EINSTEIN-MAXWELL FIELD EQUATIONS-NULL FIELD

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In this paper, the propagation vector and the polarisation vector have been studied when the electromagnetic tensor field is a null field. Non-index notation has been used.

INTRODUCTION

Let us consider a four dimensional space time V_4 with the gravitational symmetric tensor field g, which is assumed to be of signature $+ + + -$. Then we can define

$$(G(X))(Y) = (G(Y))(X) \overset{\text{def}}{=} g(X, Y) \quad .. \quad (1a)$$

$$Go\,{}^{-1}G = {}^{-1}GoG = I_4 \quad .. \quad (1b)$$

$$^{-1}g(\alpha, \beta) \overset{\text{def}}{=} \alpha\ ({}^{-1}G(\beta)) = \beta\ ({}^{-1}G(\alpha)). \quad .. \quad (1c)$$

Agreement 1—In this and in what follows, the equations containing $X, Y, Z, \ldots$ and $\alpha, \beta, \gamma, \ldots$ hold for arbitrary vector fields $X, Y, Z, \ldots$ and arbitrary l-forms $\alpha, \beta, \gamma, \ldots$.

Let K be the curvature tensor of g and 'Ric' tensor. Let k be the electromagnetic tensor field of type (0, 2). Then it is skew-symmetric. Let us write

$$g(k(X), Y) \overset{\text{def}}{=} {}'k(X, Y) \quad .. \quad (2)$$

$$4K = -C_1^1\, k(k), \quad .. \quad (3)$$

where C_1^1 is the operation of contraction with respect to first contravariant and first convariant slot. The stress energy tensor F of type (1, 1) associated with k is given by

$$F(X) = \overline{\overline{X}} + K X \quad .. \quad (4a)$$

where

$$\overline{X} \overset{\text{def}}{=} k(X). \quad .. \quad (4b)$$

We assume the basic system of field equations in the form

$$R'(X) - \tfrac{1}{2} r'\, X = \mu u\,(X)\, U - F(X) + \zeta\, X, \quad .. \quad (5)$$

where

$$g(R'(X), Y) \overset{\text{def}}{=} Ric(X, Y) \quad .. \quad (6a)$$

$$r' \overset{\text{def}}{=} C_1^1\, R' \quad .. \quad (6b)$$

μ is the density and ζ the pressure of the medium. U is the time-like velocity vector of the particle moving in the medium, such that

$$u(X) \stackrel{\text{def}}{=} g(U, X) \qquad .. \quad (7a)$$

$$u(U) = -1. \qquad .. \quad (7b)$$

The electromagnetic tensor field k is said to be of the (a) first class if $K|k| \neq 0$, where $|k| \stackrel{\text{def}}{=} \det(k)$), (b) second class if $|k| = 0$, $K \neq 0$ and (c) third class if $|k| = 0 = K$, $k^2 \neq 0$.

The electromagnetic tensor field of the third class is said to be a null field.

The equation of continuity is

$$\operatorname{div}(\mu U) = 0. \qquad .. \quad (8)$$

Maxwell equations in the idealized form are given by

$$(\operatorname{div} k)(X) = -\epsilon\, u(X) \qquad .. \quad (9)$$

where ϵ is the energy, and

$$(D'_X k)(Y, Z) + (D_Y{}'k)(Z, X) + (D_Z{}'k)(X, Y) = 0, \qquad .. \quad (10)$$

D being Riemannian connexion. Eq. (10) is a statement of the fact that $'k$ is closed.

In order that the basic system of Eqs. (5), (9) and (10) should be compatible with the continuity Eq. (8), it is necessary and sufficient that the pressure is constant along U (Mishra 1962).

It has been shown by Hlavaty (1961) using Line Geometry and by Mishra (1963) by another method, that when k is a null field, we can always find a set of linearly independent null vectors $[P, Q, R, S]$, P, Q being real and R, S being complex conjugate and the set of l-forms $\{p, q, r, s\}$ dual to $\{P, Q, R, S\}$ such that

$$g(X, Y) = -p(X)\,q(Y) - q(X)\,p(Y) + r(X)\,s(Y) + s(X)r(Y) \qquad .. \quad (11)$$

and $\bar{X} = p(X)M + m(X)\,Q$, $\qquad .. \quad (12)$

where M, called the polarisation vector is given by

$$\sqrt{2}\; M \stackrel{\text{def}}{=} R + S \qquad .. \quad (13a)$$

and

$$\sqrt{2}\; m(X) \stackrel{\text{def}}{=} r(X) + s(X). \qquad .. \quad (13b)$$

The vector Q is called propagation vector.

From Eq. (12) we get

$$(a)\ \bar{\bar{X}} = p(X)Q \text{ and } (b)\ \bar{\bar{\bar{X}}} = O. \qquad .. \quad (14)$$

The last is the necessary and sufficient condition for the null electromagnetic field (Mishra 1958).

A vector field T is said to be a Killing vector field if

$$(D_X\ t)\ (Y) + (D_Y\ t)\ (X) = 0, \qquad \text{(15a)}$$

where

$$t(X) \stackrel{\text{def}}{=} g(T,\ X), \qquad \text{(15b)}$$

and D is Riemannian connexion in V_4. From (15*a*), we get

$$\operatorname{div}\ T = 0. \qquad \text{(16)}$$

A vector field T is said to be a Harmonic field if

$$(a)\ (D_X\ t)\ (Y) = (D_Y t)\ (X) \text{ and } (b) \operatorname{div}\ T = 0. \qquad \text{(17)}$$

In the following, the notation

$$(\square\ \alpha)\ (X) \stackrel{\text{def}}{=} C_2^1\ {}^{-1}G(\nabla\ \nabla\ \alpha\ )\ (X), \qquad \text{(18)}$$

where ∇, the operator of covariant derivative, will be used.

We will now prove the following theorems.

Theorem 1—*Let the polarisation vector M be a Killing vector. Then the accelerations of P, Q and M along M vanish. Also $D_P\ Q + D_Q\ P$, $D_P\ M + D_M\ P$, $D_Q\ M + D_M\ Q$* are all orthogonal to M.

Proof : Since M is a Killing vector

$$(D_X\ m)\ (Y) + (D_Y\ m)\cdot(X) = 0.$$

Writing P for X and Y in this equation, it is found that

$$(D_P\ m)\ (P) = 0,$$

which is equivalent to

$$m(D_P\ P) = 0.$$

This equation proves that the acceleration of P along M vanishes. The remaining part can be proved similarly.

Theorem 2—*Let the propagation vector Q or the vector P be a Killing vector. Then the accelerations of P, Q, M along Q vanish. Also $D_P\ Q + D_Q\ P$, $D_P\ M + D_M\ P$, $D_Q\ M + D_M\ Q$ are all orthogonal to Q or P respectively.*

Proof : The proof follows the pattern of the proof of Theorem (1).

Theorem 3—*Let M, Q or P be a Harmonic vector. Then $D_P\ Q - D_Q\ P$, $D_P\ M - D_M\ P$, $D_Q\ M - D_M\ Q$ are orthogonal to M, Q, P respectively.*
The proof is obvious.

Field Equations

For the null electromagnetic field, the system of field equations (5) reduces, in consequence of (14*a*), to

$$R'(X) - (\tfrac{1}{2}r' + \zeta)\ X = \mu\ u(X)U - p(X)\ Q. \qquad \text{(19)}$$

This equation shows that $R-(\frac{1}{2}r+\zeta)\,I_4$ is of rank 2. Let us write

$$\sqrt{2}\;U \stackrel{\text{def}}{=} \eta\, P + \frac{1}{\eta} Q \quad .. \quad (20)$$

This form has been chosen, so that (*7a* and *b*) may be satisfied.

From Eqs. (*7a*), (20) and (11), we immediately get

$$\sqrt{2}\,u(X) = -\left\{\eta\, q\,(X) + \frac{1}{\eta} p(X)\right\}. \quad .. \quad (21)$$

Contracting Eq. (19), we get

$$\mu = r' + 4\zeta. \quad .. \quad (22)$$

Substituting from Eqs. (20) and (22) in (19), we get

$$2R'(X)=(\mu-2\zeta)X-\mu\{\eta^2 q(X)+p(X)\}\;P-\mu\left\{q(X)+\left(\frac{1}{\eta^2}+\frac{2}{\mu}\right)p(X)\right\}Q. \quad .. \quad (23)$$

Theorem 4—The necessary and sufficient conditions that the Killing or the Harmonic propagation vector travels with the fundamental velocity are $\zeta=0$, $\mu=0$.

Proof : If Q is a Killing or Harmonic vector, we have (Yano 1970).

$$(\square\, p)(X) + Ric\,(Q,\,X) = 0. \quad .. \quad (24)$$

The necessary and sufficient condition that Q moves with fundamental velocity is $\square\, p = 0$. Consequently $Ric\;(Q,\;X) = 0 \Leftrightarrow R'\,(Q) = 0$. Substituting Q for X in Eq. (23) and putting $R'(Q) = 0$, we get $\zeta = 0 = \mu$. Hence we have the statement.

Theorem 5—Let P be Killing or Harmonic and let the density be non-negative. Then P cannot travel with fundamental velocity.

Proof: Writing P for X in Eq. (23) and putting $R'\,(P) = 0$, we get $\zeta = 0$ and $\mu = -2\eta^2$.

But from Eq. (20) and $X = p(X)P + q(X)Q + r(X)\,R+s(X)S$

we have $\eta = \sqrt{2}\;p(U) = real.$

Therefore $\mu = -2\eta^2$ is negative, which contradicts our assumption. Hence we have the statement.

Theorem 6 : Let the polarisation vector M be Killing or Harmonic. The necessary and sufficient condition that it moves with fundamental velocity is

$$\mu = 2\,\zeta.$$

Proof: Putting M for X in Eq. (23) and $R(M) = 0$, and using (A) we get Eq. 25.

Remark 1: Theorem (6) holds not only for the polarisation vector M but also for the vectors R and S when they are Killing or Harmonic.

Maxwell Equations

We will first consider Maxwell equation (9).

From Eq. (12), we have

$$(D_Y k)(X) = (D_Y p)\;(X)M + p(X)\;D_Y\;M + (D_Y m)\;(X)\;Q + m(X)D_Y\;Q.$$

Hence

$$(div\ k)\ (X) = (D_M p)\ (X) + p(X)\ div\ M + (D_Q m)\ (X) + m\ (X)\ div\ Q. \quad .. \quad (26)$$

Substituting from this equation and Eq. (21) in Eq. (9), we obtain the Maxwell equation in the form

$$(D_M p)\ (X) + p(X) \Big\{ (div\ M - \frac{\epsilon}{\sqrt{2}\eta} + (D_Q\ m)\ (X) + m(X)\ div\ Q$$
$$- \frac{\epsilon}{\sqrt{2}}\eta\ Q\ (X) = 0 \quad .. \quad (27)$$

Theorem 7: The necessary condition that Maxwell equations (9) are compatible is

$$div\ Q = p(D_M\ M) + m(D_Q\ M). \quad .. \quad (28)$$

When this condition is satisfied e and η *are given by*

$$\frac{1}{\sqrt{2}}\ \frac{\epsilon}{\eta} + p(D_M\ P) + m(D_Q\ P) = div\ M. \quad .. \quad (29)$$

$$\frac{1}{\sqrt{2}}\ \frac{\epsilon}{\eta} + p(D_M\ Q) + m(D_Q\ Q) = 0. \quad .. \quad (30)$$

Proof : Putting M for X in Eq. (27), we obtain Eq. (28). Putting P and Q for X in Eq. (27), we obtain Eqs. (29) and (30) respectively.

Remark 2 : The null vector Q is a tangent to the congruence of light-like geodesies. Therefore, we can assume without loss of generality (Sacks 1960), $D_Q\ Q = 0$.

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