ON THE TORSIONAL WAVE PROPAGATION IN A TWO-LAYERED CIRCULAR CYLINDER WITH IMPERFECT BONDING

by R. C. BHATTACHARYA,* *Department of Electronics and Telecommunication Engineering, Bengal Engineering College, Howrah-3, W. Bengal.*

(*Received 4 July 1975; after revision 3 September 1975*)

Torsional wave propagation in a composite circular cylinder consisting of two elastic concentric cylinders joined together with a thin flexible bond has been studied from theoretical standpoint. A suitable simplified description for the behaviour of the bond whose thickness is small compared to the wavelength has been assumed. A few simple dispersion relations have been derived for some particular values of certain parameters from the complicated transcendental frequency equation obtained from the present analysis. A few propagation characteristics evaluated numerically in terms of a number of purely non-dimensionalised parameters have been produced graphically. The wave propagation characteristic for the present system have been found to be governed by bond properties apart from by other parameters which are usually involved in similar composite structures.

INTRODUCTION

The studies of wave propagation in various composite structures and in layered media which are bonded together have received some attention recently, chiefly because the need of a complete understanding of different bond characteristics with respect to mechanical shocks and vibrations is often felt in modern technology (Payton 1965). One of the major reasons which impose importance on this type of study lies with the requirement for a thorough knowledge about the measures against possible bond failures. Payton (1965) considered the case of propagation of stresses along a bond which joins two elastic rods along their generators. Jones and Whittier (1967) studied plane strain wave propagation along the interface of two semi-infinite media joined together with a thin flexible bond. Scott (1973) worked out a solution for the case of transient compressional wave propagation in a two-layered cylinder with an imperfect bonding.

In the present paper a solution for the torsional wave propagation in a composite cylinder consisting of two concentric cylinders joined together with a thin flexible bond has been worked out. The elastic properties of the material of the bond are taken into account following a somewhat simplified description of the behaviour of the bond introduced by Jones and Whittier (1967).

THEORY

A cylindrical polar co-ordinate system (r, θ, z) is chosen for the present study with the z-axis along the symmetry axis of the concentric cylinders, $r = a$ and $r = b$,

**Present address :* P. R. W., Planning and Development Division, Fertilizer Corporation of India, Sindri, Dhanbad, Bihar.

indicating the two contact surfaces of the bond material ($b > a$) with the inner solid cylinder and the outer cylindrical shell respectively and $r = d$ being the outside surface of the outer cylindrical shell. Let all the material constants of the outer cylindrical shell and of the inner solid cylinder be designated by the superscripts 1 and 2 respectively. The equations of motion for the system are (Love 1944):

$$\rho_i \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{rri}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta i}}{\partial \theta} + \frac{\partial T_{rzi}}{\partial z} + \frac{T_{rri} - T_{\theta\theta i}}{r}$$

$$\rho_i \frac{\partial^2 v_i}{\partial t^2} + \frac{\partial T_{r\theta i}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta i}}{\partial \theta} + \frac{\partial T_{\theta zi}}{\partial z} + \frac{2T_{r\theta i}}{r}$$

$$\rho_i \frac{\partial^2 w_i}{\partial t^2} = \frac{\partial T_{rzi}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta zi}}{\partial \theta} + \frac{\partial T_{zzi}}{\partial z} + \frac{T_{rzi}}{r}$$

where $i = 1$ or 2, and T_{rri}, $T_{\theta\theta i}$, T_{zzi}, $T_{r\theta i}$, $T_{\theta zi}$ and T_{rzi} are the corresponding stress components in their conventional sense, u_i, v_i, w_i are the displacement components in radial circumferential and axial directions respectively.

For the present problem we confine ourselves to the study of torsional were propagation. Following the usual procedure for the problems having θ symmetry it can easily be seen that the first and the third equations of motion given above are automatically satisfied as

$u_i = w_i = 0$ for $i = 1$ or 2

The remaining equation of motion using the usual stress-strain relations for each of the two media becomes

$$\frac{\partial^2 v_i}{\partial r^2} + \frac{1}{r} \frac{\partial v_i}{\partial r} - \frac{v_i}{r^2} + \frac{\partial^2 v_i}{\partial z^2} \quad \frac{1}{c_i^2} - \frac{\partial^2 v_i}{\partial t^2} = 0 \tag{1}$$

where $c_i^2 = \mu_i/p_i$, for $i = 1$ or 2,

μ_i and ρ_i are the shear elastic constant and the density of the material of the respective cylinder.

Assuming harmonic wave solution $e^{(i\omega t - iK_0 Z)}$, the complete solution for the circumferential displacement in the outer cylinder becomes

$$v_1 (r, z, t) = [AJ_1(K_1 r) + BY_1(K_1 r)]\, e^{(i\omega t - iK_0 Z)} \tag{2}$$

The solution for the inner solid cylinder is

$$v_2(r,z,t) = DJ_1(K_2 r)\, e^{(i\omega t - iK_0 Z)} \tag{3}$$

where J_1 and Y_1 denote Bessel functions of the first order and of the first and second kind respectively and

$$K_i = \left(\frac{\omega^2}{c_i^2} - K_0^2 \right)^{\frac{1}{2}}$$

A part of the general solution is Eq. (2) is left out from the solution for the solid cylinder for wellknown physical reasons (Redwood 1962).

The bonding model utilized here, after Jones and Whittier (1967), consists in an inertialess thin material in which the shear stress produced is proportional to the relative displacements of its end surfaces. Accordingly we obtain the two relations.

$$T_{r\theta b} = \frac{G}{b-a}\left[v(r_1 z_1 t) - v_2(r,z,t)\right], \text{ at } r = r_0 \tag{4}$$

$$(T_{r\theta})_1 = (T_{r\theta})_2 , \qquad \text{at } r = r_0 \tag{5}$$

where $r_0 = (a + b)/2$, $T_{r\theta b}$ is the stress component inside the bond, and G is the shear elastic constant for the bond material.

The above assumptions do not oversimplify the case under study. On the contrary the model approximates to the actual experimental conditions to a high degree of accuracy (Jones & Whittier 1967).

Combining (2), (3) and (5) one obtains

$$A\chi_1(r_0) + B\chi_2(r_0) - D\chi_3(r_0) = 0 \tag{6}$$

where $\chi_1(x) = \mu_1 K_1 J_0(K_1 x) - \frac{2\mu_1}{x} J_1(K_1 x)$

$$\chi_2(x) = \mu_1 K_1 Y_0(K_1 x) - \frac{2\mu_1}{x} Y_1(K_1 x)$$

$$\chi_3(x) = \mu_2 K_2 J_0(K_2 x) - \frac{2\mu_2}{x} J_1(K_2 x)$$

Similarly combining (2), (3) and (4) one obtains

$$A\left\{\chi_1(r_0) - \frac{2G}{b-a} J_1(K_1 r_0)\right\} + B\left\{\chi_2(r_0) - \frac{2G}{b-a} Y_1 K_1 r_0)\right\} + D\left\{\chi_3(r_0) + \frac{2G}{b-a} J_1(K_2 r_0)\right\} = 0 \tag{7}$$

The boundary conditions for the outer free surface of the composite cylinder is

$$[T_{r\theta 2}]_{r=d} = 0$$

This yields the equation.

$$A\chi_1(d) + B\chi_2(d) = 0 \tag{8}$$

From (6), (7) and (8) the frequency equation for the composite cylinder can be written in the determinantal form

$$\begin{vmatrix} x_1(r_0) & x_2(r_0) & x_3(r_0) \\ x_1(r_0) - 2G'\mu_1[J_1(K_1 r_0)]/r_0 & x_2(r_0) - 2G'\mu_1[Y_1(K_1 r_0)]/r_0 & x_3(r_0) + 2G'\mu_1[J_1(K_2 r_0)]/r_0 \\ x_1(d) & x_2(d) & 0 \end{vmatrix} = 0 \tag{9}$$

where $G' = Gr_0/\mu_1(b-a)$

Results and Discussions

The equation (9) above is the desired frequency equation for the composite system under study. It is difficult to extract numerical results from this complicated transcendental equation without resorting to further approximations.

Let us derive a few special results from the equation (9) for large values of K_1 and K_2. Making asymptotic expansions of the Bessel functions for large arguments and taking help of first order approximations, only one obtains the simplified frequency equation for large values of K_1 and K_2 as

$$K_1 tan\, K_1(r_0-d)\left[1 - \frac{2}{K_2 r_0} tan\,(K_2 r_0 - \pi/4) + \frac{G}{\mu_2 K_2(b-a)}\left(\frac{d}{r_0}\right)^{\frac{1}{2}} tan\,(K_2 r_0 - \pi/4)\right] = \frac{2}{d} - \frac{2}{r_0} - \frac{G}{\mu_1(b-a)}\left(\frac{d}{r_0}\right)^{\frac{1}{2}} \quad (10)$$

A few more simple and interesting results can be derived for some particular values of the various parameters from the above equation (10).

Case I : $G = 2\mu_1(b-a)(r_0-d)/d^{3/2}r_0^{1/2}$

For the above particular value of G, the equation (10) yields two sets of partially uncoupled modes of propagation expressed by the following two equations

$$tan\, K_1(r_0-d) = 0 \quad (11a)$$

$$tan\,(K_2 r_0 - \pi/4) = \frac{1}{\dfrac{2}{K_2 d} - \dfrac{G}{\mu_2(b-a)K_2}\left(\dfrac{d}{r_0}\right)^{\frac{1}{2}}} \quad (11b)$$

Thus one can conclude that for this particular case the sets of dispersion modes break up into two separate series of modes which resemble uncoupled torsional wave dispersion modes in the respective cylinders. The bond stiffness, however, affects the dispersion curves corresponding to the inner cylinder.

Case II : $G = \dfrac{2\mu_2(b-a)}{(r_0 d)^{1/2}}$

The equation (10) in the present case assumes a simple form

$$tan\, K_1(r_0-d) = \frac{1}{K_1}\left[\frac{2}{d} - \frac{2}{r_0} - \frac{G}{(b-a)\mu_1}\left(\frac{d}{r_0}\right)^{1/2}\right] \quad (12)$$

The dispersion curves corresponding to the equation (12) should have similarities with those of hollow cylinders for corresponding large values of K. One of the major differences, however, arises from the dependence of the former on the bond characteristics.

Case III : $\dfrac{d}{r_0} >> 1$

The equation (10) under above approximation becomes

$$tan\, K_1(r_0-d) tan\,(K_2 r_0 - \pi/4) = -\frac{K_2}{K_1}\frac{\mu_2}{\mu_1} \quad (13)$$

The propagation in this particular case becomes independent of the bond stiffness. All the above simple results are, however, valid for large values of K_1 and K_2, which in turn signifies that the above results will hold good for short wave-lengths and for large values of phase velocities compared to the distorsional wave velocities in the infinite media.

In order to perform some numerical calculations from the trancendental equation (9) for small values of K_1 and K_2 let us consider the case $\mu_1 = \mu_2 = \mu$ (say) and $\rho_1 = \rho_2 = \rho$ (say). Then one obtains $C_1 = C_2 = C$ (say) and $K_1 = K_2 = K$ (say). In addition we resort to approximations of expanding the Bessel functions in series forms for small values of their arguments (Whittaker & Watson 1962). Under this assumptions the frequency equation (9) concedes the curves shown in Fig. 1 which depicts a plot of Kr_0 against G' computed for three different values of r_0/d. Fig. 2 shows a plot of non-dimensionalised phase velocity for the torsional waves in the composite system against non-dimensionalised propagation constant along the symmetry axis of the system for $r_0/d = 0.10$ and for two separate values of G'. These curves correspond to the lowest order modes of the system. For information about higher order modes more terms are to be retained in the series-expansions stated above.

The above numerical results for the composite system both for large and small values K_1 and K_2 reveal clearly that the non-dispersive lowest order mode so familiar in hollow and solid cylinder cases gives way to a dispersive mode due to the presence of the bounding material. The apparent non-dispersive lowest order mode which seems to be predicted by equation (11a) is actually non-existent even in the partially uncoupled state of propagation corresponding to case 1 as the conditions of expansions utilized during the derivation of the above said equation do not conform to this solution. The dispersion curves of the composite system assume in general concave upward shapes resembling in form the corresponding dispersion curves for solid and hollow cylinders. But this agreement is only in shapes and the quantitative

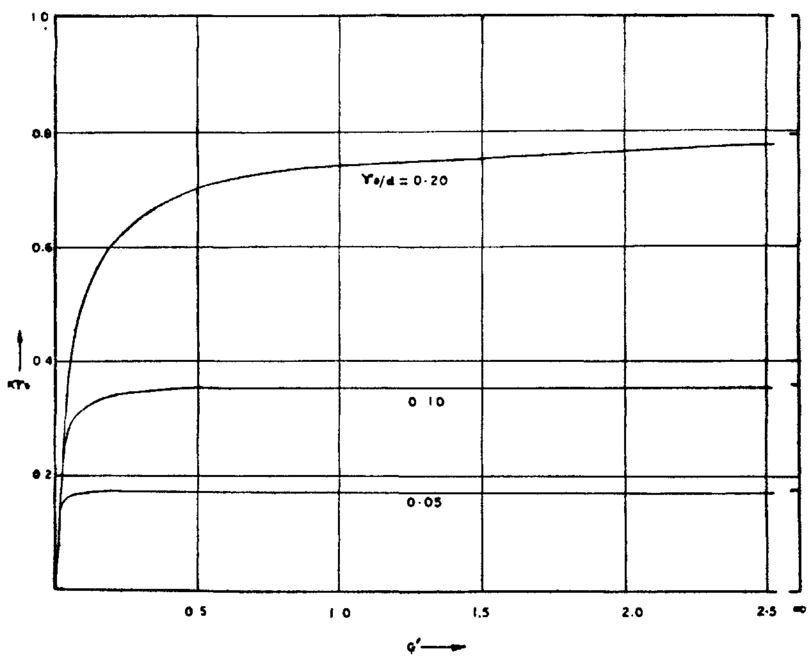

FIG. 1. Plot of Kr_0 against G'.

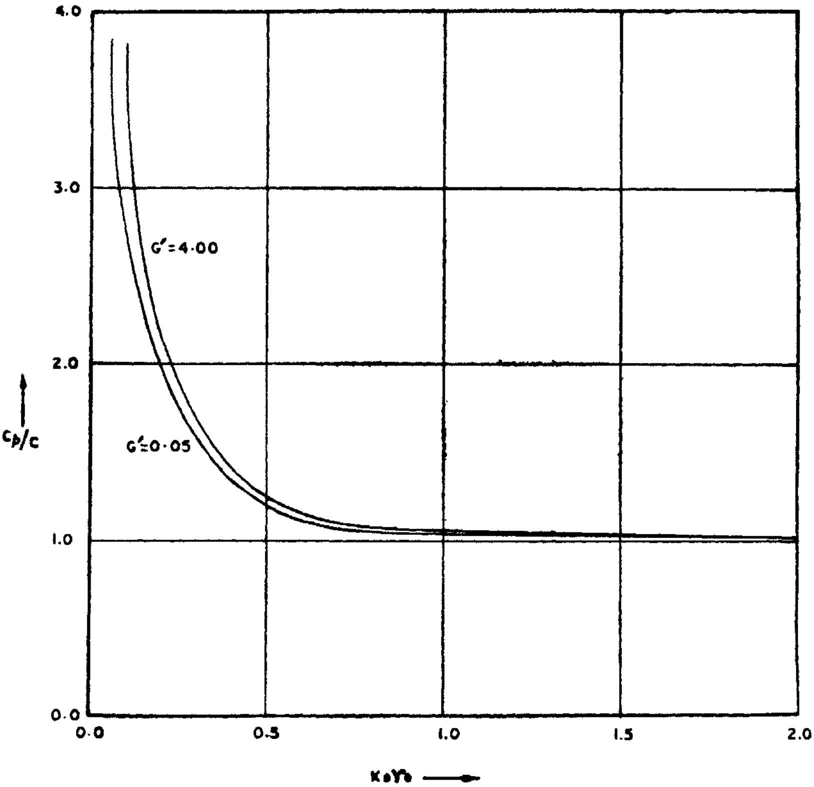

FIG. 2. Plot of non-dimensionalised phase velocity against non-dimensionalised propagation constant for $r_{0/d} = 0.10$.

variations are wide, apart from the dependence of the dispersion characteristics of the composite case on the bond stiffness.

The frequency equation (9) predicts cut-off frequencies for the different modes of propagation in the composite system. These cut-off frequencies are dependent on the bond stiffness apart from on other parameters which are involved in similar structures. The equation predicts cut-off frequencies for the lowest order mode of propagation also. Table I shows certain numerical values for non-dimensionalised cut-off frequencies for this lowest order mode for a number of values of bond stiffness and bond dimension.

In conclusion it can be said that the dispersion characteristics for torsional wave propagation are affected appreciably by the elastic properties of a bonding material and any sort of trial of interpretation of experimental findings is bound to produce totally misleading results in bonded layered problems if proper care is not taken to take into account the elastic properties of the bonding material. Further, since the phase velocity of the torsional waves in the composite system studied in the present paper is a function of the elastic parameters of the bond the apparent flexibility or softness of the bond depends upon the wavelength of the propagating waves. This aspect deserves full interest from the investigators in any study involving bond failures.

TABLE I

G'	r_0/d	$\omega r_0/c$
	0.05	0.163
0.05	0.10	0.279
	0.20	0.390
	0.05	0.168
0.10	0.10	0.311
	0.20	0.495
	0.05	0.170
0.20	0.10	0.333
	0.20	0.595
	0.05	0.173
0.50	0.10	0.348
	0.20	0.693

ACKNOWLEDGEMENTS

The author wishes to thank Prof. S. S. Baral for his help and encouragement during the work. The author also wishes to thank the referee for his important comments.

REFERENCES

Jones, J.P., and Whittier, J.S.(1967). Waves at a flexibly bonded interface. *J. appl. Mech.*, **34**, 905–909.

Love, A.E.H.(1944). *The Mathematical Theory of Elasticity*. 4th Ed., Cambridge University Press.

Payton, R.G.(1965). Dynamic bond stresses in a composite structure subjected to a sudden pressure rise. *J. appl. Mech.*, **32**, 643–650.

Redwood, M.(1962). *Mechanical Wave-guides*. Pergamon Press.

Scott, R.A. (1973). Transient compressional wave propagation in a two-layered circular rod with imperfect bonding. *J. Sound Vibration*, **26**, 321-335.

Whittaker, E. T., and Watson, G. N. (1962). *A Course of Modern Analysis*. 4th Ed., Cambridge University Press.