PROPAGATION OF SONIC DISCONTINUITIES THROUGH THERMALLY CONDUCTING AND DISSOCIATING GASES

by RISHI RAM *and* MADHUKAR GAUR, *Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi-221005*

(*Received 18 April 1975*)

The propagation of sonic discontinuities through thermally conducting and dissociating gases is studied. It is found that the velocity of propagation relative to the normal gas velocity is the effective isothermal velocity of sound. The differential equations governing the growth and decay of sonic discontinuities are obtained and solved for various wave-fronts. It is concluded that if the sonic discontinuity is a compressive wave, then it terminates into a shock wave after a critical time t_c. But on the other hand, if it is an expansion wave, it will continuously decay and will be damped out ultimately. An expression for critical time t_c is obtained and it is found that the effects of thermal conduction and dissociation are to increase the critical time t_c. In the case of an expansion wave the thermal conduction and dissociation cause more rapid damping effects. The results of Thomas (1957*a*) are obtained as a particular case.

INTRODUCTION

Thomas (1957*a*) discussed the propagation of sonic waves in an ideal gas flow. Nariboli and Secrest (1967) extended the analysis of Thomas to Magneto-gasdynamics of finite electrical conductivity by taking into account the effects of thermal conduction under the restriction that the magnetic field and its first derivatives are both continuous across a sonic discontinuity. Using the ray theory, Upadhyaya (1967) obtained the growth equation for a sonic discontinuity propagating through thermally conducting gases, but he did not discuss its growing or decaying tendency. The main object of the present paper is to investigate the effects of thermal conduction and dissociation on the growth and decay of sonic discontinuities propagating through thermally conducting and dissociating gases. In the following analysis, we shall neglect the molecular transport effects leading to viscosity and diffusion. The temperature range is assumed from 1000°K to 7000° K so that the only chemical reaction involved is that of dissociation and hence the contributions of energy from electronic excitation and ionization are neglected (Clarke 1960).

A simple dissociating gas is defined as a mixture resulting from a dissociation reaction in a symmetrical diatomic gas A_2, each A_2 molecule being made up from $2A_1$ atoms. The reaction is

$$A_2 + X \underset{k_r}{\overset{k_f}{\rightleftarrows}} 2A_1 + X,$$

where the species X can be either A_2 or A_1 and K_f and K_r are the reaction rate constants for the forward and reverse reactions. The rate of production of an atom mass per unit volume is given by Lighthill (1957) in the form :

$$\frac{\partial \alpha}{\partial t} + U_i\, \alpha_{,i} = \frac{4\rho D^2 k_r}{R^2 T_d^2} (1 + \alpha) \left\{ (1 - \alpha)\, \rho_d \exp\,(-\, T_d/T) - \rho\alpha^2 \right\}, \quad \dots \quad (1)$$

where ρ_d, T_d, D, α and R are respectively the characteristic density for dissociation, the characteristic temperature for dissociation, the dissociation energy per unit mass, the atom mass fraction or degree of dissociation and the gas constant for A_2. ρ and U_i are respectively the density and velocity components of the ideal dissociating gas and a comma followed by an index denotes partial differentiation with respect to corresponding co-ordinate. Although ρ_d is a function of temperature T, it has been seen that the variation of ρ_d over the temperature range for dissociation *i.e.*, from 1000° K to 7000° K is very slight. Hence for practical purposes the useful simplification of regarding ρ_d as a constant has been made, which should lead to negligible errors.

Propagation of Sonic Discontinuities

The basic equations governing the continuous flow of a thermally conducting and simple dissociating gas (Lighthill 1957) are

$$\frac{\partial \rho}{\partial t} + U_i\, \rho_{,i} + \rho U_{i,i} = 0, \quad \dots \quad (2)$$

$$\rho \frac{\partial U_i}{\partial t} + p_{,i} + \rho U_j U_{i,j} = 0, \quad \dots \quad (3)$$

$$\rho \frac{\partial h}{\partial t} + \rho U_i h_{,i} - U_i\, p_{,i} - (kT_{,i})_{,i} = 0, \quad \dots \quad (4)$$

where h is the effective enthalpy and K is the coefficient of thermal conduction.

The caloric and thermal equations of states of an ideal dissociating gas are given by (Clarke 1960)

$$h = (4 + \alpha)\, RT + \alpha D \quad \dots \quad (5)$$

and $p = (1 + \alpha)\, \rho RT,$ ... (6)

where p is the gas pressure. By virtue of Eqs. (1), (5) and (6) the energy Eq. (4) can be written in the form :

$$\frac{\partial p}{\partial t} \cdot \rho U_i \frac{\partial U_i}{\partial t} - \rho U_i U_j U_{ij} + \gamma_e p U_{i,i} + F\,(p, \rho, \alpha) - \frac{k\,(1+\alpha)}{3} T_{,ii} = 0, \quad \dots \quad (7)$$

where

$$F\,(p, \rho, \alpha) = \frac{4\rho D^2 k_r}{3R^2 T_d^2} \left\{ 3p - \rho D\, (1 + \alpha)^2 \right\} \left\{ \rho_d\, (1 - \alpha) \exp\,(-\, T_d/T) - \rho\alpha^2 \right\}$$

γ_e being $\dfrac{4 + \alpha}{3}$ as the effective exponent of heat for the gas mixture. A moving surface $\Sigma\,(t)$ of a sonic discontinuity across which the flow variables are continuous but their first and higher derivatives are discontinuous, is defined as a sonic wave or a wave of order 1 (Thomas 1957*a*). We shall denote the jump in the parameter Z across the wave by $[Z]$. Taking jump in the Eqs. (1), (2), (3) and making use of the geometrical and kinematic compatibility conditions of Thomas (1957*a*), we obtain

$$(U_n - G)\,[\alpha_{,i}]\,n_i = 0, \qquad \dots \quad (8)$$

$$(U_n - G)\,\tau + \rho\lambda_n = 0, \qquad \dots \quad (9)$$

$$\rho\,(U_n - G)\,\lambda_i + \mu_{ni} = 0, \qquad \dots \quad (10)$$

where G is the velocity of propagation of the wave front $\Sigma\ (t)$ and n_i are the components of the unit normal to the wave surface. The scalar functions λ_i, τ and U_n are defined over the wave surface by the relations :

$$\lambda_i = [U_{i,j}]\,n_j,\ \mu = [p_{,j}]\,n_j,\ \tau = [\rho_{,j}]\,n_j,\ U_n = U_i n_i.$$

From the law of conservation of energy across $\Sigma(t)$, we have (Pant & Mishra 1963).

$$[T_{,i}]\,n_i = 0 \qquad \dots \quad (11)$$

Differentiating Eq. (6) with respect to x_i and then taking jump across the wave front we get the relation

$$\mu = a_e\,\tau, \qquad \dots \quad (12)$$

where a_e is the effective isothermal velocity of sound.

Multiplying Eq. (10) by n_i and substituting the value of μ from Eq. (12) we get

$$\rho(U_n - G)\,\lambda_n + a_e^3\,\tau = 0. \qquad \dots \quad (13)$$

Using Eq. (13) in Eq. (9) we obtain

$$\left\{\left(U_n - G\right)^2 - a_e^2 \right\}\tau = 0. \qquad \dots \quad (14)$$

Since $\tau \neq 0$ over the wave surface, we have

$$(U_n - G)^2 = a_e^2\ . \qquad \dots \quad (15)$$

The relation (15) shows that the velocity of propagation relative to the normal gas velocity is the effective isothermal velocity of sound. From Eqs. (9), (10) and (11), we get

$$\lambda_i = \lambda n_i\ , \qquad \tau = \frac{\rho\lambda}{G - U_n} = \frac{\mu}{(G - U_n)^2}\ . \qquad \dots \quad (16)$$

If the medium ahead of a sonic discontinuity is uniform and at rest, the sonic discontinuity will propagate with a constant isothermal speed of sound. In this case the relations (16) will reduce to the following forms :

$$\mu = \rho G\lambda = G^2\tau. \qquad \dots \quad (17)$$

THE GROWTH AND DECAY OF SONIC DISCONTINUITIES

Differentiating Eqs. (2) and (3) with respect to x_k and making use of the geometrical and kinematical compatibility conditions of Thomas (1957*b*) we obtain

$$\rho \frac{\delta \lambda}{\delta t} + \bar{\mu} - \rho G \bar{\lambda}_i n_i = 0, \qquad \dots (18)$$

$$\frac{\delta \tau}{\delta t} + 2\tau\lambda - 2\rho\lambda\Omega - G\bar{\tau} + \rho\bar{\lambda}_i n_i = 0, \qquad \dots (19)$$

where

$$\bar{\lambda}_i = [U_{i,jk}]\, n_j n_k, \quad \bar{\mu} = [p_{,ij}]\, n_i n_j, \quad \bar{\tau} = [\rho_{,ij}]\, n_i n_j,$$

and Ω is the mean curvature of the wave surface $\Sigma(t)$.

Taking the jump of Ihe Eq. (7) we get

$$\frac{1+\alpha}{3} k \left[T_{,ii} \right] = \left(a_e C_e^2 - a_e^2 \right) \tau, \qquad \dots (20)$$

where $C_e = \gamma_e \dfrac{p}{\rho}$.

Now differentiating Eq. (1) and making use of geometrical and kinematical compatibility conditions of Thomas (1957*a*), we g t

$$(U_n - G)\, \bar{\theta} = A\, \tau, \qquad \dots (21)$$

where

$$A = \frac{4D^2 (1+\alpha) K_\gamma}{R^2 - T_d^2} \left\{ (1-\alpha)\, \rho_d \exp\left(-T_d/T\right) - 2\rho\alpha^2 \right\}$$

and

$$\bar{\theta} = [\alpha_{,ij}]\, n_i n_j.$$

Differentiating Eq. (6) twice with respect to x_i and x_j and then taking jumps, we get

$$\bar{\mu} = \frac{3\rho R}{K} \left(C_e^2 - G^3 \right) \tau + G^2 \bar{\tau} - \frac{Ap}{(1+\alpha)G} \tau. \qquad \dots (22)$$

Using Eq. (22) and (23) in Eqs. (18) and (19), we obtain

$$\frac{\delta \lambda}{\delta t} = -\lambda^2 + G\Omega(t)\lambda - \left\{ \frac{3\rho R}{2k} \left(C_e^2 - G^2 \right) - \frac{Ap}{2(1+\alpha)G^2} \right\} \lambda, \qquad \dots (23)$$

$$\frac{\delta \tau}{\delta t} = -\frac{G}{\rho} \tau^2 + G\Omega(t)\, \tau - \left\{ \frac{3\rho R}{2k} \left(C_e^2 - G^2 \right) - \frac{Ap}{2(1+\alpha)G^2} \right\} \tau \qquad \dots (24)$$

$$\frac{\delta \mu}{\delta t} = -\frac{\mu^2}{\rho G} + G\Omega(t)\mu - \left\{ \frac{3\rho R}{2k} \left(C_e^2 - G^2 \right) - \frac{Ap}{2(1+\alpha)G^2} \right\} \mu. \qquad \dots (25)$$

The relations (23), (24) and (25) are the fundamental differential equations for the variations of the quantities λ, τ and μ along the direction of propagation.

These equations govern the growth and decay of the sonic discontinuities propagating in an ideal dissociating and thermally conducting gas at rest.

Let $\Sigma\ (t_0)$ represent the initial wave front at time t_0 and let σ represent the distance measured from $\Sigma\ (t_0)$ along the normal trajectory to the family of surfaces $\Sigma\,(t)$. Then $\sigma = G\ (t—t_0)$ and the scalar functions λ, τ and μ can be regarded as functions of σ. Hence

$$\frac{\delta\lambda}{\delta t} = G\frac{d\lambda}{d\sigma}\,,\ \frac{\delta\tau}{\delta t} = G\frac{d\tau}{d\sigma},\ \frac{\delta\mu}{\delta t} = G\frac{d\mu}{d\sigma}\,. \qquad \ldots\ (26)$$

Substituting from Eqs. (26) in (23) (24) and (25), we get

$$\frac{d\lambda}{d\sigma} - \Big[\,\Omega - \Big\{\frac{3\rho R}{2kG}\Big(C_e^2 - G^2\Big) - \frac{Ap}{2(1+\alpha)G^3}\Big\}\,\lambda + \frac{1}{G}\,\lambda^2 = 0, \qquad \ldots\ (27)$$

$$\frac{d\tau}{d\sigma} - \Big[\,\Omega - \Big\{\frac{3\rho R}{2kG}\Big(C_e^2 - G^2\Big) - \frac{Ap}{2(1+\alpha)G^3}\Big\}\Big]\,\tau + \frac{1}{\rho}\,\tau^2 = 0, \qquad \ldots\ (28)$$

$$\frac{d\mu}{d\sigma} = \Big[\,\Omega - \Big\{\frac{3\rho R}{2kG}\Big(C_e^3 - G^2\Big) - \frac{Ap}{2(1+\alpha)G^3}\Big\}\Big]\,\mu + \frac{1}{\rho G^2}\,\mu^2 = 0. \qquad \ldots\ (29)$$

In view of the relations (15) and (17), the Eqs. (27) and (29) are derivable from (28). Therefore, Eq. (28) is sufficient to predict the growth or decay of the sonic discontinuity associated with the wave $\Sigma\ (t)$.

The mean curvature Ω of the sonic wave surface $\Sigma(t)$ is a function of distance σ and has been calculated (Lane 1940) in the form

$$\Omega = \frac{\Omega_0 - K_0\sigma}{1 - 2\Omega_0\sigma + K_0\sigma^2}\,,$$

where Ω_0 and K_0 are respectively the Gaussian and mean curvature of the surface $\Sigma(t_0)$ at time t_0. Substituting for Ω in Eq. (28) and integrating, we get

$$\frac{1}{\tau} = \frac{1}{\tau_0}\sqrt{1 - 2\Omega_0\sigma + K_0\sigma^2}\ e^{L\sigma} + \frac{1}{\rho}\,e^{L\sigma}\sqrt{1 - 2\Omega_0\sigma + K_0\sigma^2}\int_0^{\sigma}\frac{e^{-L\sigma}}{\sqrt{1 - 2\Omega_0\sigma + K_0\sigma^2}}\,, \qquad \ldots\ (30)$$

where $L = \Big\{\dfrac{3\rho R}{2KG}\Big(C_e^2 - G^2\Big) - \dfrac{Ap}{2(1+\alpha)G^3}\Big\}$,

and τ_0 is the value of τ at the initial time t_0. In order to predict the physical situation more clearly, we consider particular case of a plane wave front, for which $\Omega = \Omega_0 = 0$ and $K_0 = 0$. For this case, Eq (30) reduces to the form :

$$\frac{1}{\tau} = \frac{1}{\tau_0}\,e^{L\sigma} + \frac{1}{\rho L}\Big(e^{L\sigma} - 1\Big),$$

which can be written as

$$\tau = \frac{\tau_0}{\left[[e^{L\sigma} + \frac{1}{\rho L}\left(e^{L\sigma} - 1 \right) \tau_0 \right]} . \qquad \dots (31)$$

Eq (31) suggests that if τ_0 is negative, the discontinuity τ_0 grows continuously till it tends to infinity as

$$\sigma \to \frac{1}{L} \log \left\{ \frac{|\tau_0|}{|\tau_0| - \rho L} \right\} .$$

In such a situation, the continuity of density across $\Sigma(t)$ will break down and consequently the sonic wave will terminate into a shock wave. However, for the real value of σ, the following inequality should be satisfied by $|\tau_0|$:

$$|\tau_0| > \rho L, \qquad \dots (32)$$

which is automatically satisfied in the absence of dissociation and thermal conduction effects. Hence we arrive at an interesting conclusion that when a compressive wave of order 1 propagates in a thermally conducting and dissociating gas, then the critical time t_c for its termination into a shock wave from the instant t_0 is given by

$$t_c = t_0 + \frac{1}{LG} \log \left\{ \frac{|\tau_0|}{|\tau_0| - \rho L} \right\} .$$

Hence in this case a sonic discontinuity will grow and after time t_c, it will terminate into a shock wave. The thermal conduction and dissociation effects are to increase the critical time t_c. But on the other hand when the sonic discontinuity is an expansion wave of order $1 (\tau_0 > 0)$, the discontinuity will continuously decay and will be damped out ultimately. The thermal conduction and dissociation effects will cause more rapid damping effects. From the similarity of the Eqs. (27), (28) and (29), it is obvious that the other discontinuities λ and μ will also behave in similar manner. When we neglect the contributions of thermal conduction and dissociation ($L=0$), the fundamental Eqs. (27), (28) and (29) reduce to the forms derived by Thomas (1957*a*) Hence we can recover all the conclusions of Thomas (1957*a*) as a particular case of our results.

When the sonic wave surface $\Sigma(t)$ consists of a family of concentric spheres, the mean curvature Ω is $\frac{1}{R}$, where R denotes the radius of the sphere of the family. This follows from the fact that $\frac{\delta n_i}{\delta t} = 0$ for the family of surface $\Sigma(t)$. Since the wave propagates along the normal trajectories, we can replace σ by the radius R. Hence the Eqs. (27), (28) and (29) assume the following forms:

$$\frac{d\lambda}{dR} + \left(L - \frac{1}{R} \right) \lambda + \frac{\lambda^2}{G} = 0 \qquad \dots (33)$$

$$\frac{d\tau}{dR} + \left(L - \frac{1}{R} \right) \tau + \frac{\tau^2}{\rho} = 0 \qquad \dots (34)$$

$$\frac{d\mu}{dR}+\left(L-\frac{1}{R}\right)\mu+\frac{\mu^2}{\rho G^2}\ 0. \qquad \ldots \quad (35)$$

Solving Eqs. (33), (34) and (35) we obtain

$$\lambda=\frac{\lambda_0}{\frac{R_0}{R}\,e^{L(R-R_0)}+\frac{\lambda_0}{GL^2R}\left\{\left(1+LR_0\right)e^{L(R-R_0)}-\left(1+LR\right)\right\}}, \qquad \ldots \quad (36)$$

$$\tau=\frac{\tau_0}{\frac{R_0}{R}\,e^{L(R-R_0)}+\frac{\tau_0}{GL^2R}\left\{\left(1+LR_0\right)e^{L(R-R_0)}-\left(1+LR\right)\right\}}, \text{ and} \qquad \ldots \quad (37)$$

$$\mu=\frac{\mu_0}{\frac{R_0}{R}\,e^{L(R-R_0)}+\frac{\mu_0}{GL^2R}\left\{\left(1+LR_0\right)e^{L(B-R_0)}-\left(1+LR\right)\right.}, \qquad \ldots \quad (38)$$

where R_0, λ_0, τ_0, and μ_0 are respectively the value of R, λ, τ and μ at the initial time.

The relation (37) suggests that when τ_0 is negative, the sonic discontinuity grows in intensity till it terminates into a shock wave at the critical time T_c given by

$$T_c=\frac{GT_0-R_0}{G}+\frac{R}{G},$$

where R is the real root of the equation

$$GL^2R_0\,e^{L(R-R_0)}-\mid\tau_0\mid\left\{\left(1+LR_0\right)e^{L(R-R_0)}-\left(1+LR\right)\right\}=0.$$

When τ_0 is positive, the sonic discontinuity goes on decaying and ultimately vanishes. The thermal conduction and dissociation effects are of the same nature as discussed in the case of plane waves.

References

Clarke, J. F. (1960). The linearized flow of a dissociating gas. *J. Fluid Mech.*, **7**, 577–595.

Lane, L. P. (1940). *Metric Differential Geometry of Curves and Surfaces*, p. 209.

Lighthill, M. J. (1957). Dynamics of dissociating gases. *J. Fluid Mech.*, **2**, 1–32.

Nariboli, G. A., and Secrest (1967). Weak discontinuities in magneto gasdynamics in the presence of a dissipative mechanism. *Tensor* (*N.S.*), **18**, 22–25.

Pant, J. C., and Mishra, R. S. (1963). Shock waves of finite thickness in maneto-gasdynamics. *Rend. Circolo Mat.*, Tomo **XII**, 1–23.

Thomas, T. Y. (1957*a*). The growth and decay of sonic discontinuities in ideal gasses. *J. Math. Mech.*, **6**, 455–469.

——— (1957). Extended compatibility conditions for the study of surface of discontinuity in continuum mechanism. *J. Math. Mech.*, **6**, 311–312.

Upadhyaya, K. S. (1967). Propagation of weak discontinuities through thermally conducting gases. *Tensor* (*N.S.*), **21**, 296–300.