MAGNETISED CYLINDERS IN GENERAL RELATIVITY

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Presented here are some interesting results on test particles in the fields of magnetised cylinders for which a solenoid and a straight wire carrying current have been considered.

INTRODUCTION

KRORI AND BARUA (1975) recently obtained some results on escape of photons from magnetised cylinders. Here we have studied the behaviour of test particles in the fields of a solenoid and a straight wire carrying current.

The present authors have found from their study that the attractive force on a test particle at rest is effective at all distances from a solenoid carrying current. But this is so in the case of a straight wire carrying current only when the mass per unit length of the straight wire is greater than a certain limit. If, however, the mass per unit length is less than this limit, the radius of the straight wire should be equal to or greater than a critical value depending upon the mass per unit length and the current strength.

It has also been found that a test particle can move in a circular orbit around a solenoid carrying current if the mass per unit length is less than a certain limit and the radius of orbit is equal to or less than a critical value depending upon the mass per unit length and current strength. No circular motion is however possible around the solenoid if the mass per unit length is greater than that limit. On the other hand a test particle can move in a circular orbit of any radius around a straight wire carrying current only if the mass per unit length is less than a certain limit. No circular motion is possible for mass per unit length greater than that limit.

CALCULATIONS

I. *A solenoid carrying current*—The line element we use here is due to Safko and Witten (1972) and is given by

$$ds^2 = r^{2c+2c^2}(1+k_1 r^{2+2c})^2 A(dt^2 - dr^2) - r^{2c+2}(1+k_1 r^{2+2c})^{-2} d\phi^2 - r^{-2c}(1+k_1 r^{2+2c})^2 dz^2, \quad ...(1)$$

where c is related to mass per unit length of the solenoid and k_1 is associated with magnetic field produced by the current in the solenoid.

The geodesic equation is

$$\frac{d^2x^\gamma}{ds^2}+\Gamma^\gamma_{\mu\nu}\frac{dx^\mu}{ds}\;\frac{dx^\nu}{ds}=0,$$

where $\Gamma^\gamma_{\mu\nu}$ is the Christoffel symbol.

The surviving components of $\Gamma^\gamma_{\mu\nu}$ (with $A=1$) in this case are

$$\Gamma^1_{11}=\Gamma^4_{14}=\Gamma^1_{44}=\frac{c+c^2}{r}+\frac{(2+2c)\,k_1r^{2+2c}}{r\{1+k_1r^{2+2c}\}}$$

$$\Gamma^2_{12}=\frac{c+1}{r}-\frac{2(c+1)k_1r^{2+2c}}{r\{1+k_1r^{2+2c}\}}$$

$$\Gamma^1_{22}=-\frac{c+1}{\{1+k_1r^{2+2c}\}^4\,r^{2c^2-1}}+\frac{2(c+1)k_1}{r^{2c^2-2c-3}\{1+k_1r^{2+2c}\}^5}$$

$$\Gamma^3_{13}=-\frac{c}{r}+\frac{2(1+c)\,k_1r^{2+2c}}{r\{1+k_1r^{2+2c}\}}$$

$$\Gamma^1_{33}=\frac{c}{r^{2c^2+4c+1}}-\frac{2(1+c)k_1}{r^{2c+2c^2-1}\{1+k_1r^{2+2c}\}}\;.$$

with these values of $\Gamma^\gamma_{\mu\nu}$ we get the following geodesic equations

$$\frac{d^2r}{ds^2}+\Gamma^1_{44}\left[\left(\frac{dr}{ds}\right)^2+\left(\frac{dt}{ds}\right)^2\right]+\Gamma^1_{22}\left(\frac{d\phi}{ds}\right)^2+\Gamma^1_{33}\left(\frac{dz}{ds}\right)^2=0, \quad \ldots(2)$$

$$\frac{d^2\phi}{ds^2}+2\;\Gamma^2_{12}\frac{d\phi}{ds}\,\frac{dr}{ds}=0, \quad \ldots(3)$$

$$\frac{d^2z}{ds^2}+2\;\Gamma^3_{13}\frac{dz}{ds}\,\frac{dr}{ds}=0 \quad \ldots(4)$$

and

$$\frac{d^2t}{ds^2}+2\,\Gamma^4_{14}\frac{dt}{ds}\,\frac{dr}{ds}=0 \quad \ldots(5)$$

(*i*) When the particle is at rest, we have

$$\frac{dr}{ds}=\frac{dz}{ds}=\frac{d\phi}{ds}=0.$$

From the geodesic equations, we have

$$\frac{d^2\phi}{ds^2}=\frac{d^2z}{ds^2}=\frac{d^2t}{ds^2}=0.$$

From the line-element (1)

$$\frac{dt}{ds}=\frac{1}{r^{c+c^2}(1+k_1r^{2+2c})}.$$

From equation (2) we get the force experienced by a particle at rest

$$\frac{d^2r}{dt^2} = -\left[\frac{c+c^2}{r} + \frac{2(c+1)k_1 r^{2+2c}}{r\{1+k_1 r^{2+2c}\}}\right]$$

(*ii*) Radial motion

$$\frac{d\phi}{ds} = \frac{dz}{ds} = 0$$

From equation (5)

$$\frac{dt}{ds} = \frac{\chi}{r^{2(c+c^2)}\{1+k_1 r^{2+2c}\}^2},$$

where χ is a constant

Using the line-element (1), we get the radial velocity

$$\frac{dr}{dt} = \pm\left[1 - \frac{r^{2(c+c^2)}\{1+k_1 r^{2+2c}\}^2}{\chi^2}\right]^{1/2}. \qquad \ldots(6)$$

This shows that $\frac{dr}{dt}=0$ when $r_0^{c+c^2}\left\{1+k_1 r_0^{2(1+c)}\right\}=\chi$ and then the particle will experience an inward force [according to equation (6)] and it will begin its return journey.

(*iii*) When the particle is rotating about z–axis we have

$$\frac{dr}{ds} = \frac{dz}{ds} = 0$$

For the line-element

$$ds^2 = e^{\alpha}(dt^2 - dr^2) - r^2 e^{\beta} d\phi^2 - e^{\gamma} dz^2 \qquad \ldots(7)$$

the geodesic equation is

$$\Gamma^1_{22}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dt}{ds}\right)^2 = 0. \qquad \ldots(8)$$

Equation (8) gives

$$-\frac{2r+r^2\beta'}{2\,e^{\alpha-\beta}}\left(\frac{d\phi}{dt}\right)^2 + \frac{\alpha'}{2} = 0, \qquad \ldots(9)$$

where $\alpha' = \frac{d\alpha}{dr}$ and $\beta' = \frac{d\beta}{dr}$. Using (7) in the equation (9), we have

$$\left(\frac{ds}{dt}\right)^2 = e^{\alpha}\left(1 - \frac{r\alpha'}{2+r\beta'}\right). \qquad \ldots(10)$$

Equation (10) shows that the condition that the geodesic will be time-like is

$$\frac{r\alpha'}{2+r\beta'}<1. \qquad \ldots(11)$$

Now from the equation (1) and condition (11), we get

$$r<\left[\frac{1-c}{k_1(c+3)}\right]^{\frac{1}{2c+2}} \qquad \ldots(12)$$

(*vi*) Motion in $z=0$ plane

$$\frac{dz}{ds}=0.$$

From equation (11), we have

$$\frac{d^2z}{ds^2}=0.$$

Therefore the motion will always be in this plane.

From equations (3) and (5) we have

$$\frac{d\phi}{ds}=\frac{(1+k_1 r^{2+2c})^2 h}{r^{2(1+c)}} \text{and} \frac{dt}{ds}=\frac{\chi}{r^{2(c+c2)}(1+k_1 r^{2+2c})^2}$$

Using these values of $\frac{dt}{ds}$ and $\frac{d\phi}{ds}$ we get from equation (1) the differential equation for the orbit

$$\left(\frac{du}{d\phi}\right)^2=\frac{1}{h^2}\left[\chi^2 u^{4c^2}\left\{1+k_1 u^{-2-2c}\right\}^{-8}-u^{2c^2-2c}\left\{1+k_1 u^{-2-2c}\right\}^{-6}\right]$$

$$-u^{2+c2^2}\{1+k_1 u^{-2-2c}\}^{-4}, \qquad \ldots(13)$$

where $u=\frac{1}{r}$.

The differential equation for the light rays is obtained under the condition $ds=0$

$$\left(\frac{du}{d\phi}\right)^2=\frac{\chi^2}{h^2}u^{4c^2}\left\{1+k_1 u^{-2-2c}\right\}^{-8}-u^{2+2c^2}\left\{1+k_1 u^{-2-2c}\right\}^{-4}. \qquad \ldots(14)$$

II. *A straight wire carrying current*—The line-element we use here is due to Safko and Witten (1972) and is given by

$$ds^2=r^{2c^2-2c}(k_2+r^{2c})^2(dt^2-dr^2)-r^{2-2c}(k_2+r^{2c})^2 d\phi^2$$

$$-r^{2c}(k_2+r^{2c})^{-2}dz^2, \qquad \ldots(15)$$

where c is related to the mass per unit length and k_2 is associated with the magnetic field produced by the current in the wire.

Since the line-element (15), is similar in structure to the time-element (1), the geodesic equations for (15) can be studied in a similar manner. We only record below the final results.

(*a*) When the test particle is at rest, we get the force experienced by a particle at rest.

$$\frac{d^2r}{dt^2}=-\left[\frac{c^2-c}{r}+\frac{2\,c\,r^{2c-1}}{k_2+r^{2c}}\right] \qquad ...(16)$$

(*b*) In radial motion; we have

$$\frac{dt}{ds}=\frac{\chi}{r^{2(c^2-c)}\,(k_2+r^{2c})^2}\,.$$

Using the line-element (15), we get the radial velocity

$$\frac{dr}{dt}=\left[1-\frac{r^{2c^2-2c}(k_2+r^{2c})^2}{\chi^2}\right]^{1/2}.$$

This shows that $\frac{dr}{dt}$ is zero when $r_0^{c^2-c}(k_2+r_0^{2c})=\chi$ and then the particle experience an inward force (according to equation (16) and it will will begin its return journey.

(*c*) Also in circular motion about z-axis, the condition (11) can be applied. From line-element (15) and condition (11), we have

$$(c^2-1)\,(r^{2c}+k_2)<0. \qquad ...(17)$$

(*d*) Relating to motion in $z=0$ plane, we have

$$\left(\frac{d\phi}{ds}\right)=\frac{h}{r^{(21-c)}\,(k_2+r^{2c})^2}\ \text{and}\ \left(\frac{dt}{ds}\right)=\frac{\chi}{r^{2(c^2-c)}\,(k_2+r^{2c})^2}\,.$$

Using the values in equation (15), we get the differential equation of the orbit as

$$\left(\frac{du}{d\phi}\right)^2=\frac{1}{h^2}\left[\chi^2u^{4c^2}-u^{2c+2c^2}\left(k_2+u^{-2c}\right)^2\right]-u^{2+2c^2}, \qquad ...(18)$$

where $u=\frac{1}{r}$ and for the light ray the orbit is obtained under the condition $ds=0$,

$$\left(\frac{du}{d\phi}\right)=\frac{\chi^2}{h^2}\,u^{4c^2}-u^{2+2c^2}\,. \qquad ...(19)$$

Discussion and Conclusions

We discuss here (*A*) the case of a test particle at rest [case (*i*)] and (*B*) the case of a test particle rotating in a circular path in the fields of a solenoid and a straight wire carrying current [case (*iii*)].

(A) In the case of a solenoid carrying current we find from (6) that the expression within the brackets is positive. This shows that the attractive force on a test particle at rest is effective at all distances from the solenoid.

On the other hand (16) shows for a straight wire carrying current that the attractive force on a test particle is effective at distances given by

$$r^{2c} > \left(\frac{1-c}{1+c}\, k_2\right) \qquad \ldots(20)$$

Evidently r can have all possible values for $c>1$. But for $c<1$, the radius of the straight wire should be equal to or greater than the critical value

$$r_0 = \left(\frac{1-c}{1+c} k_2\right)^{1/2c} \qquad \ldots(21)$$

so that the attractive force may be effective at all distances outside the wire.

(B) In the case of a test particle moving around a solenoid carrying current if $c<1$, then the radius of the circular motion should according to equation (12) be less than the value

$$r_0^2 = \left(\frac{1-c}{3+c}\, \frac{1}{k^1}\right)^{1/2(1+c)} .$$

It is evident from equation (12) that no circular motion is possible around the solenoid if c is greater than unity. In the case of a test particle moving around a straight wire the condition (17) has to be satisfied. Obviously no circular motion is possible for $c>1$.

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REFERENCES

Krori, K. D., and Barua, J. (1975). Escape of photons from magnetised cylinders. *J. Phys.*, *A*, **8**, 186.

Safko, J. L., and Witten, L. (1972). Models of static, cylindrically symmetric solutions of the Einstein—Marwell field. *Phys. Rev.*, D, **5**, 293-300.