USE OF 2 × 2 MATRICES IN MAGNETOTELLURIC ANALYSIS FOR LAYERED EARTH

by SRI NIWAS, VIRENDRA K. PAINULY, *Department of Geology & Geophysics, University of Roorkee, and* J. B. SRIVASTAV, *Rajasthan Ground Water Board, Jodhpur, Rajasthan*

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This paper deals with the theory of magnetotelluric effect of a multilayered anisotropic earth. A generalized matrix relation is obtained for finding out the layer constant Ao, Bo; hence the Cagniard impedance. This relation can also be utilized for layered isotropic earth, embedded in inhomogeneous layer, embedded layer having inclined anisotropy and layered earth with each layer possessing different degree of inclined anisotropy.

INTRODUCTION

Following the classical work of Cagniard (1953) several investigators have considered the stratified homogeneous isotropic earth for further study in magnetotelluric analysis. But at all times these considerations do not closely represent the actual earth conditions. In the recent past, some realistic representative earth models have been studied assuming the anisotropy as well as the inhomogeneity in the layers (cf. Chetaev 1969; O'Brien & Morrison 1967; Praus & Petr 1969; Reddy & Rankin 1971; Sinha 1969; and Mullick 1970). O'Brien and Morrison (1967) and Praus and Petr (1969) studied the effect of anisotropy in horizontal planes, for a multilayered medium. Chetaev (1960) considered a half-space having dipping anisotropy. Sinha (1969) extended Chataev's work to a two-layered model, the bottom layer having dipping anisotropy. Reddy and Rankin (1971) took up a more general case of a multilayered earth where all the layers having different degrees of dipping anisotropies. Mullick (1970) studied the effect of a transition zone embedded between the two homogeneous layers.

In the present study we have developed a generalized 2 × 2 matrix relation to calculate the cagniard impedance in the case of multilayered anisotropic earth, each layer having dipping anisotropy. The developed relation herein can also be utilized for theoretical calculation of impedances in every possible configurations of layered earth model.

STATEMENT OF THE PROBLEM

Consider an oscillating plane electromagnetic wave of angular frequency ω is incident normally over a layered anisotropic earth model as given in Fig. 1. The origin of the co-ordinate system (X', Y', Z') coincides with the surface of the earth and the Z'-axis extends vertically downward. The longitudinal and transverse conductivities of layers is taken to be σ_{l_j} and σ_{t_j} respectively ($j = 0, 1, 2, \ldots, n+1$)

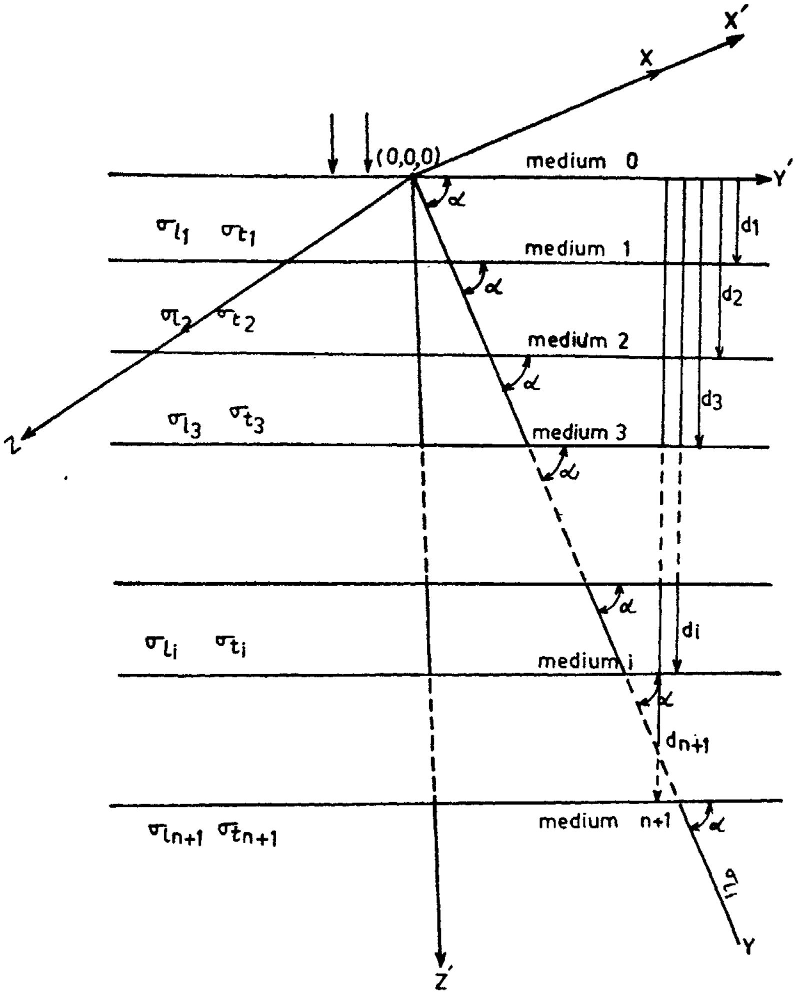

FIG. 1. Layered anisotropic earth model.

and the layer thicknesses are taken as d_k ($k = 1$ to n). $\sigma_{l_0} = \sigma_{t_0} = \sigma_0$ is the conductivity of the air. α is angle which axis of anisotropy makes with the horizontal axis in each layer. The co-ordinate system (X, Y, Z) in Fig. 1 is oriented along the axis of anisotropy.

Since the air is isotropic, the fields in medium 0 will therefore be $E_{y'}$ and $H_{x'}$ only. However we will have components E_y, E_z and H_x for mediums 1 to $n+1$ since the anisotropic effect will come into play. Medium 1 to $n+1$ will satisfy the Maxwell's equation (say for jth medium).

$$\frac{\partial^2 H_{x_j}}{\partial z^2} + k_j^2 \frac{\partial^2 H_{x_j}}{\partial y^2} - \nu_j^2 H_{x_j} = 0 \qquad \dots (1)$$

where

$k_j^2 = \sigma_{l_j} \sigma_{t_j}$ and $\nu_j^2 = i\omega\mu_j\sigma_{l_j}$; μ_j is the permeability of the medium and $i = \sqrt{-1}$.

The solution of Eq. (1) can be written as

$$H'_{x_j} = H_{x_j} = A_j \, exp\left[-q_j \left(z' - z'_{j-1}\right)\right] + B_j \, exp\left[q_j \left(z' - z'_{j-1}\right)\right] \quad \dots (2)$$

where $z' = z \cos \alpha + y \sin \alpha$, A_j and B_j are layer constants and are complex in nature, and q is an attenuating factor;
Re $q > 0$.

Putting the Eq. (2) into (1) we get

$$q_j = \frac{i\omega\mu_j\sigma_{l_j}}{1 + \sin^2\alpha \, (k^2_j - 1)} \quad \dots (3)$$

The Magnetic field, H_{x_j} is related with the electric fields E_{y_j} and E_{z_j} in the following way:

$$\frac{\partial^2 H_{x_j}}{\partial z} = \sigma_{l_j} E_{y_j} \quad \dots (4)$$

$$\frac{\partial^2 H_{x_j}}{\partial y} = -\sigma_{t_j} E_{z_j} \quad \dots (5)$$

From Eqs. (4), (5) and (2) we have

$$E_{y_j} = -\frac{q_j \cos\alpha}{\sigma_{l_j}}\left[A_j \, exp\left\{-q_j(z' - z'_{j-1})\right\} - B_j \, exp\left\{q_j(z' - z'_{j-1})\right\}\right] \quad \dots (6)$$

$$E_{z_j} = \frac{q_j \sin\alpha}{\sigma_{t_j}}\left[A_j \, exp\left\{-q_j(z' - z'_{j-1})\right\} - B_j \, exp\left\{q_j(z' - z'_{j-1})\right\}\right] \quad \dots (7)$$

To apply the boundary conditions successfully it is required to get the horizontal component of the electric field E_{y_j} in (x', y', z') system. Thus we have

$$E'_{y_j} = E_{y_j} \cos\alpha - E_{z_j} \, i \sin\alpha$$

or

$$E'_{y_j} = -\frac{q_j}{\sigma_{l_j}}\left[1 + \sin^2\alpha(k_j^2 - 1)\right]\left[A_j \, exp\left\{-q_j(z' - z'_{j-1})\right\} - B_j \, exp\left\{q_j(z' - z'_{j-1})\right\}\right] \quad \dots (8)$$

Since medium 0 ($j = 0$) is air, it will satisfy the Helmholtz equation

$$\nabla^2 E_0 = \nu_0^2 E_0 \quad \dots (9)$$

where

$$\nu_0^2 = i\omega\mu_0\sigma_0$$

We are considering only E_y' component for the medium 0, so Eq. (9) reduces to

$$\frac{\partial^2 E_{y_0}'}{\partial z'^2} = \nu_0 E_{y_0}' \qquad \ldots (10)$$

whose solution is

$$E_{y_0}' = A_0 \, exp\,(-\nu_0 z') + B_0 \, exp\,(\nu_0 z') \qquad \ldots (11)$$

The corresponding magnetic field for air may be obtained from Maxwell's equation as

$$i\omega\mu_0 \, H_{x_0}' = \frac{\partial E_{y_0}'}{\partial z^1} \qquad \ldots (12)$$

Thus

$$H_{x_0}' = \frac{1}{\eta_0}\Big[-A_0 \, exp\,(-\nu_0 z') + B_0 \, exp\,(\nu_0 z')\Big] \qquad \ldots (13)$$

where $\eta_0 = \sqrt{i\omega\mu_0/\sigma_0}$; characteristic impedance.

Thus the tangential electric and magnetic field components in all the mediums are expressed in suitable forms in Eqs. (2), (8), (11) and (13).

THEORY

The boundary conditions are: the continuity of the tangential electric and magnetic field components at each interface. Thus

$$\left.\begin{aligned} E_{y_j} &= E_{y_{j+1}} \\ H_{x_j} &= H_{x_{j+1}} \end{aligned}\right\} \text{ at } z = 0 \text{ and } z = d_k \ (k=1 \text{ to } 4) \qquad \ldots (14)$$

Also as the radiation condition specifies that at

$z' \to \infty$, $H_{x_{n+1}} \to 0$ gives

$$B_{n+1} = 0 \qquad \ldots (15)$$

Applying the conditions (14) and (15) to Eqs. (2), (8), (11) and (13) we have,

$\underline{j=0;\ z=0}$

$$A_0 + B_0 = -\frac{q_1}{\sigma_{l_1}}\Big[1+\sin^2\alpha(k_1^2-1)\Big]\Big[A_1 - B_1\Big]$$

$$\frac{1}{\eta_0}\Big[A_0 - B_0\Big] = -\Big[A_1 + B_1\Big] \qquad \ldots (16\,a)$$

$\underline{j=1;\ z=d_1}$

$$\frac{q_1}{\sigma_{j_1}}\Big[1+\sin^2\alpha(k_1^2-1)\Big]\Big[A_1 e^{-q_1 d_1} - B_1 e^{q_1 d_1}\Big] = \frac{q_2}{\sigma_{l_2}}\Big[1+\sin^2\alpha(k_2^2-1)\Big]\Big[A_2 e^{-q_2 d_1} - B_2 e^{q_2 d_1}\Big]$$

$$A_1 e^{-q_1 d_1} + B_1 e^{q_1 d_1} = A_2 e^{-q_2 d_1} + B_2 e^{q_2 d_1} \qquad \ldots (16\,b)$$

$\underline{j=2;\ z=d_2}$

$$\frac{q_2}{\sigma_{l_2}}\left[1+\sin_2^2\alpha(k_2^2-1)\right]\left[A_2\,e^{-q_2d_2}-B_2\,e^{q_2d_2}\right]=\frac{q_3}{\sigma_{l_3}}\left[1+\sin^2\alpha(k_3^2-1)\right]\left[A_3\,e^{-q_3d_2}-B_3\,e^{q_3d_2}\right]$$

$$A_2\,e^{-q_2d_2}+B_2\,e^{q_2d_2}=A_3\,e^{-q_3d_2}+B_3\,e^{q_3d_2} \quad \text{... (16 c)}$$

$j=n+1;\quad z=dn$

$$\vdots$$

$$\frac{q_n}{\sigma_{l_n}}\left[1+\sin^2\alpha(k_n^2-1)\right]\left[A_n\,e^{-q_nd_n}-B_n\,e^{q_nd_n}\right]=\frac{q_{n+1}}{\sigma_{l_{n+1}}}\left[1+\sin^2\alpha(k_{n+1}^2-1)\right]A_{n+1}\;e^{-q_{n+1}d_n}$$

$$A_n\,e^{-q_nd_n}+B_n\,e^{q_nd_n}=A_{n+1}\,e^{-q_{n+1}d_n} \quad \text{... (16 n)}$$

Each set of Eqs. (16a, 16b.... 16n) can be expressed into 2 × 2 matrix form as follows:

$$\begin{bmatrix} 1 & 1 \\ \eta_0^{-1} & -\eta_0^{-1} \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}=\begin{bmatrix} -\frac{q_1}{\sigma_{l_1}}\left[1+\sin^2\alpha(k_1^2-1)\right] & \frac{q_1}{\sigma_{l_1}}\left[1+\sin^2\alpha(k_1^2-1)\right] \\ -1 & -1 \end{bmatrix}\times\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \quad \text{... (17 a)}$$

$$\begin{bmatrix} \frac{q_n}{\sigma_{l_n}}\left[1+\sin^2\alpha(k_n^2-1)\right]e^{-q_nd_n} & -\frac{q_n}{\sigma_{l_n}}\left[1+\sin^2\alpha(k_n^2-1\right]e^{q_nd_n} \\ e^{-q_nd_n} & e^{q_nd_n} \end{bmatrix}\begin{bmatrix} A_n \\ B_n \end{bmatrix}$$

$$=\begin{bmatrix} \frac{q_{n+1}}{\sigma_{l_{n+1}}}\left[1+\sin^2\alpha(k_{n+1}^2-1)\right] & 0 \\ e^{-q_{n+1}d_n} & 0 \end{bmatrix}\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} \quad \text{... (17n)}$$

If we adopt to write Eq. (17*a*) as

$$\begin{bmatrix} P_{11} \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}=\begin{bmatrix} P_{12} \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \quad \text{... (18)}$$

(and similarly the other matrix relations can also be written).
Eq. (18) can be written as

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} P_{11} \end{bmatrix}^{-1} \begin{bmatrix} P_{12} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \qquad \dots (19)$$

In equation (19) the value of $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ can be substituted in terms of $\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$, giving

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} P_{11} \end{bmatrix}^{-1} \begin{bmatrix} P_{12} \end{bmatrix} \begin{bmatrix} P_{21} \end{bmatrix}^{-1} \begin{bmatrix} P_{22} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \qquad \dots (20)$$

Finally

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} P_{11} \end{bmatrix}^{-1} \begin{bmatrix} P_{12} \end{bmatrix} \begin{bmatrix} P_{21} \end{bmatrix}^{-1} \begin{bmatrix} P_{22} \end{bmatrix} \dots \begin{bmatrix} P_{n+21} \end{bmatrix}^{-1} \times \begin{bmatrix} P_{n+12} \end{bmatrix} \begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} \qquad \dots (21)$$

Denoting the product of matrices

$\begin{bmatrix} P_{11} \end{bmatrix}^{-1} \begin{bmatrix} P_{12} \end{bmatrix} \begin{bmatrix} P_{21} \end{bmatrix}^{-1} \dots \begin{bmatrix} P_{n+12} \end{bmatrix}$ as $\begin{bmatrix} \mathrm{M} \end{bmatrix}$ we get

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ M_{21} & 0 \end{bmatrix} \begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} \qquad \dots (22)$$

where M_{11} and M_{21} are the elements of matrix [M]. Thus

$$A_0 = M_{11} A_{n+1} \qquad \dots (23)$$

$$B_0 = M_{21} A_{n+1} \qquad \dots (24)$$

From Eqs. (23) and (24) we get

$$\frac{A_0}{B_0} = \frac{M_{11}}{M_{21}} \qquad \dots (25)$$

The cagniard impedance on the surface of the earth is

$$Z = \left. \frac{E_{y_0}}{i\omega\mu_0 H_{x_0}} \right|_{z=0} = \frac{A_0 + B_0}{\nu_0[B_0 - A_0]} \qquad \dots (26)$$

Putting the value of A_0 in terms of B_0 from Eq. (25) in (26) we have

$$Z = -\frac{1}{\nu_0} \frac{M_{11}+M_{21}}{M_{11}-M_{21}} \qquad \dots (27)$$

Eq. (27) is general equation of Cagniard impedance. By calculating the elements M_{11} and M_{21} one can get the analytical expression for Z for desired layered configuration.

Conclusion

The significant feature of this study is that it brings out analytical result for a very general type of geological situation where each layer is anisotropic with inclined anisotropy. From the general solution (Eq. 27) one can readily obtain the simpler results for isotropic layered system, embedded anisotropic and/or inhomogeneous layers and layered anisotropic earth where each layer having different inclined anisotropy compared to other layers. Utilizing the value of Z so obtained in any desired layer configuration using

$$\rho_a = - \frac{i}{\omega\mu} \left| Z^2 \right| \qquad \text{... (A)}$$

Utilizing the expression (A), master curves can conveniently be prepared for desired layer configuration. The general Eq. (27) can be utilized for inverse method of magnetotelluric analysis (Nabetani & Rankin 1969) also.

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