NON-LINEAR STABILITY OF A WEAKLY IONISED CYLINDRICAL PLASMA

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An asymptotic non-linear theory of a weakly ionised plasma in a magnetic field is presented using the vector eigen-function expansion method. The inclusion of non-linear terms leads to a limiting value of the amplitude (A). It is also found that if the amplitude of the initial value of the disturbance is greater than (A), a super critical instability sets in a region which is stable in the linear theory.

INTRODUCTION

The stability of a weakly ionised plasma with a toriodial magnetic field has been examined by Kadomstev (1965) and Simon (1968). An instability similar to the flute type in a fully ionised plasma, sets in when the magnetic field increases beyond a certain determinate critical value H_c. Eckhaus (1965) has developed a method of studying non-linear stability problems in hydrodynamics by using eigen function expansion. The method has been used successfully by DiPrima (1967) who has studied the non-linear growth of Taylor vortices in the flow between rotating cylinders. We use Eckhaus method to study the stability of weakly ionised cylindrical plasma in a toriodial magnetic field. The problem is formulated in terms of a matrix partial differential equations with quadratic non-linearities. The amplitude of instability takes on a limiting value, when the quadratic non-linear terms are included in the stability theory. The procedure adopted in the present analysis is mathematically a formal one and is considerably different from the earlier work and does show the formulation of this plasma stability problem as an initial value problem and provides some details about the class of initial disturbances that can be considered.

EQUILIBRIUM AND LINEARISED THEORY

Consider a plasma occupying the regions between two earthed conducting cylinders of radii R_1 and R_2, with constant density (Simon 1968).

S and $S-s$, $(s<S)$ in the inner and outer cylinders respectively. The plasma is *imbedded* in a study toroidal magnetic field $H_\theta = (\alpha/r)$, where α is a constant, which is assumed to be so strong that

$(\Omega\tau)^2 \gg 1$ for electrons and ions. Here Ω and $(1/\tau)$ are the gyrofrequency and average collision frequency respectively with the back ground gas. For the problem at hand the equations of motion lead to following equation in cylindrical coordinates (Simon 1968),

$$\frac{\partial n}{\partial \tau}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-D\frac{\partial n}{\partial r}\pm nbE_r\right)\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left(-\frac{D}{r}\frac{\partial n}{\partial \theta}\pm nbE_\theta\right)$$
$$+\frac{\partial}{\partial z}\left(-D\frac{\partial n}{\partial z}+nbE_z\right)\pm\left(\Omega\tau\right)\frac{\partial}{\partial z}\left(-D\frac{\partial n}{\partial \gamma}\pm nbEr\right)$$
$$-\frac{\partial}{dr}\left(-D\frac{\partial n}{\partial z}\pm nbE_z\right)=0, \qquad \ldots(1)$$

where

$$D=\frac{T\tau}{m(1+\Omega^2\tau^2)},\ b=\frac{|e|\tau}{m(1+\Omega^2\tau^2)},\ \Omega=\frac{e|H|}{mc}. \qquad \ldots(2)$$

The temperature T is assumed spatially uniform, and all the other symbols have their usual meanings. Eqns. (1), (2) consist of two equations, one for the electrons and other for the ions. We assume quasi-neutrality and the electric field is assumed to be electrostatic. Let the equilibrium density be $N(r)$ and $E=-(\partial\phi/\partial r)$. The solution of eqn. (1) using the boundary condition viz. : (*i*) $N(R_1)=S$ and $N(R_2)=S-s$, (*ii*) $\Phi=0$ at $r=R_1$ and $r=R_2$, we get

$$N(x)=\delta\left[1-\frac{2(2+\delta^2)s}{S}\left(1-\frac{4(2-\delta)^2}{(1+\delta x)^2}\right)\right],$$
$$\Phi=0, \qquad \ldots(3)$$

where

$$r=R_0+xd,\ 2R_0=R_1+R_2,\ d=R_2-R_1,\ \delta=d/R_0. \qquad \ldots(4)$$

Let $n'(x, z, t)$ and $\phi'(x, z, t)$ denote the perturbation in the density and electrostatic potential. We shall only consider the interchange modes with no variation along the magnetic field as these modes are the ones which are most unstable. Considering a general periodic disturbance with period $2\pi/a$. We may write

$$n'(x, \xi, t)=\sum_{n=-\infty}^{\infty}\exp\left(-ina\xi\right)\rho_n(x, t), \qquad \ldots(5)$$

where

$$\rho_n(x, t)=\frac{a}{2\pi}\int_{-\pi/a}^{\pi/a}\exp\left(ina\xi\right)\rho(x, \xi, t)\,d\xi. \qquad \ldots(6)$$

If we use eqns. (5) – (6) into eqn. (1), we obtain in small gap approximation (i.e., $\delta\to 0$) :

$$\left(\frac{\partial}{\partial\tau}M_n+L_n\right)Q_n=S_n,\ n\geqslant 0,$$

where

$$L_n=\begin{bmatrix} -\Delta-2ina\delta\Omega_i\tau_i & -\Delta+ina\frac{s}{S}\Omega_i\tau_i \\ -\frac{D_e}{D_i}\Delta+2ina\,\delta\Omega_e\tau_e & -\frac{b_e}{b_i}\Delta+ina\frac{s}{S}\Omega_e\tau_e \end{bmatrix},\qquad Q_n=\begin{pmatrix}\rho_n(x,\tau)\\ \phi_n(x,\tau)\end{pmatrix},$$

$$D^2=\frac{d^2}{dx^2},\quad t=\frac{d^2}{D_i}\tau,\quad z=\xi d,$$

$$\Delta=(D^2-n^2a^2),\ \phi'=\frac{b_i\phi}{d^2D_i}\ ;\ M_n=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

The column vector S for $n=0,1$ and $n\geqslant 2$ is given by

$$S_o=\begin{Bmatrix} h_0^{(1)}+\sum_{q=1}^{\infty}\left(h_0^{(4)}-ia\Omega_i\tau_ih_0^{(7)}\right) \\ -\frac{b_e}{b_i}h_0^{(1)}+\sum_{q=1}^{\infty}-\frac{b_e}{b_i}\left(h_0^{(4)}+ia\Omega_e\tau_eh_0^{(7)}\right) \end{Bmatrix},$$

$$S_1=\begin{Bmatrix} h_1^{(2)}-\Omega_i\tau_ih_1^{(5)}+\sum_{q=1}^{\infty}\left(h_1^{(4)}-ia\Omega_i\tau_ih_1^{(7)}\right) \\ -\frac{b_e}{b_i}h_1^{(2)}-\frac{b_e}{b_i}\Omega_e\tau_eh_1^{(5)}+\sum_{q=1}^{\infty}-\frac{b_e}{b_i}\left(h_1^{(4)}+ia\Omega_e\tau_eh_1^{(7)}\right) \end{Bmatrix}, \qquad \ldots(7)$$

and

$$Sn=\begin{Bmatrix} h_n^{(2)}-\Omega_i\tau_ih_n^{(5)}+\sum_{q=1}^{n-1}\left(h_n^{(3)}-ia\Omega_i\tau_ih_n^{(6)}\right)+\sum_{q=1}^{\infty}\left(h_n^{(4)}-ia\Omega_i\tau_ih_n^{(7)}\right) \\ -\frac{b_e}{b_i}h_n^{(2)}-\frac{b_e}{b_i}\Omega_e\tau_eh_n^{(5)}+\sum_{q=1}^{\infty}-\frac{b_e}{b_i}\left(h_n^{(3)}+ia\Omega_e\tau_eh_n^{(6)}\right)+ \\ \sum_{q=1}^{\infty}-\frac{b_e}{b_i}\left(h_n^{(4)}+ia\Omega_e\tau_eh_n^{(7)}\right) \end{Bmatrix},$$

where

$$h_0^{(1)}=D(\rho_0D\phi_0),\quad h_n^{(2)}=D(\rho_nD\phi_0+\rho_0D\phi_n)-n^2a^2\phi_n\rho_0,$$

$$h_n^{(3)}=D(\rho_qD\phi_{n-q})-a^2n\left(q\overline{\phi}_q\rho_{n+q}-(n+q)\phi_{n+q}\overline{\rho}_q\right),$$

$$h_n^{(4)} = D(\rho_{n+q} D\overline{\phi}_q + \rho_q D\phi_{n+q}) + a^2 n(q\overline{\phi}_q \rho_{n+q} - (n+q)\phi_{n+q}\rho_q),$$

$$h_n^{(5)} = -ina[\phi_n(D+2\delta)\rho_0 - \rho_n D\phi_0)],$$

$$h_n^{(6)} = (n-q)[(D+2\delta)\rho_q\phi_{n-q} - \rho_{n-q} D\phi_q]$$

and

$$h_n^{(7)} = q\overline{\phi}_q(D+2\delta)\rho_{n+q} - (n+q)\phi_{n+q}(D+2\delta)\rho_q$$
$$- q\overline{\rho}_q D\phi_{n+q} + (n+q)\rho_{n+q} D\overline{\phi}_q.$$

The boundary conditions are

$$\rho_0 = 0,\ \phi = 0;\ \rho_n = \phi_n = D\phi_n = 0 \text{ at } x = \pm\tfrac{1}{2},\ n \geqslant 1, \qquad \ldots(8)$$

and the initial conditions are

$$\rho_0(x, 0) = \rho_0^{(0)}(x),\ \phi_0(x, 0) = \phi_0^{(0)}(x);\ \rho_n(x, 0) = \rho_n^{(0)}(x),\ \phi_n(x, 0) = \phi_n^{(0)}\, x. \qquad \ldots(9)$$

As the operators L_n and M_n are independent of time, the eigen-value problem admits solution of the form

$$Q_n(x, \tau) = \sum_{m=0}^{\infty} A_{nm}\, e^{-\sigma_{nm}\tau} \begin{pmatrix} 1 \\ \beta_{nm} \end{pmatrix} \sin(m+1)\pi x, \qquad \ldots(10)$$

where

$$Re(\sigma_{nm}) = \frac{\left(b_i D_e + b_e D_i\right)\left[\left\{\left(m+1\right)^2 \pi^2 + n^2 a^2\right\}^2 - 2\delta\left(na\right)^2 \Omega_i\tau_i\Omega_e\tau_e \frac{s}{S}\right]}{D_i(b_i + b_e)[(m+1)^2\pi^2 + n^2a^2]}, \qquad \ldots(11)$$

$$I_m(\sigma_{nm}) = \frac{sna(\Omega_i\tau_i D_e b_i - \Omega_e\tau_e D_i b_e) + 2\delta na(\Omega_i\tau_i b_e D_i - \Omega_e\tau_e b_i D_e)}{SD_i(b_i + b_e)},$$

and

$$\beta_{nm} = \left[D_e - D_i + \frac{2i\delta na(\Omega_i\tau_i D_i + \Omega_e\tau_e D_e)}{(m+1)^2\pi^2 + n^2a^2}\right] \frac{b_i}{D_i(b_e + b_i)} \qquad \ldots(12)$$

We note that σ_{0m} is always real. The eigen-values, are obtained under the assumptions that there exists an infinite denumerable sequence of eigen values σ_{nm} with no cluster point in the finite plane and eigen values may be ordered in such a way that

$$R_e(\sigma_{n,\, m+1}) > R_e(\sigma_{n,\, m}),\ m \geqslant 0. \qquad \ldots(13)$$

The real part of σ_{nm} becomes negative at large values of magnetic field resulting in an instability. The neutral point occurs at

$$\left(\Omega_i\tau_i\right)\left(\Omega_e\tau_e\right)=\frac{s(\pi^2+a^2)^2}{2a^2S\delta}, \qquad \ldots(14)$$

where the critical point of instability is chosen for the critical value of the parameter 'a' with $n=1$. The right hand side of eqn. (14) is minimum for $a_{cr}=\pi$. Hence the critical value of H is defined by

$$H_c^2=\frac{c^2m_im_e}{|e|^2\tau_i\tau_e}\left(\frac{2\pi S}{\delta s}\right). \qquad \ldots(15)$$

For the second harmonic component ($n=2$) H_c is twice as large. As $R_e(\sigma_{10})<0$ for $H \geqslant H_c$, we have instability and linear convection develops in the plasma. It is to be noted from eqns. (10) and (15) that

$Q_n \to 0$ as $\tau \to \infty$ for $n \geqslant 2$.

Non Linear Stability

We will now examine the effect of quadratic non-linearity of instability using the eigen functions of the linear stability problem. The linear theory predicts $a_{cr}=\pi$ and $H=H_c$ at which instability occurs. If the point (a, H) is in the neighbourhood of (a_c, H_c), that is, if $|a-a_c|$ and $|H-H_c|$, are small, than $R_e(\sigma_{10})\to 0$ as $(a, H)\to(a_c, H_c)$. We will construct the asymptotic solutions of eqns. (7) with respect to small parameter ϵ, valid for $\epsilon\to 0$.

We consider only disturbances which are of $O(\epsilon)$. It is convenient to use the transformation,

$$Q_n=\epsilon\delta_n\psi_n, \qquad \ldots(16)$$

where ψ_n are $O(1)$ and δ_n is a suitable scale factor $|\delta_n|\leqslant 1$ and ϵ is to be taken as the level of perturbation.

We now expand

$$\psi_n(x,\tau)=\sum_{m=0}^{\infty}A_{nm}(\tau)\ q_{nm}(x).=\sum_{m=0}^{\infty}A_{nm}(\tau)\begin{pmatrix}\rho_{nm}\\ \phi_{nm}\end{pmatrix}. \qquad \ldots(17)$$

On substituting eqn. (17) into eqn. (7), taking the inner product with q^{+}_{nm}, we obtain an infinite system of ordinary non linear differential equations

$$\frac{d}{d\tau}A_{nm}+\sigma_{nm}A_{nm}=\left(S_n\,|q^{+}_{nm}\right), \qquad \ldots(18)$$

subject to initial conditions given by eqn. (9). On carrying out systematic asymptotic analysis of above equation, Eckhaus (1965) we obtain

$$\delta_0=\delta_2=\epsilon,\ \delta_1=1 \quad \text{and } \delta_n=\epsilon^{|n-1|} \text{ for } n\geqslant 3. \qquad \ldots(19)$$

The scaling can be explained on physical grounds. The fundamental mode has no life of its own and the harmonics are generated by the results of fundamentals through quadratic non-linearities. The first harmonic and correction to mean motions are O (ϵ^2) and the second and higher harmonics are of order greater than O (ϵ^2).

Consider eqn. (7) for the fundamental mode $n=1$, using eqns. (16), (17) and (19), expanding in term of the small parameter ϵ , we get

$$\left(L_1 + \frac{d}{d\tau}M_1\right)Q_1 = \epsilon^2 S_{10} + O(\epsilon^4),$$

where S_{10} is a column matrix $\left(l_1 + ia\Omega_i\tau_i l_2; \; -\frac{b_e}{b_i}(l_1 - ia\Omega_e\tau_e l_2) \right)$,

$$l_1 = D[\rho_{10}D\phi_{00} + \rho_{00}D\phi_{10} + \rho_{20}D\bar{\phi}_{10} + \bar{\rho}_{10}D\phi_{20}] + a^2[2\bar{\phi}_{10}\rho_{20} - 2\bar{\rho}_{10}\phi_{20} - \phi_{10}\rho_{00}], \quad \ldots(20)$$

$$l_2 = [\phi_{10}(D+2\delta)\rho_{00} - \rho_{10}D\phi_{00} - \bar{\phi}_{10}(D+2\delta)\rho_{20} + 2\phi_{20}(D+2\delta)\bar{\rho}_{10} + \bar{\rho}_{10}D\phi_{20} - 2\rho_{20}D\bar{\phi}_{10}].$$

On taking the inner product with $q^+_{1m}(x)$, we get

$$\frac{dA_{1m}}{d\tau} + \sigma_{1m}A_{1m} = \epsilon^2 \left(S_{10} \,|\, q^+_{1m} \right) + O(\epsilon)^4. \quad \ldots(21)$$

For the case $m=0$, eqn. (21) leads to

$$\frac{d}{d\tau}|A_{10}|^2 + 2R_e(\sigma_{10})|A_{10}|^2 = \epsilon^2[\bar{A}_{10}B + \bar{B}A_{10}], \quad \ldots(22)$$

where

$$B = \left(S_{10} | q^+_{10} \right) = \left[l_1 \begin{pmatrix} 1 \\ -\dfrac{b_e}{b_i} \end{pmatrix} + l_2 \begin{pmatrix} ia\Omega_i\tau_i \\ ia\dfrac{b_e}{b_i}\Omega_e\tau_e \end{pmatrix} \middle| \begin{pmatrix} \sin \pi x \\ \sin \pi x \end{pmatrix} \right] \quad \ldots(22a)$$

which suggests,

$$\epsilon^2 = |R_e(\sigma_{10})| \quad \ldots(23)$$

In order to calculate the right hand side of eqn. (21), we have to find the column matrices $\psi_0=(\rho_{00}, \phi_{00})$ and $\psi_2=(\rho_{20}, \phi_{20})$ respectively. These are obtained by eqn. (7) by putting $n=0$, 2 and on using eqns. (16), (17) and (19), we find that

$$\psi_0(x, \tau) = \epsilon^2 |A_{10}(\tau)|^2 \sum_{m=0}^{\infty} \frac{\left(S_{00}(x) \,|\, q^+_{1m}(x) \right)}{\sigma_{om}} q_{1m}(x) + O(\epsilon)^4 \quad \ldots(24)$$

$$\psi_2(x,\tau)=\epsilon^2[A_{10}(\tau)]^2\sum_{m\ 0}^{\infty}\frac{\left(S_{00}(x)\,|\,q_{2m}^{+}(x)\right)}{\sigma_{2m}-2i\,I_m(\sigma_{10})}q_{2m}(x)+O(\epsilon)^4 \quad ...(25)$$

where

$$S_{00}(x)=\begin{bmatrix} l_3-i\Omega_i\tau_i l_4 \\ -\frac{b_e}{b_i}\left(l_3+i\Omega_e\tau_e l_4\right)\end{bmatrix},\quad S_{20}(x)=\begin{bmatrix} l_5-i\Omega_i\tau_i l_6 \\ -\frac{b_e}{b_i}\left(l_5+i\Omega_e\tau_e l_6\right)\end{bmatrix} \quad ...(26)$$

$$l_3=D(\rho_{10}D\bar{\phi}_{10}+\bar{\rho}_{10}D\phi_{10}),\quad l_4=(D+2\delta)(\bar{\phi}_{10}\rho_{10}-\bar{\rho}_{10}\phi_{10}), \quad ...(27)$$

$$l_5=D(\rho_{10}D\phi_{10})-2a^2\rho_{10}\phi_{10},\quad l_6=(D+2\delta)(\rho_{10}\phi_{10}), \quad ...(28)$$

On using eqn. (17) into eqns. (24)–(25), we get on evaluating the inner products

$$\psi_0(x,\tau)=|A_{10}(\tau)|^2G_0(x),$$
$$\psi_2(x,\tau)=[A_{10}(\tau)]^2G_2(x), \quad ...(29)$$

where

$$G_0(x)=\pi\left(\frac{S}{s}\right)\begin{pmatrix}\sin 2\pi x\\ \sin 2\pi x\end{pmatrix},\qquad G_2(x)=0.$$

On substituting eqns. (29) to (30) into eqn. (22*a*), and using eqn. (14), we obtain

$$R_e(B)=\frac{2\pi^4(b_iD_e+b_eD_i)}{D_i(b_i+b_e)}\left(\frac{S}{s}\right)^2A_{10}\,|\,A_{10}\,|^2, \quad ...(31)$$

and

$$I_m(B)=\frac{\pi^3}{2}\left(\frac{S}{s}\right)\left[\frac{(D_e-D_i)b_i}{D_i(b_i+b_e)}-1\right]\left(\Omega_i\tau_i+\Omega_d\tau_e\frac{b_e}{b_i}\right)A_{10}\,|\,A_{10}\,|^2. \quad ...(30)$$

From the eqn. (21) with $m=0$, we obtain the Landau's equation of stability,

$$\frac{dA_{10}}{d\tau}+\sigma_{10}A_{10}=\epsilon^2B\quad ,$$

where

$$\epsilon^2=|\,R_e(\sigma_{10})\,|\,. \quad ...(32)$$

Eqn. (32) with $m=0$ admits harmonic solutions of the form

$$A_{10}(\tau)=A\exp(i\omega\tau), \quad ...(33)$$

where $|A|^2=\frac{1}{R_e(B)}\left(\frac{R_e\sigma_{10}}{\epsilon^2}\right)+O(\epsilon)^2$

$$=\frac{(H/H_c-1)\,sD_i(b_i+b_e)}{2\pi^4S\,|\,H/H_c-1\,|\,(D_ib_e+D_eb_i)}, \quad ...(34)$$

$$\omega=-I_m(\sigma_{10})-\frac{\epsilon^2A^2\pi^3S}{2s}\left(\Omega_i\tau_i+\Omega_e\tau_e\frac{b_e}{b_i}\right)\left(\frac{(D_e-D_i)b_i}{D_i(b_i+b_e)}-1\right) \quad ...(35)$$

The solution given by eqn. (34), exists if $R_e(B)$ and $R_e(\sigma_{10})$ have the same sign and from eqn. (34), we have

$$\epsilon^2 |A|^2 = \frac{1}{\pi^2} \left(\frac{s}{S}\right)^2 \left(H/H_c - 1\right). \qquad ...(36)$$

Eqn. (21), on integration gives

$$A_{10} = \frac{a_0 \exp\{-R_e(\sigma_{10})\tau\}}{\left[1 + \frac{|a_0|^2}{|A|^2}\left\{\exp\left(-2R_e(\sigma_{10})\tau\right) - 1\right\}\right]^{1/2}}, \qquad ...(37)$$

where a_0 is the initial value of the amplitude.

An analysis in phase plane shows that when permanent solution (34) exists for $R_e(\sigma_{10}) < 0$, then $\lim_{\tau \to \infty} |A_{10}|^2 = A^2$. The static case $a_0 = 0$ is unstable but for $a_0 \neq 0$ this dynamic case possesses a strong stability. The disturbance which grows exponentially in linear theory tends to solution

$$Q(x, \xi, \tau) = 2\epsilon A R_e\left[\exp\{-i(na\xi - \omega\tau)\}\right] + \epsilon^2 A^2 G_0(x) + O(\epsilon)^4 \qquad ...(38)$$

$$as\ \tau \to \infty$$

The saturation value of the amplitude in non-linear regime is given by eqn. (34) and is proportional to $(H/H_c - 1)^{1/2}$.

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