$ and u_0 is joined to all points of $B-C$. These imply that $\bar{G}_2-C$ is connected, a contradiction. Hence $k(\bar{G}_2) \geqq 2n$ and the equality follows since $\bar{G}_2$ is a regular graph of degree $2n$. This completes the proof of the lemma.

Lemma 3: Let $p = 4n+3$ $(n \geqq 1)$ and G_3 be the graph of order p defined as follows : G_3 has vertices $A \cup B \cup \{u_0\}$ where $A = \{1, 2, .., 2n+1\}$; $B = \{1', 2', .., (2n+1')\}$ and u_0 is a point not in $A \cup B$. G_3 restricted to A and also to B is the complete graph and the only other lines of G_3 are

(i, i'),	$1 \leqslant i \leqslant 2n+1$;
$[i, (i+1)']$.	i even and $1 \leqslant i \leqslant 2n+1$;
$(u_0, 1')$	
(u_0, i),	i odd and $1 \leqslant i \leqslant 2n+1$;
(u_0, i'),	i even and $1 \leqslant i \leqslant 2n+1$.

[see Figure 1 for $G_3(15)$].

Then, (i) $k(G_3) = 2n+2$ and (ii) $k(\bar{G}_3) = 2n$. Thus $k(G_3).\ k(\bar{G}_3) = (2n+2).\ 2n = M(p)$.

Proof: Clearly, $G_1(4n+2)$ of Lemma 1 is a spanning subgraph of G_3-u_0 and u_0 is joined to $2n+2$ points of G_3-u_0. Thus by Lemma 1, $k(G_3) \geqq 2n+1$. Further, G_3 is a regular graph of degree $2n+2$. So to show that $k(G_3) = 2n+2$, it is enough to prove that G_3 has no cut set C with $|C| = 2n+1$. Let, if possible, C be a cut set of G_3 with $|C| = 2n+1$. Since $k(G_3-u_0) \geqq 2n+1$, we have $u_0 \notin C$. Let now $i_1', i_2', \ldots, i_\theta'$ be the points of $B \cap C$. Since $G_3 - A$ is connected it follows that $\theta \geqq 1$. Further, $i_j \in C$ whenever $j \notin \{1, 2, \ldots, \theta\}$ for otherwise (i_j, i_j') is a line of $G_3 - C$ and also (u_0, i_j) or (u_0, i_j') is a line of $G_3 - C$ for some $j \notin \{1, 2, \ldots, \theta\}$ so that $G_3 - C$ is a connected graph. Now since $|C| = 2n+1$, we have $C = \{i_1', i_2', \ldots, i_\theta'; i_{\theta+1}, \ldots, i_{2n+1})\}$ where $\pi = \{i_1, i_2, \ldots, i_{2n+1}\}$ is a permutation of $[1, 2, \ldots, 2n+1]$. Here we consider two cases.

Case (1): $1 \in C$. Then $1' \notin C$ and this together with the fact that $(u_0 1')$ is a line of $G_3 - C$ implies that $\{3, 5, 7, \ldots, 2n+1\}$ is a subset of C which in turn implies that

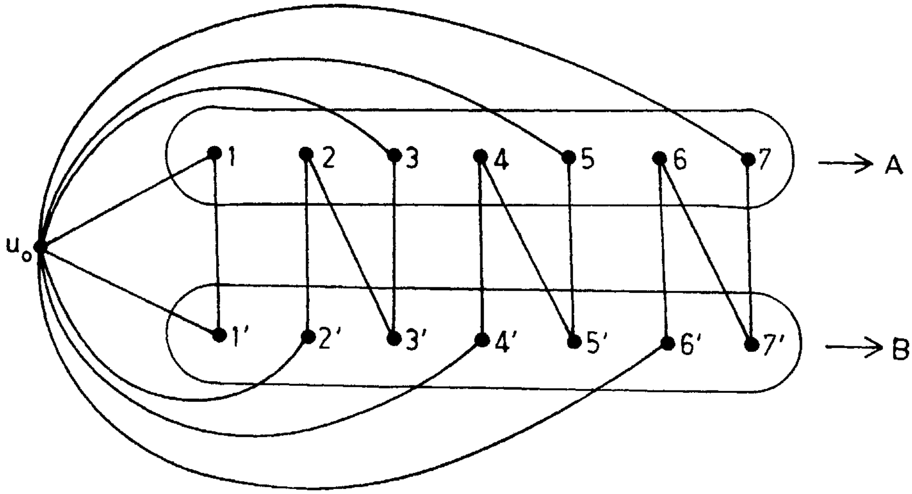

FIG. 1. The graph G_3 (15), where A, B are complete

no point of $\{3', 5', \ldots, (2n+1)'\}$ is in C. Now since $\{(i+1)', i\}$ is a line of G_3 whenever i is even, we get that $[2, 4, \ldots, 2n]$ is also a subset of C. Thus $C = A$. But then $G_3 - C$ is connected, a contradiction.

Case (2): $1 \notin C$. This case is similar to case (1) above.

Thus $k(G_3) = 2n + 2$ and this completes the proof of (i).

To show that $k(\bar{G}_3) = 2n$, we proceed as follows :

Call points $2, 4, \ldots, 2n$; $3', 5', \ldots, (2n+1)'$ of $\bar{G}_3$ the *type* 2 points and the rest, excluding u_0, the *type* 1 points. Clearly, type i point in A or B is unjoined to exactly i points of the other set, $i = 1, 2$. Further, u_0 is joined to all type 2 points. If $n = 1$, then $\bar{G}_3$ is the 7-cycle and hence $k(\bar{G}_3) = 2$. So assume that $n \geq 2$. Let, if possible C be a cut set of $\bar{G}_3$ with $|C| \leqslant 2n-1$. Since the roles of A and B in $\bar{G}_3$ are symmetric we can assume that $|A-C| \leqslant |B-C|$. We consider two cases and in each case we first show that any two nonadjacent points in $(A \cup B)-C$ are connected by a path and later conclude that $\bar{G}_3 - C$ is connected.

Case (3): $|A-C| \geq 3$. Hence $|B-C| \geq 3$.

Since $|C| \leqslant 2n-1$ and $n \geq 2$ it follows that $|B-C| \geq 4$. Let now u_1, u_2 be distinct points of $A-C$. If a point v_1 exists such that (u_1, v_1) and (u_2, v_1) are lines, then u_1 and u_2 are connected. If there is no such point v_1, then $B-C$ consists of the four points joined to u_1 and u_2 in G_3 and since $A-C$ contains a vertex other than u_1 and u_2 it is easy to see that u_1 and u_2 are connected by a path of length 4. Let now $u_1 \in A-C$ and $v_1 \in B-C$. then since $|A-C| \geq 3$, v_1 is joined to at least one point u_2 (say) of $A-C$. If $u_2 \neq u_1$, then by the above argument u_1, u_2 are connected in $\bar{G}_3 - C$. Hence u_1, v_1 are also connected in $\bar{G}_3 - C$. Since any two points of $A-C$ are connected and any point of $B-C$ and any point of $A-C$ are connected, it also follows that any two points of $B-C$ are connected.

Case (4): $| A-C | = 2$. Now since $| C | \leqslant 2n-1$ and $n \geq 2$ it follows that C is a subset of A, $| B | \geq 5$ and $u_0 \notin C$.

Let u_1, u_2 be the two distinct points of $A-C$, then since $| B | \geq 5$, there is a point v_1 of B such that (u_1, v_1), (v_1, u_2) are lines. Hence u_1, u_2 are connected in $\bar{G}_3-C$. Let $v_1 \in B$. If v_1 is joined to u_1 or u_2, then by the above argument u_1, v_1 are connected. Otherwise v_1 and one of u_1, u_2, are type 2 points and these type 2 points are connected through u_0. Hence u_1, v_1 are connected in $\bar{G}_3-C$. As in the preceding case, any two points of $B-C$ are also connected.

Clearly, $| A-C |$ cannot be less than 2. So, by cases (3), (4) and the fact that not every point adjacent to u_0 in $\bar{G}_3$ is in C we get that $\bar{G}_3-C$ is connected. Thus in $\bar{G}_3$ there is no cut set with less than $2k$ elements or equivalently $k(\bar{G}_3) \geq 2n$ and the equality follows since $\bar{G}_3$ is a regular graph of degree $2n$. This completes the proof.

Proof of Theorem 1: That $k(G). k(\bar{G}) \leqslant M(p)$ for any graph G of order p was proved in Alavi and Mitchem (1971). The bestness of the upper bound in the case $p \neq 4$ follows from Lemma 1, 2 and 3. If now $p=4$, then $M(p) = 2$. $k(G) = 2$ for a graph G means that G has at least four lines and hence $\bar{G}$ has at most two lines. Thus $\bar{G}$ is disconnected. Hence if $p = 4$, $k(G). k(\bar{G}) \leqslant 1$ and the bestness oi the upper bound follows from the fact that the path of order 4 is a self complementary graph. This completes the proof.

The Line Core Number (Point Core Number) of a Graph and its Complement

A point and a line are said to *cover* each other if they are incident. A set of points which covers all the lines of a graph G is called a *point cover* of G, while a set of lines which covers all the points is a *line cover*. The smallest number of points in any point cover of G is called its *point covering number* and is denoted by $\alpha_0(G)$ or α_0, Similarly, $\alpha_1(G)$ or α_1 is the smallest number of lines in any line cover of G and is called its *line covering number.*

The *line core* $C_1^*(G)$ of a graph G is the subgraph of G induced by the union of all independent sets Y of lines (if any) such that $| Y | = \alpha_0(G)$. This concept was introduced by Dulmage and Mendelsohn who made it an integral part of their theory of decomposition for bipartite graphs. The *line core number* $C_1(G)$ of a graph G is the number of lines in the subgraph $C_1^*(G)$ and is 0 if $C_1^*(G)$ does not exist. The point core $C_0^*(G)$ is the subgraph of G induced by the union of all independent sets S of $\alpha_1(G)$ points (if one exists). The *point core number* $C_0(G)$ of a graph G in the number of lines in the subgraph $C_0^*(G)$ and is 0 if $C_0^*(G)$ does not exist. For many theorems on line core and point core refer Harary (1969), p. 101.

In this section we derive bounds for the sum and product of the line core number (point core number) of a graph and its complement.

Theorem 2 : For any graph G of order p,

$$0 \leqslant C_1(G)+C_1(\bar{G}) \leqslant \begin{cases} \dfrac{p^2+2p-3}{4} & \textit{if } p \textit{ is odd} \\ \dfrac{p^2+4p-8}{4} & \textit{if } p \textit{ is even} \end{cases} \quad \cdots (1)$$

Further, the bounds are the best possible.

Proof: Let G be a graph of order p. If both G and $\bar{G}$ have no line core, then the bounds are trivial. So assume that $C_1(G)>0$. It is already known (Harary 1969, p. 101) that the components of the line core are bipartite subgraphs of G. Thus the line core is a graph without triangles, so by Turan's theorem (Harary 1969, p. 17), $C_1(G) \leqslant \left[\frac{p^2}{4}\right]$ If now $C_1(\bar{G}) = 0$, then the upper bound in (1) follows. So let $C_1(\bar{G}) > 0$. Without loss of generality assume that $\alpha_0(G) \geqq \alpha_0(\bar{G})$. Let $A = [u_1, u_2, \ldots, u_k]$ be a point cover of G, where $k = a_0(G)$. Clearly, $V-A$ is an independent set in G. Hence $V-A$ is complete in $\bar{G}$ so that $\alpha_0(\bar{G}) \geqq p-k-1$. But $\alpha_0(\bar{G}) \leqslant \alpha_0(G) = k$, which implies that $p \leqslant 2k+1$. Since $C_1(G)>0$, there is a set of $\alpha_0(G)=k$ independent lines of G so that $p \geqq 2k$. Thus we have $p=2k+1$ or $2k$. We consider these two cases separately.

Case (5): $p = 2k+1$. Since $V-A$ is complete in $\bar{G}$ and has $k+1$ points, any point cover B of $\bar{G}$ with exactly k points is a subset of $V(G)-A$. Thus A is independent in $\bar{G}$ and there is a point v in $V(G)-A$ which is not joined to any point of A in G. So A is complete in G and (v, u_i), $1 \leqslant i \leqslant k$, are lines of G. Since $V-A$ is independent in G and A has k points, no set of k independent lines of G contains a line of the form (u_i, u_j). Thus $C_1(G) \leqslant k + m(G, A, B)$, where $m(G, A, B)$ denotes the number of lines in G with one end point in A and the other in B. Since $A \cup [v]$ is independent and $B \cup [v]$ is complete in $\bar{G}$, it follows, as above, that $C_1(\bar{G}) \leqslant k + m(\bar{G}, A, B)$. Thus $C_1(G) + C_1(\bar{G}) \leqslant 2k + k^2 = \frac{p^2+2p-3}{4}$. This proves the upper bound in (1) in the case p is odd. Now $C_1(G)$. $C_1(\bar{G})$ is maximum whenever $m(G, A, B) = m(\bar{G}, A, B) = \frac{k^2}{2}$. Thus if $\frac{k}{2}$ is even, then the maximum value of $G_1(G)$. $G_1(\bar{G}) \leqslant \left(k+\frac{k^2}{2}\right)^2 = \left(\frac{p^2+2p-3}{8}\right)^2$. If k is odd, then the maximum value $C_1(G)$. $C_1(\bar{G}) \leqslant \left(\frac{p^2+2p+1}{8}\right) \cdot \left(\frac{p^2+2p-7}{8}\right)$.

Case (6): $p = 2k$. Clearly $C_1(G) \leqslant m(G, A, V-A)$. Let B be any point cover of $\bar{G}$ with $\alpha_0(\bar{G})$ elements. Then $\alpha_0(\bar{G}) = k$ or $k-1$, if $\alpha_0(\bar{G}) = k$ and B is a subset of $V(G)-A$, then $B = V(G)-A$ and $C_1(\bar{G}) \leqslant m(\bar{G}, A, V-A)$ so that

$$C_1(G)+C_1(\bar{G}) \leqslant k^2 = \frac{p^2}{4} < \frac{p^2+4p-8}{4}.$$

If $|B| = k$ and B is not a subset of $V-A$, then B contains exactly one element of A, say u_1. Let v_1 be the unique point of $V-A$ not in B. Then v_1 is not joined to any point of $A-(u_1)$ in $\bar{G}$. Now the line core of $\bar{G}$ cannot contain any line in $B-u_1$ and $A-u_1$ is independent in $\bar{G}$, so $C_1(\bar{G}) \leqslant 2k-2+m(\bar{G}, A, V-A)$. Thus

$$C_1(G) + C_1(\bar{G}) \leqslant 2k-2+k^2$$
$$= p-2+\frac{p^2}{4} = \frac{p^2+4p-8}{4}.$$

If $|B| = k-1$, then B is a subset of $V-A$, find $V-B$ is independent in $\bar{G}$. Hence $C_1(\bar{G}) \leqslant k-1+m(\bar{G}, A, B)$, so that $C_2(G) + C_1(\bar{G}) \leqslant k-1+k^2 \leqslant \frac{p^2+4p-8}{4}$. This proves the upper bound in (1) whenever p is even.

The maximum value of $C_1(G)$. $C_1(\bar{G})$ is $\leqslant \left(\frac{p^2+4p-8}{8}\right)^2$, if k is even and $\leqslant \frac{p^2+4p-4}{8} \cdot \frac{p^2+4p-12}{8}$ if k is odd.

To show that the bounds are the best possible, we consider cases (5) and (6) separately.

Case (5): $p = 2k + 1$. Let $A = \{1, 2, \ldots, k\}$, $B = \{1', 2', \ldots, k'\}$. The points of G are $A \cup B \cup \{u_0\}$. G restricted to A is complete and G restricted to B is independent. The only other lines of G are (i, i'), (u_0, i) for every i, $1 \leqslant i \leqslant k$. Then $C_1(G) = 2k$ and $C_1(\bar{G}) = k^2$.

Case (6): $p = 2k$. Let $A = \{1, 2, \ldots, k\}$ $B = \{1', 2', \ldots, k'\}$. The points of G are $A \cup B$. G restricted to A is complete and G restricted to $B-\{1'\}$ is independent. The only other lines of G are (i, i'), $1 \leqslant i \leqslant k$ and $(1', i')$, $2 \leqslant i \leqslant k$. Then $C_1(G) = 2k - 2 + k$ and $C_1(\bar{G}) = k^2 - k$.

The upper bounds for $C_1(G)$. $C_1(\bar{G})$ are also the best possible whenever $p \geqq 10$. We illustrate only the case $p = 2k + 1$ and k even. The other cases are similar. Let H be a regular graph of degree $k/2$ and bipartite with $\{1, 2, \ldots, k\}$ and $\{1', 2', \ldots, k'\}$ as the bipartition, construct a new graph G of order $2k + 1$ by adjoining all the lines (i, j) to H, for $1 \leqslant i \leqslant j \leqslant k$; and joining a new point u_0 by lines (u_0, i), $1 \leqslant i \leqslant k$. Then $C_1(G) = k + \frac{k^2}{2} = C_1(\bar{G})$. Thus $C_1(G)$. $C_1(\bar{G}) = \left(\frac{p^2+2p-3}{8}\right)^2$.

The point core number of a graph and its complement: We will state (without proofs the main results.

Lemma 4 : For any graph G of order p ($\geqq 6$)

$$C_0(G) \leqslant \begin{cases} \frac{p^2}{4} & \text{if } p \text{ is even} \\ \frac{(p-3)^2}{5} & \text{if } p \text{ is odd.} \end{cases}$$

Theorem 3 : For any graph G of order p($\geqq 6$)

$$C_0(G) + C_0\bar{G} \leqslant \begin{cases} \frac{p^2}{4} & \text{if } p \textit{ is even} \\ \frac{(p-3)^2}{4} & \text{if } p \text{ i odd.} \end{cases}$$

Further $K_{p/2,\, p/2}$ is the only graph attaining the upper bound if p is even.

Some Unsolved Problems in the Nordhaus-Gaddum Class

Problem 1: Let $\alpha_{00}(G)$ denote the minimum number of points required to cover all the points of G. This α_{00} is called the *point-point covering number* of G (also called the *external stability number* of G). The problem : (Vizing) is it true that for any graph of order p, $\alpha_{00}(G)$. $\alpha_{00}(\bar{G}) \leqslant p$?

Remarks: If G or $\bar{G}$ has diameter > 2, then the inequality holds. If $p=9$, then the latin square graph of order 3, $I_2(3)$ show that the bound is the best possible. If G_0 is regular of degree 1, then equality holds (p is even).

Conjecture : $L_2(3)$, $\overline{L_2(3)}$, G_0, $\bar{G}_0$ are the only *regular* graphs for which equality holds in the above inequality.

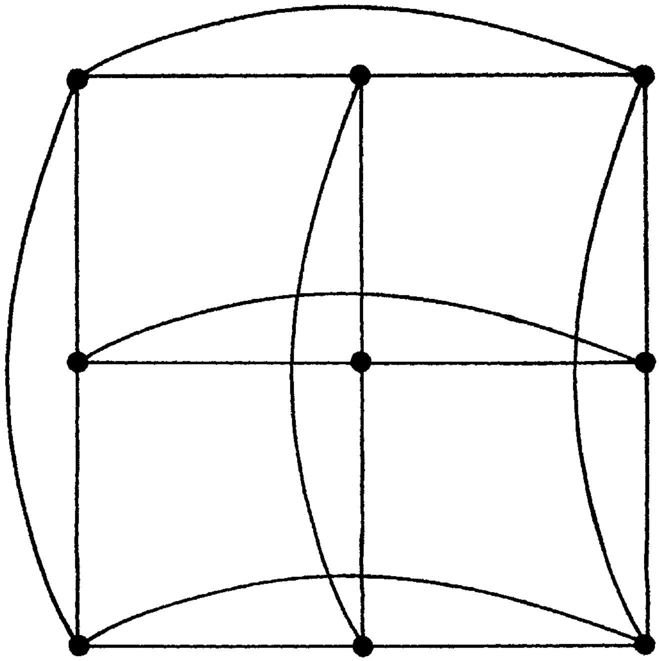

FIG. 2. The graph $L_2(3)$

Problem 2: (Chartrand & Mitchem 1971) Find bounds for $f(G) + f(\bar{G})$ when f is the *total chromatic* number of a graph.

Problem 3: For which values of s, $\bar{s}$ satisfying $s.\bar{s} \leqslant M(p)$, is there a graph G of order p such that

$$k(G) = s, \; k(\bar{G}) = \bar{s},$$

where $k(G)$ is the point connectivity of G and $s \leqslant \bar{s}$?

Remarks: If p is even and $(s, \bar{s}) \neq (1, n-2)$ the answer to problem 3 is in affirmative. For p odd the answer is unknown.

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