BROWNIAN MOTION OF A PARTICLE WITH FREQUENCY DEPENDENT FRICTION

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(*Received 5 June 1976*)

The time evolution of the probability distribution function of the Brownian particle in the velocity space is calculated when the finite relaxation time of the random force is taken into account.

INTRODUCTION

The problem of Brownian motion of a free particle based on Langevin equation has been discussed in detail by many workers (Uhlenbeck & Ornstein 1930; Chandrasekhar 1943; and Wang & Uhlenbeck 1945). The random force appearing in the Langevin equation is assumed to be delta correlated. The analysis based on this assumption implies that the random force relaxation time is negligible as compared to the velocity relaxation time. Kubo (1966) has considered the generalization of the Langevin equation to account for the finite relaxation time of the random force. Such a generalization implies the introduction of retarded, or frequency dependent frictional force. Case (1971) evaluated the velocity correlation function for frequency dependent frictional force.

In this paper, we calculate the probability distribution function of the Brownian particle in velocity space at time t when the finite correlation time of the random force is taken into account. The random force is assumed to be a centered Gaussian stationary process with an exponentially decaying auto-correlation function.

EQUATION OF MOTION

The generalized Langevin equation of motion of the Brownian particle of mass m can be written as

$$m \frac{d}{dt} u(t) + m \int_0^t f(t-t')\, u(t')dt' = F(t), \quad t \geqslant 0. \qquad \ldots(1)$$

Here u is the velocity, $f(t)$ is the friction function and $F(t)$ is the random force being characterized by

$$<F(t)> = 0; \; <F(t_1)\, F(t_2)> = mkTf_0 \exp[-\alpha \mid t_1 - t_2 \mid], \qquad \ldots(2)$$

where T is the temperature of the heat bath and k is Boltzmann's constant. The friction function appearing in Eq. (1) can be determined by using the fluctuation-dissipation theorem (Kubo 1966) which gives

$$mf(\omega) = \frac{1}{kT} \int_0^\infty < F(t_0)\, F(t_0 + \tau) > e^{-i\omega\tau}\, d\tau. \qquad ...(3)$$

Using Eqs. (2) and (3), we find

$$f(t) = f_0\, e^{-\alpha t}. \qquad ...(4)$$

CALCULATIONS

The solution of Eq. (1) can be written as

$$u(t) = m\, \chi(t) + \int_0^t \psi(t-s)\; F(s)\, ds, \qquad ...(5)$$

where $\psi(t)$ is the Green's function of the random differential equation (1) and $\chi(t)$ is the deterministic function which is to be calculated from the initial conditions. The functions $\psi(t)$ and $\chi(t)$ are given by

$$\psi(t) = \frac{e^{-\alpha t/2}}{\gamma}\left[\alpha \sin \frac{\gamma t}{2} + \gamma \cos \frac{\gamma t}{2}\right] \qquad ...(6)$$

and

$$\chi(t) = \frac{u_0 e^{-\alpha t/2}}{\gamma}\left[\alpha \sin \frac{\gamma t}{2} + \gamma \cos \frac{\gamma t}{2}\right], \qquad ...(7)$$

where

$$\gamma = (4f_0 - \alpha^2)^{1/2}. \qquad ...(8)$$

Using Eqs. (2) and (5), the ensemble average of the velocity is found to be

$$<u(t)> = \chi(t). \qquad ...(9)$$

Defining $U(t) = u(t) - <u(t)>$, Eq. (5) gives

$$U^2(t) = \frac{1}{m^2\gamma^2} \int_0^t \int_0^t \psi(t-s_1)\, \psi(t-s_2) < F(s_1)\, F(s_2) > ds_1\, ds_2. \qquad ...(10)$$

Substituting the force correlation function from Eq. (2) and $\psi(t)$ from Eq. (6), then Eq. (10) reduces to

$$<U^2(t)> = \frac{kT}{\text{m}}\left[1 - \left\{\frac{e^{-\alpha t/2}}{\gamma}\left(\alpha \sin \frac{\gamma t}{2} + \gamma \cos \frac{\gamma t}{2}\right)\right\}^2\right]. \qquad ...(11)$$

Since $F(t)$ is assumed to be a Gaussian process in our treatment, the probability distribution function $W(u,\ t;\ u_0)$ can be expressed in terms of first two velocity moments. From Eqs. (9) and (11), we find

$$W(u,\ t;\ u_0) = \left[\frac{m}{2\pi kT\left(1-\left\{\frac{e^{-\alpha t/2}}{\gamma}\left(\alpha\sin\frac{\gamma t}{2}+\gamma\cos\frac{\gamma t}{2}\right)^2\right\}\right)}\right]^{1/2}\times$$

$$\exp.\left[-\frac{m\left\{u-\frac{u_0e^{-\alpha t/2}}{\gamma}\left(\alpha\sin\frac{\gamma t}{2}+\gamma\cos\frac{\gamma t}{2}\right)\right\}^2}{2\,kT\left(1-\left\{\frac{e^{-\alpha t/2}}{\gamma}\left(\alpha\sin\frac{\gamma t}{2}+\gamma\cos\frac{\gamma t}{2}\right)\right\}^2\right)}\right]du. \quad ...(12)$$

It is found that the time evolution of velocity distribution function differs considerably from that observed in the case of delta correlated force. The time dependent terms contained in Eq.(12) depict damped oscillatory behaviour when $f\alpha_0 > \alpha^2/4$. In case $f_0 < \alpha^2/4$, the expression for probability density reduces to

$$W\ (u,\ t;\ u_0) = \left[\frac{m}{2\,\pi kT\left(1-\left\{\frac{e^{-\alpha t/2}}{\xi}\left(\alpha\sinh\frac{\xi t}{2}+\xi\cosh\frac{\xi t}{2}\right)^2\right\}\right)}\right]^{1/2}$$

$$\exp.\left[-\frac{m\left\{u-\frac{u_0e^{-\alpha t/2}}{\xi}\left(\alpha\sinh\frac{\xi t}{2}+\xi\cosh\frac{\xi t}{2}\right)\right\}^2}{2\,kT\left[1-\left\{\frac{e^{-\alpha t/2}}{\xi}\left(\alpha\sinh\frac{\xi t}{2}+\xi\cosh\frac{\xi t}{2}\right)^2\right\}\right]}\right] \quad ...(13)$$

where

$$\xi = (\alpha^2 - 4f_0)^{1/2}. \quad ...(14)$$

In the marginal case, $f_0 = \alpha^2/4$, the expression for the probability density function becomes

$$W(u,\ t;\ u_0)\,du = \left[\frac{m}{2\pi kT\left[1-\left\{e^{-\alpha t/2}\left(1+\frac{\alpha t}{2}\right)^2\right\}\right]}\right]^{1/2}\times$$

$$\exp.\left[-\frac{m\left\{u-u_0\,e^{-\alpha t/2}\left(1+\frac{\alpha t}{2}\right)\right\}^2}{2\,kT\left\{1-\left(e^{-\alpha t/2}\left(1+\frac{\alpha t}{2}\right)\right)^2\right\}}\right] \quad ...(15)$$

From Eqs. (12), (13) and (15), it is clear that the Brownian particle reaches the equilibrium state for asymptotically large times as the velocity distribution tends to Maxwellian distribution with increasing time (after a time interval of order α^{-1}). The usual result when the random force is delta correlated can be derived in the limiting cases when both α and f_0 tend to infinity such that $f_0/\alpha = \beta$, the friction coefficient.

ACKNOWLEDGEMENTS

The author wishes to thank Prof. L. S. Kothari and Dr. S. P. Talwar for helpful discussions.

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