DISTURBANCES IN $LiNbO_3$ RODS WITH DISSIPATIVE MECHANICAL PARAMETERS

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(Communicated by Prof. B. Sen, FNA)

(*Received 7 May 1974*)

This paper is an attempt to investigate disturbances along three directions of piezoelectric $LiNbO_3$ crystals characterised by dissipative mechanical parameters. The results have been compared with those determined by Kirchner (1968) for $LiNbO_3$ rods of non-dissipative mechanical parameters. These expressions have also been used to determine the group velocities of propagation.

INTRODUCTION

The recent years have witnessed a variety of problems of disturbances in piezoelectric media, *vide,* Roy (1968), Sinha, (1965) and Chakravarty (1970). In particular, problems on velocities of plane wave propagation in piezoelectric crystals have engaged the attention of many researchers, *vide,* Kyame (1949), and Kirchner (1968). But none of them has taken into account the dissipative nature of material parameters, save the attempts by Sinha (1967) and that of the author which is under print. This present paper is an attempt in this direction and it seeks to study the expressions for the velocities along the x, y and z-axes of piezoelectric $LiNbO_3$ crystals whose mechanical parameters are characterised by dissipative features that are somewhat akin to those of viscoelasticity of Voigt type. These results are then compared with results obtained by Kirchner (1968) for $LiNbO_3$ rods with non-dissipative mechanical parameters. The group velocities also were determined. The analysis presented here proceeds on the lines shown by Kirchner (1968).

PROBLEM AND BASIC EQUATIONS

Let us consider rods of $LiNbO_3$ crystals whose mechanical parameters are characterised by dissipative features of the form $(C + C'\frac{\partial}{\partial t})$, where C is the coefficient of elastic stiffness and C' is the same associated with viscous terms. The problem is to determine the velocities of wave propagation along the cartesian coordinate axes of the crystal rods.

Using the summation convention for repeated tensor, the mechanical equation of motion is

$$\frac{\partial T_{ij}}{\partial x_i} = \rho \ddot{u}_j, \quad \quad (1)$$

The Maxwell's equation for the electric displacement, when no free charges are present, is given by

$$\frac{\partial D_i}{\partial x_i} = 0, \qquad (2)$$

and the constitutive relations of the material are taken to be

$$T_{ij} = \left(C_{ijkl} + C'_{ijkl} \frac{\partial}{\partial t} \right) S_{kl} - e_{kij} E_k, \qquad (3)$$

$$D_i = e_{ikl} S_{kl} + \epsilon_{ik} E_k, \qquad (4)$$

where $E_k = -\dfrac{\partial \varphi}{\partial x_k}$, (5)

T_{ij} is the stress tensor,

x_i (i = 1, 2, 3) are the spatial coordinates defining a rectangular Cartesian co-ordinate system,

ρ is the mass density,

u_j are the components of the mechanical displacement,

D_i are the components of the electric displacement,

S_{kl} are the strain components,

e_{kij} are the piezoelectric coefficients,

E_k are the electric field components,

ϵ_{jk} are the dielectric constants,

φ is the electric potential,

and the dot notation is for the differentiation with respect to time.

Solution of the Problem

Following Kirchner (1968), using the above equations and seeking the plane wave solutions of the form

$$u_j = u_j^0 \, e^{i \frac{2\pi}{\lambda} \left(n_k x_k - vt \right)} \quad (j = 1, 2, 3) \qquad (6)$$

and $\varphi = \varphi^\circ \, e^{i \frac{2\pi}{\lambda} \left(n_k x_k - vt \right)}$

the equations for any piezoelectric crystal become

$$\overline{C}_{jk} u_k - \rho v^2 u_j = 0, \qquad (7)$$

where $\overline{C}_{jk} \equiv \overline{C}_{ijkl} n_i n_l$, (8)

$$\overline{C}_{ijkl} = C_{ijkl} - i \frac{2\pi}{\lambda} v C'_{ijkl} + \frac{n_p n_q e_{pkl} e_{qij}}{n_r n_s \epsilon_{rs}} \qquad (9)$$

λ is the wavelength,

n_k are the direction cosines of the normal to the wavefront,

v is the phase velocity of the wave, and

i stands for $\sqrt{-1}$.

Eliminating U_j's from equation (7), the equation, for the velocities for any piezoelectric crystal is given by

$$\begin{vmatrix} \bar{C}_{11} - \rho v^2 & \bar{C}_{12} & \bar{C}_{13} \\ \bar{C}_{21} & \bar{C}_{22} - \rho v^2 & \bar{C}_{23} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} - \rho v^2 \end{vmatrix} = 0 \quad \ldots \quad (10)$$

Equation (10), in general, will have six solutions for the velocity v of the plane elastic wave travelling in a piezoelectric medium. The polarization of the mechanical displacement can be determined for each of the velocities using equation (7). If the polarization is normal to the wave vector, the mode is a pure shear mode. If the polarization is in the direction of the wave vector, the mode is a pure longitudinal mode. In general, for an arbitrary direction of propagation, the elastic wave will be neither a pure shear nor pure longitudinal mode and then the waves are called quasi-shear or quasi-longitudinal, *vide,* Kirchner (1968).

Now let us consider the case of $LiNbO_3$ crystal; it is a piezoelectric crystal belonging to the symmetry point group 3m (Trigonal system, class 19). The coefficients of elastic stiffness, piezoelectric and dielectric constants are taken from Cady (1962). Three cases have been discussed for three directions of propagation.

X-axis propagation—Let us first consider the mechanical wave propagation along the x-axis. For this case $n_1 = 1$, $n_2 = n_3 = 0$. and hence (10) reduces to

$$\begin{vmatrix} C_{11} - i\frac{2\pi}{\lambda} C'_{11}v - \rho v^2 & 0 & 0 \\ 0 & \frac{1}{2}\left(C_{11}-C_{12}\right) - i\frac{\pi}{\lambda}\left(C'_{11}-C'_{12}\right)v + \frac{e_{22}^2}{\epsilon_{11}} - \rho v^2 & C_{14} - i\frac{2\pi}{2}C'_{14}v - \frac{e_{22}e_{15}}{\epsilon_{11}} \\ 0 & C_{14} - i\frac{2\pi}{\lambda}C'_{14}v - \frac{e_{22}e_{15}}{\epsilon_{11}} & C_{44} - i\frac{2\pi}{\lambda}C'_{44}v + \frac{e_{15}^2}{\epsilon_{11}} - \rho v^2 \end{vmatrix} = 0$$

which separates into

$$C_{11} - i\frac{2\pi}{\lambda} C'_{11} v - \rho v^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

and
$$\left\{\tfrac{1}{2}\left(C_{11} - C_{12}\right) - i\frac{\pi}{\lambda}\left(C'_{11} - C'_{12}\right)v + \frac{e_{22}^2}{e_{11}} - \rho v^2\right\}$$

$$\left\{C_{44} - i\frac{2\pi}{\lambda} C'_{44} v + \frac{e_{15}^2}{\epsilon_{11}} - \rho v^2\right\} - \left(C_{14} - i\frac{2\pi}{\lambda} C'_{14} v - \frac{e_{22}e_{15}}{\epsilon_{11}}\right)^2 = 0. \quad (12)$$

Equation (11) gives)

$$v = -i\frac{\pi}{\rho\lambda} C'_{11} + \left(\rho C_{11} - \frac{\pi^2}{\lambda^2} C'^2_{11}\right)^{\frac{1}{2}}/\rho \tag{13}$$

where $PC_{11} > \frac{\pi^2}{\lambda^2} C'^2_{11}$ may be taken for all practical purposes. These values, when substituted in (7), give $u_2 = u_3 = 0$. The mechanical waves with these velocities are obviously longitudinal, as in the case without dissipative characteristics (Kirchner 1968). The equation (6) implies a decaying effect to the displacement, the decaying factor being $e^{-2\pi^2 C'_{11}/\rho\lambda^2}$. Equation (12), being too complicated, let us solve this for a given wave length λ and for small velocities v, we have

$$\begin{aligned} v = \Bigg\{ &-i\frac{2\pi}{\lambda}\Bigg\{\tfrac{1}{2}\left(C'_{11} - C'_{12}\right)\left(C_{44} + \frac{e^2_{15}}{\epsilon_{11}}\right) + C'_{44}\left(\frac{C_{11}-C_{12}}{2} + \frac{e^2_{22}}{\epsilon_{11}}\right) \\ &-2C'_{14}\left(C_{14} - \frac{e_{22}e_{15}}{\epsilon_{11}}\right)\Bigg\} \pm \Bigg[\Bigg\{\rho\left(\frac{C_{11}-C_{12}}{2} + \frac{e^2_{22}}{\epsilon_{11}} + C_{44} + \frac{e^2_{15}}{\epsilon_{11}}\right) + \frac{4\pi}{\lambda^2} \\ &\left(\frac{C'_{11}-C'_{12}}{2} C'_{44} - C'^2_{14}\right)\Bigg\}\left\{\left(\frac{C_{11}-C_{12}}{2} + \frac{e^3_{22}}{\epsilon_{11}}\right)\left(C_{44} + \frac{e^2_{15}}{\epsilon_{11}}\right) - \left(C_{14} - \frac{e_{22}e_{15}}{\epsilon_{11}}\right)^2\right\} \\ &-\frac{\pi^2}{\lambda^2}\Bigg\{\tfrac{1}{2}\left(C'_{11} - C'_{12}\right)\left(C_{44} + \frac{e^2_{15}}{\epsilon_{11}}\right) + C'_{44}\left(\frac{C_{11}-C_{12}}{2} + \frac{e^2_{22}}{\epsilon_{11}}\right) - 2C'_{14} \\ &\left(C_{14} - \frac{e_{22}e_{15}}{\epsilon_{11}}\right)\Bigg\}^2\Bigg]\Bigg\}\Big/\Bigg\{\rho\left(\frac{C_{11}-C_{12}}{2} + \frac{e^2_{12}}{\epsilon_{11}} + C_{44} + \frac{e^2_{15}}{\epsilon_{11}}\right) + \frac{4\pi^2}{\lambda^2} \\ &\left(\frac{C'_{11}-C'_{12}}{2} C'_{44} - C'^2_{14}\right)\Bigg\} \end{aligned} \tag{14}$$

Substitution of these values of v in (7), subject to the above approximations, gives $u_1 = 0$; which means that the waves in this case are pure shear modes, same as that in the case without dissipative characteristics (Kirchner 1968); this also gives rise to a decaying effect to the displacement.

Y-axis propagation — Next, let us consider the mechanical wave propagation along the y-axis. For this case, $n_2 = 1, n_1 = n_3 = 0$, and hence (10) reduces to

$$\begin{vmatrix} \tfrac{1}{2}\left(C_{11} - C_{12}\right) - i\frac{\pi}{\lambda}\left(C'_{11} - C'_{12}\right)v - \rho v^2 & 0 & 0 \\ 0 & C_{11} - i\frac{2\pi}{\lambda} C'_{11}v + \frac{e^2_{22}}{\epsilon_{22}} - \rho v^2 & -C_{14} + i\frac{2\pi}{\lambda} C'_{14}v + \frac{e_{22}e_{15}}{\epsilon_{22}} \\ 0 & -C_{14} + i\frac{2\pi}{\lambda} C'_{14}v + \frac{e_{22}e_{15}}{\epsilon_{22}} & C_{44} - i\frac{2\pi}{\lambda} C'_{44}v + \frac{e^2_{15}}{\epsilon_{11}} - \rho v^2 \end{vmatrix} = 0$$

which separates into

$$\tfrac{1}{2}\left(C_{11}-C_{12}\right)-i\frac{\pi}{\lambda}\left(C'_{11}-C'_{12}\right)v-\rho v^{2}=0, \quad \ldots \quad \ldots \quad (15)$$

and

$$\left(C_{11}-i\frac{2\pi}{\lambda}C'_{11}v+\frac{e_{22}^{2}}{\epsilon_{22}}-\rho v^{2}\right)\left(C_{44}-i\frac{2\pi}{\lambda}C'_{44}v+\frac{e_{15}^{2}}{\epsilon_{22}}-\rho v^{2}\right)$$
$$-\left(-C_{14}+i\frac{2\pi}{\lambda}C'_{14}v+\frac{e_{22}e_{15}}{\epsilon_{22}}\right)^{2}=0 \quad \ldots \quad \ldots \quad (16)$$

Equation (15) gives

$$v=-i\frac{\pi}{\lambda\rho}\cdot\frac{C'_{11}-C'_{12}}{2}\pm\left\{\rho\,\frac{C_{11}-C_{12}}{2}-\frac{\pi^{2}}{\lambda^{2}}\left(\frac{C'_{11}-C'_{12}}{2}\right)^{2}\right\}^{\frac{1}{2}}\Big/\rho. \quad (17)$$

These values when substituted in (7), give $u_2=u_3=0$. Although (like the case without dissipative characteristics) the mechanical wave with these velocities are pure shear waves polarised along X — axis (Kirchner 1968), this means, from (6), a decaying effect to the displacement, the decaying factor being

$$e^{-\frac{\pi^{2}}{\rho\lambda^{2}}\left(C'_{11}-C_{12}\right)}.$$

As before, equation (16) being, too complicated, we solve this for given wave length λ, and for small values of velocities v (similar to equation 12), we have

$$v=\left\{-i\frac{\pi}{\lambda}\left\{C'_{11}\left(C_{44}+\frac{e_{15}^{2}}{\epsilon_{22}}\right)+C'_{44}\left(C_{11}+\frac{e_{22}^{2}}{\epsilon_{22}}\right)+2C'_{14}\left(-C_{14}+\frac{e_{22}e_{15}}{\epsilon_{22}}\right)\right\}\right.$$
$$\pm\left[\left\{\rho\left(C_{44}+C_{11}+\frac{e_{15}^{2}+e_{22}^{2}}{\epsilon_{22}}\right)+\frac{4\pi^{2}}{\lambda^{2}}\left(C'_{11}C'_{44}-C'^{2}_{14}\right)\right\}\left\{\left(C_{11}+\frac{e_{22}^{2}}{\epsilon_{22}}\right)\right.\right.$$
$$\left.\left(C_{44}+\frac{e_{15}^{2}}{\epsilon_{22}}\right)-\left(-C_{14}+\frac{e_{22}e_{15}}{\epsilon_{22}}\right)^{2}\right\}-\frac{\pi^{2}}{\lambda^{2}}\left\{C'_{11}\left(C_{44}+\frac{e_{15}^{2}}{\epsilon_{22}}\right)+C'_{44}\right.$$
$$\left.\left.\left(C_{11}+\frac{e_{22}^{2}}{\epsilon_{22}}\right)+2C'_{14}\left(-C_{14}+\frac{e_{22}e_{15}}{\epsilon_{22}}\right)\right\}^{2}\right]\Big/\left\{\rho\left(C_{44}+C_{11}+\frac{e_{15}^{2}+e_{22}^{2}}{\epsilon_{22}}\right)\right.$$
$$\left.+\frac{4\pi^{2}}{\lambda^{2}}\left(C'_{11}C'_{44}-C'^{2}_{14}\right)\right\}. \quad (18)$$

Substitution of these values of v in (7), subject to the above approximation, gives $u_1=0$ and so the waves are likewise polarized in the Y—Z plane, implying from (6), a decaying effect to the displacement.

Z-axis propagation—Finally, let us consider the mechanical wave propagation along the Z-axis. For this case $n_3=1$, $n_1=n_2=0$ and hence (10) reduces to

$$\begin{vmatrix} C_{44} - i\frac{2\pi}{\lambda}C'_{44}\nu - \rho\nu^2 & 0 & 0 \\ 0 & C_{44} - i\frac{2\pi}{\lambda}C'_{44}\nu - \rho\nu^2 & 0 \\ 0 & 0 & C_{33} - i\frac{2\pi}{\lambda}C'_{33}\nu + \frac{e_{33}^2}{\epsilon_{33}} - \rho\nu^2 \end{vmatrix} = 0$$

which separates into

$$C_{44} - i\frac{2\pi}{\lambda}C'_{44}\nu - \rho\nu^2 = 0 \quad \ldots \quad (19)$$

and

$$C_{33} - i\frac{2\pi}{\lambda}C'_{33}\nu + \frac{e_{33}^2}{\epsilon_{33}} - \rho\nu^2 = 0. \quad \ldots \quad (20)$$

Equation (19) gives

$$\nu = -i\frac{\pi}{\rho\lambda}C_{44} + \left(\rho C'_{44} - \frac{\pi^2}{\lambda^2}C'^2_{44}\right)^{1/2} / \rho. \quad \ldots \quad (21)$$

These values when substituted in (2) give $u_3 = 0$, which leads to the fact that mechanical wave polarisation takes place in X—Y plane and is a pure shear wave, as in the case without dissipative characteristics (Kirchner 1968) and there is a decay in the displacement, the decaying factor being $e^{-2\pi^2 C'_{44}/\rho\lambda^2}$ And equation (20) gives

$$\nu = -i\frac{\pi}{\rho\lambda}C'_{33} \pm \left\{\rho\left(C_{33} + \frac{e_{33}^2}{\epsilon_{33}}\right) - \frac{\pi^2}{\lambda^2}C'^2_{33}\right\}^{1/2} / \rho. \quad \ldots \quad (22)$$

The mechanical wave with these velocities are pure longitudinal, as in the case without dissipative characteristics (Kirchner 1968). Further, this brings out a decaying in the displacement, the decaying factor being $e^{-2\pi^2 C'_{33}/\rho\lambda^2}$.

It is to be noted that in the case of propagation in the direction of Z-axis, in calculating the values of velocities, no approximation was necessary to arrive at conclusive facts, as in the cases of X-and Y- axis.

Numerical Evaluation

The values of the coefficients of $LiNbO_3$ crystal as determined by Warner *et al.* (1967) are as follows :

ρ	4.7×10^3 kg/m^3	$C_{11} = C_{22}$	2.03×10^{11} N/m^2
$\epsilon_{11}/\epsilon_o = \epsilon_{22}/\epsilon_o$	44	C_{12}	0.53
ϵ_{33}/ϵ_o	29	C_{13}	0.75
e_{15}	3.7 C/m^3	C_{33}	2.45
e_{22}	2.5		
e_{33}	1.3	C_{44}	0.60

Taking wave length to be unity and C's also to be unity, for reasons of simplification in computation, the calculations are tabulated below and compared with those values as determined by Kirchner (1968) for non-dissipative characteristics of the crystal. Velocities are calculated from (13), (14), (17), (18), (21) and (22).

Direction of propagation	Type of wave	Velocities		Decaying factor
		Non-dissipative (Kirchner)	Dissipative	
X-axis	Pure longitudinal	6.57×10^3m/sec.	6.57×10^3m/sec.	$e^{-4.20321\times10^{-3}}$
	Pure Shear	4.80 4.07	2.63	$e^{-1.77\times10^{-3}}$
	Pure Shear	3.99	3.99	1
Y-axis	Quasi Longitudinal Quasi Shear	6.88 4.45	3.13	$e^{-3.897\times10^{-4}}$
Z-axis	Pure Shear	3.57	3.57	$e^{-4.20321\times10^{-3}}$
	Pure Longitudinal	7.32	7.32	$e^{-4.20321\times10^{-3}}$

It is to be noted that the variations in the velocities have occurred only in those cases where approximations have been used. Also it should be noted that in the case of dissipative characteristics of the material there appear decays in the displacement unlike the case of non-dissipative characteristics of the material. The decaying factors calculated here will vary surely for different materials owing to the variation of C's as well as due to the viscous terms associated with the elastic stiffness coefficient, and also due to λ, the wave length.

To estimate the propagation of energy, we next proceed to workout the group velocities in the cases of pure shear wave propagating along X-axis and of quasi-longitudinal quasi-shear wave propagating along Y-axis.

The expression for group velocity v_g is given by

$$v_g = v\left(1 - \frac{\lambda}{v}\frac{dv}{d\lambda}\right) \qquad \text{.. (23)}$$

Obtaining the expressions for $\frac{dv}{\lambda d}$ from equations (12) and (16) and putting in (23), taking C's to be unity the group velocities of pure shear wave propagating along x-axis

$$v_g = v^2 \frac{i\,2\pi\lambda\rho v - \rho a\lambda^2 + 4\pi^2 + 2\rho^2 v\lambda^3}{i\lambda\,(3\rho v^2 - b) - v\,(\rho a\lambda^2 - 4\pi^2)}. \qquad \text{.. (24)}$$

where

$$a = \tfrac{1}{2}\,(C_{11} - C_{12}) + C_{44} + \left(e_{22}^2 + e_{15}^2\right)/\,\epsilon_{11},$$

$$b = \tfrac{1}{2}\,(C_{11} - C_{12}) - 2C_{14} + e_{22}\,(e_{22} + 2e_{15})\,/\,\epsilon_{11}$$

and of quasilongitudinal, quasishear wave propagating along Y-axis

$$v_g = \rho v^2 \frac{\lambda (v^2 - a_1) + i\, 4\pi v}{\rho v \lambda (v^2 - a_1) + i\pi (6\rho v^2 - a_1 - b_1)} \tag{25}$$

where

$$a_1 = C_{11} + C_{44} + \left(e_{22}^2 + e_{15}^2 \right) / \epsilon_{22}.$$

$$b_1 = -2 (C_{14} - e_{22} e_{15} / \epsilon_{22}.$$

Using the values of the coefficients taken above, for unit wave length the group velocity of pure shear wave propagating along X-axis, we have

$$v_g = 2.63 \times 10^3 \text{ m/sec.}$$

which is the same as the phase velocity calculated above, and the group velocity of quasilongitudinal, quasi-shear wave propagating along Y-axis, we have

$$v_g = 3.13 \times 10^3 \text{ m/sec.}$$

which is also the same as the phase velocity calculated above.

It is to be mentioned that in calculating the group velocities in the above cases, no approximations have been done as in case of phase velocities. Hence in these cases the group velocities will be same as phase velocities, even when calculated without any approximation.

Acknowledgement

I express my sincerest gratitude to Professor D. K. Sinha of Jadavpur University for his active guidance in the preparation of this paper.

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