ON THE INTERSECTION NUMBER OF SOME GRAPHS

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A symbol allocation procedure which when judiciously applied leads to sharp upper bounds for ω, the intersection number of a graph, is described. Two general results which give lower bounds for ω are then obtained. Matching these lower and upper bounds, ω is determined for a wide class of graphs (including block-graphs, K_4—x free graphs etc.). $\omega(L(G))$ for a general graph G and $\omega(T(G))$ for a triangle-free graph G are also obtained (where $L(G)$ is the line graph and $T(G)$ is the total graph of G).

INTRODUCTION

The graphs considered here are finite and without loops or multiple edges. We refer to Harary (1969) for general terminology not explained here.

We write $v_i\ Av_j$ to denote vertex v_i is adjacent to vertex v_j. A vertex v is *apexed* to an edge (u_1, u_2) if vAu_1 and vAu_2. Let $V_i = \{v \in V(G) : d(v) = i\}$ and $U_i = \{v \in V(G) : d(v) \geqslant i\}$. If a triangular block of G has two of its vertices in V_2, it is called a *terminal triangle,* which is said to be incident with the vertex of degree $\geqslant 3$. If a triangle in G has only one point of degree 2 in G it is said to be a *semi-terminal triangle.* An edge (u, v) with $d(u) = 1$ and $d(v) \geqslant 2$ is called a *terminal edge* incident at v.

In recent years more and more attention is being devoted to the study of the invariant numbers of a graph. While no complete set of invariants for a graph has been found, many relations between the various invariants, mostly in the form of inequalities, have been obtained. One of these interesting invariants is the intersection number ω of a graph G which is the smallest cardinality of a set S such that there is a family F of distinct non-empty subsets S_i of S and a $(1-1)$ correspondence between these subsets and the vertices v_i of G such that v_iAv_j in G if $S_i \cap S_j \neq \phi$ $(i \neq j)$. If G is the intersection graph on a set S, we write $G = \Omega(S)$. It is well-known that every graph G is an intersection graph of some set and that, and

(*i*) if G is a (p, q)-graph with $p \geqslant 3$, then $\omega(G) \leqslant q$ and (for $p > 3$) $\omega(G) = q$ iff G has no triangles; and

(*ii*) if $p \geqslant 4$, then $\omega(G) \leqslant \left[\frac{p^2}{4}\right]$ $[x]$ = greatest integer not exceeding x.

The evolution of a closed formula for ω or even an algorithm for the determination of ω for a general graph seems to be quite a difficult problem. In this paper we

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suggest a two-way approach to tackle this problem and obtain a few particular results. On the one hand we obtain (by what we call an 'allocation procedure' described under "Allocation Procedure and Upper Bounds for ω") an upper bound for ω by finding a set S for which G is an intersection graph. With some practice and intuition it is not difficult, for any specific graph or for certain classes of graphs, to find a minimal allocation by the above procedure; but to prove that the allocation is minimal is usually not easy. In the specific cases covered in this paper we prove the minimality of the allocation by means of two results (see section "Lower Bounds for ω which give lower bounds for ω. The proof of minimality is achieved by making these lower bounds equal to the upper bounds determined by the allocation procedure, by a judicious deletion of cliques from the graph. Incidentally this throws some light on the relation between ω and other invariant for graphs : the lineclique-cover number θ_1. The specific results obtained include formulae for ω for graphs in which no two cliques intersect in more than one point, line graphs and total graphs.

Allocation Procedure and Upper Bounds for ω

We use the set of natural numbers $N = \{1, 2, 3, \ldots\}$ as the symbol store from which sets of symbols are assigned to the vertices of the graph such that no two symbol sets S_α, S_β assigned to different vertices v_α, v_β are same and $S_\alpha \cap S_\beta \neq \phi$ iff $v_\alpha A v_\beta$. We also insist that if a total of n symbols are used for a graph these are the first n natural numbers. If ω is known in advance, an allocation (in fact, a minimal allocation) can be obtained by assigning suitable subsets of the ω-set to the vertices. If ω is not known in advance we start from the symbol 1 and go on adding extra symbols as and when necessary to account for all the incidences in G and to distinguish all the vertices. If a symbol is common to all the vertices of a subgraph H, it is clear that H should be a complete subgraph (with $| V(H) | \geqslant 2$). Such a symbol is called a circulating symbol. We adopt the notation $x_i \circlearrowright H$ to indicate that x_i is a *circulating symbol* over H. A symbol which is not common to even a pair of vertices is called a *distinguish symbol.*

Even a minimal allocation need not be unique. For example for K_3, (1, 12, 123), (12, 23, 31), (12, 13, 123) are all minimal allocations. To determine $\omega(K_p)$ we allocate symbol 1 to all the vertices to account for all the incidences and go on introducing new distinguishing symbols forming the sets, 12, 13, 123, 14, 124, 134, 1234 etc. This easily leads to the upper bound : $\omega(K_p) \leqslant \{1 + \log_2 p\}$. ($\{x\}$ = smallest integer $\geqslant x$). Since the maximum number of pairwise intersecting subsets of a p-set is 2^{p-1}, (to see this note that for any subset A, only one of A and its complement A^c can be included in such a collection) we also have the lower bound $\omega(K_2^p) \geqslant p+1$, that is, $\omega(K_p) \geqslant [1+\log_2 p]$. Thus we have $\omega(K_p) = \{1 + \log_2 p\}$.

In the general case, one way of finding an allocation is to assign one symbol to each member of a line-clique cover G and add sufficient number of distinguishing symbols. A line-clique cover $\{\theta_1(G)\}$ for a graph G is a set of minimal number of cliques that cover all the edges of G. The number of cliques in such a cover is an invariant for G though the cover itself need not be unique. It can be easily demonstrated that the above procedure is not necessarily minimal in all cases. Since no algorithm for

the general case is available, in the subsequent sections we describe the particular procedures for the specific cases.

Lower Bounds for ω

The simplest lower bound for ω is, perhaps, given by the following lemma 1. The line clique cover number $\theta_1(G)$ of a graph G is the minimum number of cliques covering all the lines (and points) of G.

Lemma 1: $\omega(G) \geqslant \theta_1(G)$.

Proof: Let $G = \Omega(S)$ where $|S| = \omega(G)$. Then for each $s \in S$, consider $V_s = \{v \in V(G) : s \in S_v\}$. Then $< V_s >$ is a complete graph and $E(G) \subseteq \bigcup_{s \in S} E(<V_s>)$. Hence $\omega(G) \geqslant \theta_1(G)$.

Lemma 2: If H is an induced subgraph of a graph G, then $\omega(G) \geqslant \omega(H)$.

Proof: If a minimal allocation for G is taken, the subsets allocated to the vertices of H provide an allocation for H, since H is an induced subgraph. Therefore $\omega(H) \leqslant$ number of elements in the union of these subsets $\leqslant \omega(G)$.

A powerful tool which we will use to prove the minimality of specific allocations in the sequel is provided by the following

Theorem 1: Let $Q_1, Q_2, \ldots, Q_r$ be some of the cliques in a θ_1-cover for G, and H be the subgraph of G obtained by removing $\bigcup_{i=1}^{r} V(Q_i)$ for G. If H has the property that,

no vertex of H is apexed to an edge of a clique (1)

Q_i, $i = 1, 2, \ldots, r$

then,

$$\omega(G) \geqslant r + \omega(H) \tag{2}$$

Proof: Consider a minimal allocation set S of ω symbols to the vertices of G. Let T_i be the set of all symbols which are circulating over at least one edge of Q_i if Q_i is not a single vertex. If Q_i is a single vertex, let T_i be the set allocated to it. We now claim that a system of distinct representatives exists for the family $T_1, T_2 \ldots, T_r$. For, otherwise, by theorem 5.19 of Harary (1969), for some k, $1 \leqslant k \leqslant r$, the union of k of these sets contains at most $k-1$ elements. Let without loss of generality, $T_1 \cup \ldots \cup T_k = \{x_1, x_2, \ldots, x_\lambda\}$ where $\lambda < k$. Next, if X_j is the subgraph of G induced by the set V_j of vertices sharing the sumbol $x_j \in S$, then clearly X_j is a complete subgraph. These complete subgraphs $X_1, X_2, \ldots, X_\lambda$ are contained in some clique $C_1, C_2, \ldots, C_\lambda$ which cover $\bigcup_{i=1}^{\gamma} E(Q_i)$. Since the θ_1-k cliques in the θ_1-cover other than $Q_i, \ldots, Q_k$ cover the edges of $E(G) - \bigcup E(Q_i)$, we have $\theta_1 - k + \lambda < \theta_1$ cliques covering all the edges of G, violating the minimality of θ_1. Thus a system of distinct representatives $S' = \{x_1, x_2, \ldots, x_r\}$ exists for $T_1, T_2, \ldots, T_r$.

Also, none of the symbols $x_i \in S'$ is assigned to a vertex of H. For if x_i is assigned to $v \in V(H)$ and $x_i \in e_i = (u_i, u_j) \in E(Q_i)$, then $< \{v, u_i, u_j\} >$ is a triangle in G, violating (1).

Thus if r' is the number of distinct symbols from S used in the allocation to the vertices of H, $\omega(G) \geqslant r + r'$ and $r' \geqslant \omega(H)$, so that (2) is proved.

Theorem 2: If $Q_1, Q_2, \ldots, Q_r$ is a set of cliques in G, each having a unicliqual edge and H is the subgraph obtained by removing $\bigcup_{i=1}^{r} V(Q_i)$ then $\omega(G) \geqslant r + \omega(H)$.

Proof: Let S be a minimal allocation set for G. Let $e_i \in E(Q_i)$, $i = 1, 2, \ldots, r$ be unicliqual edges and $x_i \in S$ be a circulating symbol over e_i. Then $x_i \notin S_v$ for any $v \in V(H)$. For, $x_i \in S_v$, $v \in V(H) \Rightarrow \{v\, e_i\}$ is a triangle in G. This triangle is contained in a clique which cannot be Q_i since v is not in Q_i. If it is contained in some other clique, e_i will not be a unicliqual edge. Hence $x_i \notin S_v$ for any $v \in V(H)$. Also $e_i \neq e_j$ implies $x_i \neq x_j$. Thus $\omega(G) \geqslant r +$ the number of symbols of S used in the allocation for H. But the latter number is clearly $\geqslant \omega(H)$. Hence $\omega(G) \geqslant r + \omega(H)$.

Corollary 1: Let $C_1, C_2 \ldots, C_t$ be some of the cliques in G with unicliqual lines (hence they belong to every θ_1-cover), and let $Q_1, Q_2, \ldots, Q_r$ be some of the cliques in a θ_1-cover of G (with $\{C_1, C_2, \ldots, C_t\} \cap \{Q_1, Q_2, \ldots, Q_r\} = \phi$). Let H be the subgraph of G obtained by removing $\bigcup_{i=1}^{r} V(Q_i) \cup \bigcup_{i=1}^{t} V(C_i)$ from G. Let $H_1 \subseteq H$ be any induced subgraph of G with the property that no vertex of H_1 is apexed to a line of a clique Q_i, $i = 1, 2, \ldots, r$. Then $\omega(G) \geqslant r + t + \omega(H_1)$.

Proof: As in the above theorems 1 and 2, there exist sets R and T such that $|R| \geqslant r$ and $|T| \geqslant t$, and no $s \in R \cup T$ is assigned to any vertex $v \in V(H_1)$, It is also clear that $R \cap T = \phi$. Hence $\omega(G) \geqslant r + t + \omega(H_1)$.

Determination of ω For a Class of Graphs

Our main result in this section is the following

Theorem 3: If no two cliques of G have more than one point in common, then

$$\omega(G) = \Sigma\, \omega(Kn_i) \tag{3}$$

where the summation is over all the cliques of G and n_i is the number of unicliqual vertices in the i-th clique. By convention

$$\omega(K_0) = \omega(K_1) = 1. \tag{4}$$

Proof: We first establish an upper bound by an allocation. First circulate one symbol over each clique of G. This accounts for all the connections and also helps to distinguish all the non-unicliqual vertices. The unicliqual vertices in each clique form a complete subgraph (possibly empty) and these vertices may now be distinguished by the allocation procedure for K_p (observing that the circulating symbol has already been provided). Thus

$$\omega(G) \leqslant \Sigma\, \omega(Kn_i) \tag{5}$$

We now establish the lower bound : Let $Q_1, Q_2, \ldots, Q_r$ be the cliques of G such that each Q_i consists of non-unicliqual vertices only. Let G_1 be the subgraph obtained by removing these cliques from G. Since the θ_1-cover for G consists of all the cliques of G and each clique has an unicliqual edge, applying the theorem 2 we get

$$\omega(G) \geqslant r + \omega(G_1) \tag{6}$$

Consider now the subgraph G_2 obtained by removing from G_1, all the remaining multicliqual vertices. Clearly G_2 is the disjoint union of θ_1—r cliques (some possibly trivial) Kn_i each comprising the unicliqual vertices of the corresponding cliques in G. Since G_2 is an induced subgraph of G_1, lemma 1 applies and we have

$$\omega(G_1) \geqslant \omega(G_2) = \Sigma\, \omega(Kn_i) \tag{7}$$

Thus $\omega(G) \geqslant r + \sum_{i=1}^{\theta_1 - r} \omega(Kn_i)$. Since, for the r cliques $Q_1, Q_2, \ldots, Q_r$, $n_i = 0$, using the convention (4), the above inequality becomes

$$\omega(G) \geqslant \Sigma\, \omega\,(Kn_i) \tag{8}$$

Combining (5) and (8) we get (3).

Remark 1: Among the class of graphs covered by this theorem are Husimi trees, block graphs, cut piont graphs and $(K_4 - x)$ — free graphs.

For example, for $(K_4 - x)$ — free graphs G we get the following specific result.

Corollary 2: $\omega(G) = q - 2T + t$, where T = number of triangles in G and t = number of terminal triangles in G.

Proof: If r is the number of edge-cliques in G, by the above theorem, $\omega(G) = r + (T - t) + \Sigma\omega(K_2)$ where the summation runs over all the terminal triangles of G. Hence $\omega(G) = r + T - t + 2t = q - 2T + t$.

$$\omega\,(L\,(G))$$

A θ_1-cover for $L(G)$: Let T be the set of all terminal triangles and S the set of all semi-terminal triangles of G and $|\,T\,| = t$, $|\,S\,| = s$. Let $V_2^t = \{v \in V_2 : v \in$ a terminal triangle of $G\}$

and $V_2^s = \{v \in V_2 : v \in$ a semi-terminal triangle of $G\}$.

Let $W = U_2 - V_2^t - V_2^s$

The following can be easily verified :

(*i*) $v \in W \Rightarrow$ Image of v in $L(G)$ is a clique $K_d(v)$.

(*ii*) $\triangle \in T \Rightarrow$ Image of $\triangle$ is a clique in $L(G)$ with two unicliqual edges.

(*iii*) $\triangle \in S \Rightarrow$ Image of $\triangle$ is a clique in $L(G)$ with one unicliqual edge.

(*iv*) The cliques in (*i*) are edge disjoint.

(*v*) The above set of cliques provides a θ_1 cover for $L(G)$.

Hence we have,

Lemma 2: $\theta_1\,(L(G)) = |\,W\,| + t + s$

$= |\,U_2\,| - t = p - p_1 - t$, where $p_1 = |\,V_1\,|$.

An upper bound for $\omega(L(G))$: By the allocation procedure we construct a set S such that $\Omega\,(S) = L(G)$ so that $\omega(L(G)) \leqslant |\,S\,|$.

Let l_i and t_i be the number of terminal edges and terminal triangles incident at v_i respectively.

We define : $\delta(t_i, t_i) = \delta(t_i)$ if $l_i \leqslant 1$

$= 0$ if $l_i > 1$ and

$\delta(t_i) = 1$ if $t_i > 0$

$= 0$ if $t_i = 0$.

Let $A_1 = \{v_i \in W : l_i \leqslant 1\}$,

$A_1' = \{v_i \in W : l_i \leqslant 1, t_i > 0\}$ and

$A_2 = \{v_i \in W : l_i \geqslant 1\}$.

We circulate one symbol over each of the cliques $K_{d(vi)}$ of $L(G)$ corresdonding to $v_i \in A_1$ in G and one symbol over each of the images of triangles $\triangle$, $\triangle \in T \cup S$. We allocate symbols to the complete subgraphs K_{l_i} of the cliques $Kd(v_i)$ $v_i \in A_2$ as in "Allocation Procedure" and use the circulating symbol of K_{l_i} for circulation over $K_{d(vi)}$ also. Now, for each $e_i \in V(L(G))$ define

$Se_i = \{ x : x \bigcirc$ over a clique containing $e_i\}$.

Clearly, $Se_i \neq \phi$ and $Se_i \cap Se_j \neq \phi$ if e_i A e_j in $L(G)$.

We have so far used $\sum_{v_i \in A_1 \cup A_2} \omega(K_{l_i}) + t + s$ symbols.

Since the cliques $K_{d(vi)}$, $v_i \in A_1 \cup A_2$ are all edge disjoint by (iv), in "A θ_1 cover for $L(G)$." We have $Se_i \neq Se_j$ unless the edges corresponding to e_i and e_j in G are either (i) terminal edges incident with a vertex or (ii) the two edges incident with the vertex v_{ij} of degree $\geqslant 2$, of a terminal triangle T_{ij}.

In case (i), the sets of symbols associated with the vertices e_i and e_j of $L(G)$ are distinct. In the latter case, $Se_i = Se_j = \{x, y\}$ where x and y are the symbols circulated, respectively over the cliques $K_{d(vij)}$ and $L(T_{ij})$. If $v_{ij} \in A_2$, then there exists a distinguishing symbol, say z of K_{l_i}. Hence we add z to the set of symbols associated with e_i. If any other $\triangle \in T$ is adjacent with v_{ij} then the same z can be used to distinguish the sets associated with e_i and e_j where e_i and $e_j \in \triangle \in T$.

If $v_{ij} \in A_1$ then we use a new symbol to distinguish the sets associated with e_i and e_j. Note that the same symbol can be used for all terminal triangles at v_{ij}. That is, in all we use

$$|S| = \sum_{v_i \in A_1 \cup A_2} \omega(Kl_i) + t + s + A'_1 \text{ symbols. That is,}$$

$$|S| = \sum_{v_i \in V_1' \cup V_2'} \omega(Kl_i) - t + |A'_1|.$$

By definition, $\delta(l_i, t_i) = 1$ iff $v_i \in A'_1$. Therefore

Lemma 3: $\omega(L(G)] \leqslant \sum_{v_i \in U_2} [\omega(K_{l_i}) + \delta(l_i, t_i)] - t.$

A lower bound for $\omega\{L(G)\}$*:* To obtain a lower bound we apply the results of "Lower Bounds for ω" by removing a suitable set of cliques in $L(G)$ in two stages. First we set up the following further notations :

$$W_1 = \{v_i \in W : l_i \leqslant 1, t_i > 0\},$$
$$W_2 = \{v_i \in W : l_i > 1, t_i > 0\},$$
$$W_3 = \{v_i \in W : l_i = 0 = t_i\},$$
$$W_4 = \{v_i \in W : l_i \geqslant 1, t_i = 0\}.$$

Stage 1: For all $v_i \in W_1$, remove from $L(G)$ the images of all but one of the t_i terminal triangles incident at v_i.

For all $v_i \in W_2$, remove from $L(G)$, the images of all the t_i terminal triangles incident at v_i.

Let G_1 be the induced subgraphs of $L(G)$ after removing the above n_1 cliques. Since each such clique has a unicliqual edge, we have by 2 theorem $\omega(L(G)) \geqslant \omega(G_1) + n_1$ where $n_1 = t - |W_1|$.

Stage 2: For all $v_i \in W_3$ remove from G_1 the images $K_{d(vi)}$ of v_i in $L(G)$. Simultaneously we remove the images of the triangles of S also. The number of cliques thus removed is $n_2 = |W_3| + s$. Let G_2 be the remaining induced subgraph. Let H be the induced subgraph of G_2 obtained by removing the vertices of $L(G)$ that are images of edges (which are neither terminal edges nor edges of a terminal triangle) of G not incident with any vertex of W_3. We claim that if $v \in V(H)$, then v is not apexed to any edge of a clique $K_{d(vi)}$, $v_i \in W_3$. To prove the claim we first observe that H is the disjoint union of triangles, $(K_4 - x)'s$ and K'_{i_i}s. The triangles are the images of terminal triangles incident at $v_i \in W_3$ with $l_i = 0$, which are not removed in stage 1. Each K_4—x the image of a terminal edge and a terminal triangle incident at some $v_i \in W_1$, with $l_i = 1$, which is not removed in stage 1. The K'_{i_i}s are the images of the vertices $v_i \in W_2 \cup W_4$. Now, let $v \in V(H)$. Then the corresponding edge e in G is either a terminal edge or an edge of a terminal triangle. If v is apexed to an edge of a clique of $K_{d(vi)}$, $v_i \in W_3$, then in G either (*i*) e is incident with the vertex v_i in which case $l_i > 0$ or $t_i > 0$ and so $v_i \notin W_3$, a contradiction *or* (ii) e is an edge of a K_4 or K_3 (not a terminal triangle) and hence a contradiction. Also, as noted in (*ii*) of "A θ_1 cover for $L(G)$", image of $\triangle \in S$ is a clique in $L(G)$ with a unicliqual edge. With these observations, we apply corollary to the graph G_1 and H and obtain.

$$\omega(G_1) \geqslant |W_3| + s + \omega(H).$$

But $\omega(H) = 3|W_1| + \sum_{v_i \in W_2 \cup W_4} f(v_i)$, where $f(v_i) = \omega(K_{1_i}) + \delta(1_1, t_1)$

Hence $\omega(L(G)) \geqslant n_1 + |W_3| + s + 3|W_1| + \sum_{v_i \in W_2 \cup W_4} f(v_i)$

$$= t + |W_3| + s + 2|W_1| + \sum_{v_i \in W_2 \cup W_4} f(v_i)$$

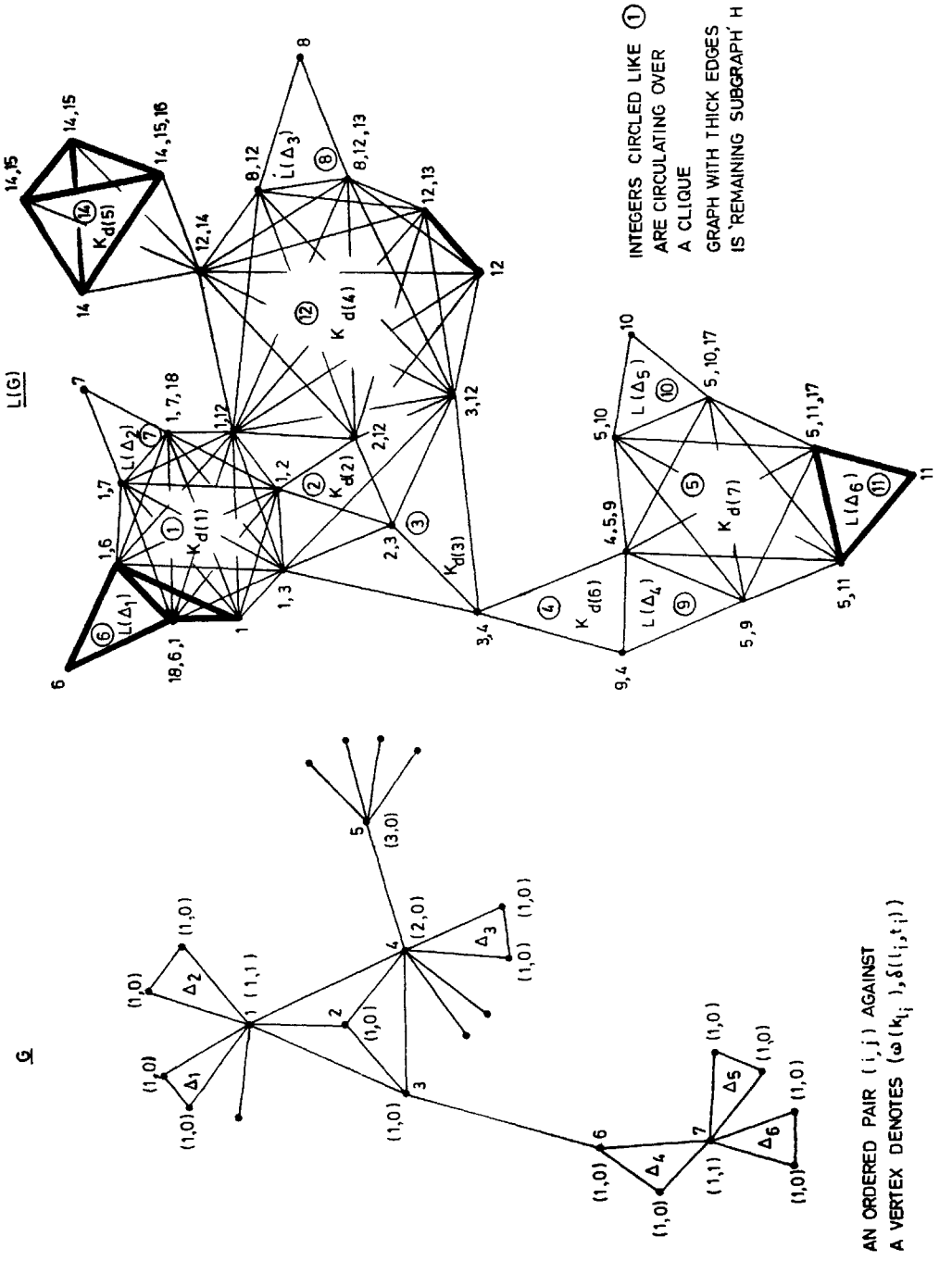
L(G)
G
INTEGERS CIRCLED LIKE ① ARE CIRCULATING OVER A CLIQUE
GRAPH WITH THICK EDGES IS 'REMAINING SUBGRAPH' H
AN ORDERED PAIR (i , j) AGAINST A VERTEX DENOTES (ω(k_ij), δ(i_j ,t_ij))

FIG. 1

$$= t + \sum_{V_i \in \bigcup_4 W_j \cup V_2^s} f(v_i)$$

$$= t + \sum_{v_i \in U_2} f(v_i) - 2t$$

Lemma 4: $\omega(L(G)) \geqslant \sum_{v_i \in U_2} [\omega(Kl_i) + \delta(i_{i'}, t_i)] - t.$

Combining Lemma (3) and Lemma (4) we get the following

Theorem 4: $\omega(L(G)) = \sum_{v_i \in U_2} \{\omega(K_{l_i}) + \delta(l_i, t_i)\} - t.$

In Fig. 1 we have illustrated the various motations and also the allocation procedure.

Corollary 3: If G is 2-connected. then $l_i = 0 = t_i$, for all $v_i \in V(G)$ and we have $\omega(L(G)) = | U_2 | = p$.

Corollary 4: If $d(v) \geqslant 3$, for all $v \in V(G)$, then $\omega(L(G)) = p$.

ω[T(G)]

Theorem 5: If G is triangle free ($p > 3$), *then* $\omega(T(G)) = q + | U_2 |$

Proof: (*i*) G is triangle free, and $e = (u, v) \in E(G)$ implies $< \{e, u, v\} >$ is a triangle-clique in $T(G)$ with (u, v) as its unicliqual edge. We denote this set of cliques by τ_1. $| \tau_1 | = q$.

(*ii*) If $v_i \in U_2$, then $< [v_i] \cup \{e_j \in E(G) : e_j$ is incident with v_i in $G\} >$ is a clique $C_{(dvi)}$ in $T(G)$ with (e_j, e_k) as a unicliqual edge where e_j and e_k belong to $E(G) \cap C_{d(vi)}$ Denote this set of cliques by τ_2. $| \tau_2 | U_2 |$.

Now it is easy to see that $\tau_1 \cup \tau_2$ is a cover of $E(T(G))$ and so a θ_1 cover of $T(G)$. Therefore by lemma 1,
$\omega(T(G)) \geqslant | \tau_1 \cup \tau_2 | = q + | U_2 |$. Now, by allocating one symbol over each of the cliques of $\tau_1 \cup \tau_2$, it is easy to prove that $(u, v) \in E(T(G))$ if and only if $S_u \cap S_v \neq \phi$, and $S_u \neq S_v$ for all $u, v, \in V(T(G))$. Therefore $\omega(T(G)) \leqslant q + | U_2 |$ and the theorem is proved.

Corollary 5: If G is triangle free and $d(v) \geqslant 2$ for all $v \in V(G)$, then $\omega[T(G)] = p + q$.

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Reference

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