PROPERTIES AND NATURE OF TIME-DEPENDENT DEFORMATION OF ROCKS WHICH ARE NOT PURELY ELASTIC

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Time-dependent deformation history of rocks which are not purely elastic, is the combination of elastic and permanent deformation gradient at constant and varying temperatures. The stress applied to the rocks in both the cases are constants. An analogy between the mathematical sequence and the mechanical models has been brought about in this paper.

INTRODUCTION

In Nature the observed deformation in rocks is due to the tectonic movement or other orogenic processes. This deformation in the rocks is the finite strained state which is the result of various superimposed incremental strains. These superimposed incremental strains are due to various stress states.

The stress-strain relationships are dependent on many significant factors viz., (1) the confining pressure, (2) the rate of strain, (3) the temperature and (4) the nature of the chemical environment. But in geological processes time is an extremely important factor and orogenic deformations take many millions of years to complete.

When the stress-strain relationships are linear or nearly linear the deformation is in the elastic region. In the elastic region if the stress is restored the body will recover its original shape and position. But some of the rocks, say, porous sedimentary rocks, do not fully recover. The rocks in this case suddenly break within the elastic limit. This effect is the brittle failure and the value of stress at which this failure occurs is called the brittle strength. If the material is not brittle the rocks will flow and undergo permanent strain when the applied stress is removed. This occurs beyond the elastic limit. The point where this limit first exceeds is the yield point and the stress at which this occurs is called the yield stress. The deformation which occurs at or above the yield stress is called the plastic deformation. When a fluid substance is subjected to a stress system with the deviatoric components, the shearing stress leads to a continuous deformation by lamellar flow. The rate of this strain is proportional to the amount of shearing stress and the constant of proportionality is called the viscosity co-efficient. In Nature, purely elastic, plastic or viscous materials are rarely found.

Ramsey (1967, p. 261) discussed that when the stress is applied to the rock materials for a short duration it is found that almost all the rock materials are brittle at room temperature and atmospheric confining pressure. If the confining pressure is increased, the rocks become more brittle. At relatively low confining pressure, some of the sedimentary rocks can flow and over a considerable range of pressure and temperature these brittle rocks become ductile in nature. Coarse-grained igneous rocks, say, pyroxenite, peridotite, granite, are stronger than the sedimentary rocks and an increase in temperature markedly reduces the yield stress of these rocks. If a constant stress is applied to a rock the deformation occurs at a constant rate. This phenomenon will occur in rocks under a stress much lower than the instantaneous failure load and the rocks will show a permanent deformation, provided the stress is applied for a long period. In Nature, the naturally deformed rocks take millions of years to deform to a present state. So this time-dependent deformation known as "creep" sets up an important problem in the study of the nature of the deformation of naturally deformed rocks.

In the present paper we consider the rock materials of the type elastic-plastic, elastoviscous which are not purely elastic. We have studied the nature of the deformation of these types of rocks when subjected to a constant stress at both constant and varying temperature and have brought out an analogy between the mathematical sequence and mechanical models.

For this, we take the help of some fundamental ideas which are stated as below:

Nature of deformation in rock medium—We consider that a rock body has the property of a continuous medium. Let x^i and y^i be the co-ordinates of a rectangular system. Let P_x be the initial position of a point P of a rock-body with the co-ordinates x^i and P_y the position of the same point with the co-ordinates y^i at the time τ. P_x is transformed to P_y with the following relation.

$$y^i = \psi^i (x; \tau) \qquad \dots (1)$$

Let us assume that the relation (1) has a unique inverse at any time τ and the function ψ^i be continuous and differentiable functions of the initial co-ordinates x^i and the time τ. Since the correspondence $y \leftrightarrow x$ is one-one it follows that the inverse function $\xi^i(y; \tau)$ such that

$$x^i = \xi^i (y; \tau)$$

is continuous and differentiable function of y^i and the time τ, provided the functional determinants

$$\left| \frac{\partial y^i}{\partial x^\alpha} \right| \neq 0 \quad \text{and} \quad \left| \frac{\partial x^i}{\partial y^\alpha} \right| \neq 0$$

and these determinants must be positive if the correspondence $y \leftrightarrow x$ gives the identical transformation at the initial time.

Deformation gradient—Let at the initial time $\tau = 0$ the co-ordinates of the point P_x be denoted by $x = (x^1, x^2, x^3)$ and at all other time τ the point P_y by $y = (y^1, y^2, y^3)$. This correspondence from $x \leftrightarrow P \leftrightarrow y$ is one-to-one at each time τ. Let R_0 and R_τ be the configurations of a deforming body at the time $\tau = 0$ and τ respectively. It is to be noted that P is an arbitrary point of the deforming body.

In Eulerian co-ordinates, the co-ordinates of P_y at each time τ may be expressed in terms of the initial co-ordinates x^i and the time τ according to the relation (1) i.e.,

$$y^i = \psi^i (x^1, x^2, x^3; \tau), \; x \in R_0, \; 0 \leqslant \tau \leqslant t \qquad \dots (2)$$

Let us assume that the function $\psi(x;\tau)$ belongs to the class of all real continuous functions in R_0 and is differentiable upto 3rd order at each time τ, i.e., $\psi(x\ \tau) \in ;c^3(R_0)$ and is piecewise smooth in τ for each $x \in R_0$ so that

$$\left|\frac{\partial y^i}{\partial x^\alpha}\right| > 0, \text{ for } x \in R_0,\ 0 \leqslant \tau \leqslant t \qquad \dots (3)$$

Then the *deformation gradient* $F(x;\tau)$ of the point P at the time τ relative to the configuration R_0 is defined as

$$F(x;\tau) = \frac{\partial y^i}{\partial x^\alpha}, \text{ in } R_0 \times [0, t]\ i, \alpha = 1, 2, 3 \qquad \dots (4)$$

$F(x;\tau)$ is continuous in R_0 and is differentiable upto second order for each time τ in R_0 i.e., $F(x;\tau) \in c^2(R_0)$ and is piecewise smooth in $\tau \in [0, t]$ for each $x \in R_0$.

Conversely, if $F(x;\tau) \in c^2(R_0)$ and is continuous and piecewise smooth in $\tau \in [0, t]$ then the Darboux theorem states that the Cauchy problem

$$F^i_\alpha = \frac{\partial \psi^i}{\partial x^\alpha},\ \psi^i\left(x^1_0, x^2_0, x^3_0 ; \tau\right) = \psi^i_0\ (\tau), \qquad \dots (5)$$

for $0 \leqslant \tau \leqslant t$,

$i, \alpha = 1, 2, 3$ and $x_0{}^j, j = 1, 2, 3$ being a fixed interior point of R_0, has a unique solution $\psi(x:\tau)$ which belongs to $C^3(R_0)$ and is continuous and piecewise smooth in $\tau \in [0, t]$ for each $x \in R_0$, provided

$$\frac{\partial F^i_\alpha}{\partial x^\beta} = \frac{\partial F^i_\beta}{\partial x^\alpha},\ i, \alpha, \beta = 1, 2, 3 \qquad \dots (6)$$

satisfies everywhere in $R_0 \times [0, t]$.

Summarising, therefore, if we know the transformation of a point P of a deforming body within a certain time interval, we can find the deformation gradient of that point at the same time interval and vice-versa.

Temperature gradient—Let $\theta(x;\tau)$ be the temperature associated with the same point P in Lagrangian co-ordinate system. From the above conceptions 1 and 2 we can write in Eulerian co-ordinate system

$$x = \psi^{-1}(y;\tau),\ 0 \leqslant \tau \leqslant t \qquad \dots (7)$$

Hence $\theta(x;\tau)$ can be expressed as $\theta(\psi^{-1}(y;\tau);\tau)$.

Let the temperature θ of the point P be varying, then the spatial temperature gradient $q(x;\tau)$ of the temperature $\theta(x;\tau)$ can be expressed as

$$q(x;\tau) = \nabla_y\, \theta(\psi^{-1}(y;\tau);\tau).$$

Let F^T be the transpose of the deformation gradient $F(x;\tau)$, then

$$F^T q(x;\tau) = \nabla_x\, \theta(x;\tau) \qquad \dots (8)$$

From the relation (8) it is evident that if the deformation gradient and the temperature gradient of the point P are known we can calculate the temperature $\theta;(x\ \tau)$ of the point P.

Since the point P is an arbitrary point in the deforming body, $F(x;\tau)$ and $\theta(x;\tau)$ are true for all points P of the deforming body hence they are true for the whole deforming body.

Deformation history

(*a*) *Varying Temperature*:

(*i*) If

$$\Gamma(x;\tau)\equiv(F(x;\tau);\theta(x;\tau)), \quad \text{for } x\in R_0,\ 0\leqslant\tau\leqslant t. \qquad \dots (9)$$

belongs to $C^2(R_0)$ and is continuous and piecewise smooth in τ and satisfies the conditions (3) and (6) everywhere in the cylinder $R_0\ x\ [0, t]$, then $\Gamma(x;\tau)$ is defined as the *deformation history of the deforming body* relative to the configuration R_0 from the time $\tau = 0$ to $\tau = t$.

(*ii*) Let x_0 be a fixed interior point of R_0 then the *deformation history of the fixed point* $(x_0, 0)$ from the time $\tau = 0$ to $\tau = t$ is

$$\Gamma(x_0;\tau)\equiv(F(x;\tau);\ \theta(x_0;\tau)) \quad \text{for } x_0\in R_0,\ 0\leqslant\tau\leqslant t. \qquad \dots (10)$$

(*b*) *Constant temperature:*

(i) If the temperature θ is constant at $\theta=\theta_0$, *the deformation history of the deforming body* relative to the configuration R_0 from the time $\tau = 0$ to $\tau = t$ is

$$\Gamma(x;\tau)\equiv(F(x_0;\tau);\ \theta_0(x;\tau)) \quad \text{for } x\in R_0,\ 0\leqslant\tau\leqslant t. \qquad \dots (11)$$

(*ii*) *Deformation history of the point* $(x_0, 0)$ from the time $\tau = 0$ to $\tau = t$ is

$$\Gamma(x_0;\tau)\equiv(F(x_0;\tau);\ \theta_0(x_0;\tau)) \quad \text{for } x_0\in R_0,\ 0\leqslant\tau\leqslant t. \qquad \dots (12)$$

Mathematical sequence of the nature of the time-dependent deformation of rocks which are not purely elastic, while subjected to a constant stress and temperature

We are considering here the rock materials of the type elastic-plastic and elasto-viscous. These types of rocks are not purely elastic.

Let D be any arbitrary part of a rock of the above type in a three-dimensional Euclidean space E_3. Let this rock consist of a number of small volumes or points P. Let us assume that R_τ and R_0 be the regions occupied by D in E_3 at the time τ and $\tau = 0$ respectively. Let the co-ordinates of the point P at the time $\tau = 0$ be denoted by $x = (x^1, x^2, x^3)$ and let all other times by $y = (y^1, y^2, y^3)$, x and y being assumed to be two co-ordinate systems in E_3.

Whenever a rock deforms the coordinates of any point in it transfer to a new position according to the relation (1). If now the deforming rock D be subjected to a certain stress at certain temperature the displacement of the rock D may be described by the Eq. (2) at each instant τ, $0\leqslant\tau\leqslant t$. If some time τ, $0\leqslant\tau\leqslant t$, elapses before the removal of the applied stress to the rock material D of the type mentioned above, while the temperature field in D is maintained, the elastic recovery of the deformed state R_τ is not complete but there remains a permanently deformed state $P(R_\tau)$ specified by the Eq.

$$z^I = z^I(x;\tau) \quad \text{for } I = 1, 2, 3;\ x \in R_0,\ 0 \leqslant \tau \leqslant t, \qquad \ldots (13)$$

relative to the undeformed state R_0.

Clearly, $(z;\tau)$, $(y;\tau)$ and $(x;0)$ identify the same material point P in D though the co-ordinate system z in (13) may be taken as arbitrary.

If now the same stress is immediately reapplied to the deforming rock D the permanently deformed state $P(R_\tau)$ will return to R_τ provided the temperature in D maintained constant. This deformation from $p(R_\tau)$ to R_τ is specified by

$$y^i = y^i(z;\tau),\quad i = 1, 2, 3 \quad \text{for } z \in P(R_\tau),\ 0 \leqslant \tau \leqslant t. \qquad \ldots (14)$$

This deformation is purely elastic in the sense of complete recovery. Now combining (2), (13) and (14) we have

$$\psi^i(x;\tau) = y^i = y^i(z;\tau) = y^i[z(x;\tau);\tau]. \quad 0 \leqslant \tau \leqslant t,\ x \in R_0 \qquad \ldots (15)$$

It is to be noted that the times required to deform from R_τ to $P(R_\tau)$ and from $P(R_\tau)$ to R_τ have not been taken into consideration.

$z^I(x;\tau)$ and $y^i(z;\tau)$ are restricted to be in class $C^3(R_0)$ for each fixed τ and are continuous and piecewise smooth in τ. Since the transformations are one-to-one and result the incremental strains,

$$\left|\frac{\partial y^i}{\partial z^I}\right| > 0 \quad \text{and} \quad \left|\frac{\partial z^I}{\partial x^\alpha}\right| > 0 \quad \text{for } 0 \leqslant \tau \leqslant t \qquad \ldots (16)$$

Now $F^i_\alpha(x;\tau)$ of (4) can be decomposed as

$$F^i_\alpha(x;\tau) = \frac{\partial y^i}{\partial x^\alpha} = \left(\frac{\partial y^i}{\partial z^I}\right)\left(\frac{\partial z^I}{\partial x^\alpha}\right) \quad i, \alpha, I = 1, 2, 3;\ 0 \leqslant \tau \leqslant t. \qquad \ldots (17)$$

Since $\left(\frac{\partial y^i}{\partial z^I}\right)$ and $\left(\frac{\partial z^I}{\partial x^\alpha}\right)$, $i, \alpha, I = 1, 2, 3$ belong to the class $C^2(R_0)$ and are continuous and piecewise smooth in τ, $F^i_\alpha(x;\tau)$ also belongs to $C^2(R_0)$ and is continuous and piecewise smooth in τ. This also satisfies the integrability conditions (6) and the impenetrability condition (3) everywhere in the cylinder $R_0 \times [0, t]$. Then, from the definition, $F^i_\alpha(x:\tau)$ is the deformation gradient throughout the deforming rock D relative to the undeformed state R_0 for all time τ.

The decomposition in (17) is uniquely determined for the specified co-ordinate system z.

Since the deformation specified in (14) is purely elastic and the deformation in (13) concerns the permanently deformed state, $\left(\frac{\partial y^i}{\partial z^I}\right)$ is the elastic deformation gradient and $\left(\frac{\partial z^I}{\partial x^\alpha}\right)$ is the permanent deformation gradient. Let them symbolise as

$$\left.\begin{aligned} \left(\frac{\partial y^i}{\partial z^I}\right) &= e(d) \\ \left(\frac{\partial z^I}{\partial x^\alpha}\right) &= p(d) \end{aligned}\right\} \quad \ldots (18)$$

Hence using (17) and (18), according to (11), the deformation history of the deforming rock D relative to the undeformed state R_0 from the time $\tau=0$ to $\tau = t$ is

$$\Gamma(x;\tau)\equiv[e(d)p(d);\ \theta_0(x;\tau)] \quad \ldots (19)$$
$$\text{for } 0 \leqslant \tau \leqslant t$$

where $\theta_0(x;\tau)$ is the constant temperature.

This deformation history of the deforming rock D is uniquely determined if the temperature and the elastic and permanent deformation gradient of the rock D is known.

Hence the conclusion is that for every deformation history $\Gamma(x;\tau)$ of a given rock material which is not purely elastic, there is a unique decomposition with $e(d)$ and $p(d)$ subjected to the condition that the rock material experiences constant temperature and pressure for a long time.

It is proved, therefore, that the total time dependent deformation of the above mentioned rocks is the combination of an elastic and permanent deformation.

Analogy

The above resembles the mechanical models of time-dependent deformation relation or the "creep" phenomenon of both the elastic-plastic and the elastoviscous rocks. This relation shows that at constant temperature and under constant stress σ applied to the rock, the total strain e_T of the rock is:

(*a*) *Rock-materials being elastic-plastic*:
(Jaeger 1962, p. 105)

$$e_T = \frac{\sigma}{E} + \frac{(\sigma-\sigma_y)}{\eta}\,t, \quad \sigma > \sigma_y \quad \ldots (20)$$

where σ_y = yield stress

$\frac{\sigma}{E}$ = elastic deformation

$\frac{(\sigma-\sigma_y)t}{\eta}$ = permanent deformation

E = Young's modulus

and η = plastic constant of proportionality.

(*b*) *Rock-materials being elastoviscous:*
(Jaeger 1962, p. 102)

$$e_T = \frac{\sigma}{E} + \frac{\sigma}{\mu} t \qquad \dots (21)$$

where μ = viscous modulus

and $\frac{\sigma t}{\mu}$ = permanent deformation.

Thus when the constant stress σ is applied at the time t_0 the rocks show an instantaneous elastic strain $\frac{\sigma}{E}$ for both the cases (1) and (2) and then the elastic-plastic rock flows at a constant rate $\frac{de}{dt} = \frac{\sigma - \sigma_y}{\eta}$ and the elastoviscous rocks at the rate $\frac{de}{dt} = \frac{\sigma}{\mu}$. On removal of the stress at any time t, the elastic strain $\frac{\sigma}{E}$ is completely recovered for both the cases (1) and (2) but a permanent strain of $\frac{\sigma - \sigma_y}{\eta}$ for the case of elastic-plastic rock material and $\frac{\sigma t}{\mu}$ for the case of elastoviscous rock material remains.

The time-strain curve (Price 1966, Fig. 17, p. 39) of elastic-plastic rocks also describes the same type of result. The behaviour of nodular limestones and a number of other specimens investigated by Price (1966, Fig. 22, p. 47) show similar strain-time relationships and he suggested that all these rocks contain stresses stored from some previous strain history of orogenic antiquity which have not been completely relaxed.

Rocks Subjected to a Constant Stress but Varying Temperature

So far we have discussed the time-dependent deformation at constant temperature and pressure applied for a long time. Now we consider the rock-materials of the type, as discussed already, are subjected to a constant stress for a long period but the temperature of the rock materials is not maintained constant. Then using (17) and (18) the deformation history $\Gamma(x; \tau)$ of the rock according to the definition (9) is

$$\Gamma(x; \tau) \equiv [e(d)\ p(d);\ \theta(x; \tau)] \qquad \dots (22)$$
$$\text{for } x \in R_0,\ 0 \leqslant \tau \leqslant t.$$

In this case also the deformation history of the deforming rock D can uniquely be determined if the elastic and permanent deformation gradient as well as the temperature gradient is known.

This temperature plays an important role in rock-deformation. An increase in temperature in some cases lowers the yield point. Peridotite, pyroxenite, granite, marble, which are brittle at room temperature become ductile and flow with the increasing temperature (Ramsey 1967, Figs. 6–4, 6–5, pp. 259–260) and show permanent deformation.

Ramsey (1967) discussed that one of the most remarkable features of creep curves is that they are broadly similar for a great range of materials of quite different make-up i.e., rocks, asphalts, metals, polymers, rubber etc. The precise mechanism of deformation must vary with each of these different materials but the bulk behaviour takes a much more restricted form.

Conclusion

Both at the constant and varying temperature the time-dependent deformation history of elastic-plastic and elastoviscous rocks is the combination of elastic and permanent deformation gradient, provided the long-term stress applied to the rocks is constant.

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