HYDROMAGNETIC DISTURBANCES IN A CONDUCTING SHALLOW LIQUID DUE TO TRANSIENT EXCITING FORCE

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The present paper is concerned with the study of magnetohydrodynamic disturbances caused by a transient exciting force in a conducting shallow-liquid. A solution is obtained by using the technique of integral transforms. It is found that the disturbances are damped exponentially enroute and lag behind their input value at $x=0$ by a time $t_1=\frac{x}{v}$. The effect of magnetic field on propagation of the disturbances is also discussed. The problem considered here turns out to be closely analogous to that of a uniform semi-infinite transmission line with zero initial current and charge where the E.M.F. $f(t)$ is applied at $x=0$ for $t>0$.

INTRODUCTION

FREE surface magneto-hydrodynamic phenomena have been considered by several investigators. Fraenkel (1960) discussed many shallow-liquid hydromagnetic problems including the wave motion analysis. Alpher, Hurwitz *et al.* (1960) considered surface wave motions along with other magneto-hydrodynamic flows. Later on, Baral (1969) extended their analysis to study the disturbances in a conducting shallow-liquid permeated by a transverse magnetic field.

Baral's solution was derived by using the Laplace transform method but he could furnish the solution of the problem only for large values of time due to the inversion difficulty involved. His result is therefore valid only after a certain time and not instantaneously. The technique as used by him does not take account of the initial condition assumed for the flow variable u and an extra condition has been used for the purpose. Moreover, the problem has been worked out without explaining the resultant physical picture.

Keeping in view the above mentioned discrepancies, we propose to reinvestigate the problem considered by Baral (1969). This problem, as formulated here, differs from his in the time character of the input function. Although the analysis of the present paper is carried out by using the Laplace transform in conjunction with Fourier sine transform, the essential results are equally valid for instantaneous motions also. The approach is direct and simple and is different from that of Baral. One feature of the method is that it demonstrates the effect of magnetic field on the qualitative nature of the solutions.

It is found that the effect of the magnetic field is to :
(*a*) introduce a damping factor; and (*b*) produce distortion in the transmission. It is observed that the disturbance is propagated with a finite velocity and its amplitude falls exponentially with distance from the boundary.

One of the purposes of the present paper is to point out that the solution obtained here has its application to another problem of a uniform semi-infinite transmission line with zero initial current and charge when the E.M.F. $f(t)$ is applied at $x=0$ for $t>0$ (Carslaw & Jaeger 1953).

Required Integral Transforms

The Laplace transform of any arbitrary function $f(t)$ is defined as

$$\bar{f}(S)=\int_0^\infty e^{-St} f(t)\, dt, \quad R_e S>0 \qquad \text{...(1)}$$

provided that the above integral exists.
Also, let the Fourier sine transform of $\varphi(x)$ be

$$F(\xi)=\int_0^\infty \varphi(x) \sin \xi x\, dx, \qquad \xi>0. \qquad \text{...(2)}$$

Then, we have (Churchill 1972)

$$\varphi(x)=\frac{2}{\pi}\int_0^\infty F(\xi) \sin \xi x\, d\xi, \qquad x>0. \qquad \text{...(3)}$$

Formulation of the Problem

Consider an infinite mass of conducting fluid which is initially at rest in the equilibrium position. We suppose the depth h of the fluid to be constant and that it extends to infinity in the horizontal direction. The disturbances are supposed to proceed in the direction of x' axis along horizontal surface of the conducting fluid due to a transient exciting force at $x'=a$. A magnetic field $\vec{B}$ is imposed in the fluid in the direction transverse to that of propagation. Our problem is to find the surface displacement due to disturbances arising out of the interaction of the hydromagnetic and electromagnetic fields. Following Baral (1969), the equations governing the propagation of small disturbances in one dimension are

$$\frac{\partial u}{\partial t}=-g\left(\frac{\partial \eta}{\partial x'}\right)-\frac{\sigma B^2}{\rho C^2}u \qquad \text{...(4)}$$

and

$$\frac{\partial \eta}{\partial t}=-\left(\frac{\partial}{\partial x'}\right)\left(hu\right). \qquad \text{...(5)}$$

where

$u = u(x', t)$, the fluid velocity in the direction of x' axis ;

$\eta = \eta(x', t)$, the surface displacement from the undisturbed depth h ;

$\sigma =$ conductivity of the fluid ; and

$C =$ velocity of light

The boundary and initial conditions of the problem are

$$\left.\begin{array}{l}(i)\ u=0 \text{ at } t=0 \text{ for } x'\geqslant 0\\ (ii)\ \eta=0 \text{ at } t=0 \text{ for } x'\geqslant 0\end{array}\right] \quad \ldots(6)$$

$$\eta=f(t) \text{ at } x'=a \quad \text{for } t>0 \quad \ldots(7)$$

$$\lim_{x'\to\infty} [\eta(x',t)]\to 0 \text{ for } t>0 \quad \ldots(8)$$

Eqns. (4) and (5) constitute the displacement equation of the motion. Eqn. (6) reflects the state of rest at $t=0$ and eqn. (7) is the statement of the boundary condition at the surface for the dynamic part of the problem. Eqn. (8) is consistent in stating that there is no disturbance at infinity. It may be noted that eqns. (4) to (8) represent the boundary value problem whose solution is given in the section that follows.

Solution of the Problem

On taking Laplace transform defined by eqn. (1). the governing eqns. (4) and (5) subject to the initial conditions (6), transform to

$$S\bar{u}=-g\frac{d\bar{\eta}}{dx'}-\frac{\sigma B^2}{\rho C^2}\,\bar{u} \quad \ldots(9)$$

$$S\bar{\eta}=-h\,\frac{d\bar{u}}{dx'} \quad \ldots(10)$$

Eliminating $\bar{u}$ from the eqns. (9) and (10), we obtain

$$\nu^2\frac{d^2\bar{\eta}}{dx'^2}-S\left(S+\frac{1}{\tau_0}\right)\bar{\eta}=0, \quad \ldots(11)$$

where $\nu=(gh)^{\frac{1}{2}}$ is the classical gravity wave velocity and

$$\tau_0=\frac{\rho c^2}{\sigma B^2}.$$

Now we shift the origin at the point $x'=a$ and replace x' by $x+a$. The eqn. (11) then becomes

$$\nu^2\,\frac{d^2\bar{\eta}}{dx^2}-S\left(S+\frac{1}{\tau_0}\right)\bar{\eta}=0 \quad \ldots(12)$$

and the transformed boundary conditions are

$$\left.\begin{array}{ll}(i)\ \bar{\eta}=\bar{f}\,(S) & \text{at } x=0\\ (ii)\ \bar{\eta}\to 0 & \text{at } x\to\infty.\end{array}\right\} \quad \ldots(13)$$

To solve eqn. (12), we employ the Fourier sine transform defined by eqn. (2). Then, since $\bar{\eta}(0, S) = \bar{f}(S)$, we find that

$$\int_0^\infty \frac{d^2\bar{\eta}}{dx^2} \sin \xi x dx = \xi \bar{f}(S) - \xi^2 F(\xi, S)$$

provided $\bar{\eta}$ and $\frac{d\bar{\eta}}{dx}$ both tend to zero as $x \to \infty$, so that on multiplying eqn. (12) by $\sin \xi x$ and integrating with respect to x between limits $(0, \infty)$, we obtain

$$v^2[\xi \bar{f}(S) - \xi^2 F(\xi, S)] - S\left(S + \frac{1}{\tau_0}\right)F(\xi, S) = 0$$

or $\left(S^2 + \frac{S}{\tau_0} + \xi^2 v^2\right)F = v^2 \xi \bar{f}(S)$

or $F = \frac{\xi \bar{f}(S)}{\alpha^2 + \xi^2}$, ...(14)

where $\alpha^2 = \frac{1}{v^2}\left(S^2 + \frac{S}{\tau_0}\right)$.

The application of Fourier inversion theorm, mentioned in eqn. (3), to eqn. (14) gives

$$\bar{\eta} = e^{-\alpha x} \bar{f}(S) \qquad (\textit{vide } \text{Churchill 1972})$$

$$= e^{-x/v \left(S^2 + \frac{S}{\tau_0}\right)^{1/2}} \bar{f}(S) \qquad ...(15)$$

Finally, Laplace inversion of (15) leads to the result (Carslaw & Jaeger 1953)

$$\eta(x, t) = \begin{cases} 0 & \text{for } 0 \leqslant t < \frac{x}{v} \\ e^{-\beta x/v} f\left(t - \frac{x}{v}\right) + \beta \frac{x}{v} \int_{x/v}^{t} f(t-\tau)\, e^{-\beta\tau} \frac{I_1(\beta R)}{R} d\tau & \text{for } t > \frac{x}{v}, \end{cases} \qquad ...(16)$$

where I_1 is the modified Bessel function of the first kind,

$\beta = \frac{1}{2\tau_0}$ and $R = \left(\tau^2 - \frac{x^2}{v^2}\right)^{1/2}$.

Now we discuss the character of the motion exhibited by the above relation. We see that first term in the second row corresponds to a propagation of the input disturbance $f(t)$ in the direction of increasing x with v. The time

$t = \frac{x}{v}$ elapses before the first excitation occurs at x, i.e., until then $\eta(x, t) = 0$. This corresponds to the first row in the above result. As it propagates, the disturbance is damped by the factor $e^{-\beta x/v}$. The second term in the second row represents the sum of all values $f(0)$ to $f\left(t - \frac{x}{v}\right)$ of the driving function which have arrived at the point x upto time t and each value of the disturbance function is multiplied by a certain weight function. It means that at a point (x, t) all the values of input disturbance which have arrived previously and remain there as residues are superimposed. This supreposition of these residual disturbances causes a distortion in the transmission.

In the nonmagnetic case $\beta = 0$, the expression (16) reduces to

$$\eta(x, t) = \begin{cases} 0 & , \text{ for } 0 \leqslant t < \frac{x}{v} \\ f\left(t - \frac{x}{v}\right), & \text{for } t > \frac{x}{v}. \end{cases} \qquad \ldots(17)$$

This solution signifies a propagation of the disturbance with a finite velocity v in the direction of x increasing. At a fixed point only a single value of the input disturbance arrives at a definite time t. There is, therefore, no superposition of values.

Comparing the two cases discussed above, it may be noticed here that the magnetic field plays the role of a distortion agency. Its another function is to introduce a damping which increases with x.

Particular Cases

(*i*) *Motion under harmonic input* : Let us assume that

$$f(t) = \eta_o \sin \omega t, \qquad \ldots(18)$$

where η_0 and ω are positive constants. Substituting it in (16), the disturbance at any point x at time t is given by

$$\eta(x, t) = \begin{cases} 0 & \text{for } 0 \leqslant t < \frac{x}{v} \\ \eta_o \left[e^{-\beta x/v} \sin \omega\left(t - \frac{x}{v}\right) + \beta \frac{x}{v} \int_{x/v}^{t} \sin \omega (t - \tau) e^{-\beta\tau} \frac{I_1(\beta R)}{R} d\tau \right] & \text{for } t > \frac{x}{v} \end{cases} \qquad \ldots(19)$$

The first term in the second row shows that the disturbance at a point $x > 0$ is of the same form as was applied on the boundary $x=0$ except that it is damped by the factor $e^{-\beta x/v}$ and has the phase shift $\frac{\omega x}{v}$. The second term may be interpreted in a similar way as in the previous section.

In the non-magnetic case, eqn. (17) gives

$$\eta(x, t)=\begin{cases} 0 & , \text{ for } 0 \leqslant t < \dfrac{x}{v} \\ \eta_0 \sin \omega\left(t-\dfrac{x}{v}\right), & \text{ for } t > \dfrac{x}{v} \end{cases} \quad \ldots(20)$$

For a deeper insight into the behaviour of $\eta\,(x, t)$ under the influence of magnetic field, we shall consider the solution (19) for large values of time. In this case the analysis can be simplified by taking $f(t)=\eta_0 e^{i\omega t}$ at $x=0$ and finally selecting the imaginary part. The surface displacement at any point x at time t is therefore the imaginary part of

$$\eta(x, t)=\eta_0\left[e^{i\omega(t-x/v)-\beta x/v}+e^{i\omega t}\,\beta\,\frac{x}{v}\int_{x/v}^{t} e^{-(i\omega+\beta)\tau}\,\frac{I_1(\beta R)}{R}\,d\tau\right]. \quad \ldots(21)$$

When t is large, the value of the integral is sensibly that with t infinite. Writing $(\beta+i\omega)$ for S, the integral is a Laplace integral with t infinite and we have (Churchill 1972),

$$\beta\frac{x}{v}\int_{x/v}^{\infty} e^{-(\beta+i\omega)\tau}.\,\frac{I_1\left[\beta\left(\tau^2-\dfrac{x^2}{v^2}\right)^{1/2}\right]}{\left(\tau^2-\dfrac{x^2}{v^2}\right)^{1/2}}\,d\tau$$

$$=e^{-x/v\sqrt{(\beta+i\omega)^2-\beta^2}}-e^{-x/v(\beta+i\omega)}$$

$$=e^{-x/v(\beta+i\omega)\sqrt{1-(\beta/\beta+i\omega)^2}}-e^{-x/v(\beta+i\omega)}$$

or $I \simeq e^{-x/v(\beta+i\omega)\{1-\beta^2/2\,(\beta+i\omega)^2\}}-e^{-x/v(\beta+i\omega)}$

if $\omega \gg \beta$. Reducing still further and multiplying by $e^{i\omega t}$ yields

$$e^{i\omega t}\,I \simeq e^{-\beta x/v}\left[e^{i\omega\{t-x/v\,(1+\beta^2/2\omega^2)\}}-e^{i\omega(t-x/v)}\right] \quad \ldots(22)$$

Inserting the value of $e^{i\omega t} I$ from (22) into (21) leads to

$$\eta\,(x, t) \simeq \eta_0\, e^{-\beta x/v}\left[e^{i\omega\{t-x/v\,(1+\beta^2/2\omega^2)\}}\right] \quad \ldots(23)$$

when t is large enough. Taking the imaginary part, the disturbance at (x, t) approximates to

$$\eta\,(x, t)=\eta_0\, e^{-\beta x/v}\sin\left[\omega\left\{t-\frac{x}{v}\left(1+\beta^2/2\omega^2\right)\right\}\right] \quad \ldots(24)$$

From the relations (20) and (24), it is observed that the effect of magnetic field is to :

(*a*) introduce the damping factor $e^{-\beta x/v}$

(*b*) reduce the phase velocity in the ratio

$$\left(1+\frac{\beta^2}{2\omega^2}\right)^{-1}\simeq\left(1-\beta^2/2\omega^2\right)_{/1}.$$

Thus we see that the amplitude of the disturbance is damped exponentially with increasing distance x, while the phase velocity is reduced by an amount $\frac{\beta^2 v}{2\omega^2}$

(ii) Motion under exponentially decaying input : We now consider the case which has been studied by Baral (1969). Let us take

$$f(t)=\eta_0 e^{-\omega t}$$

Then eqn. (16) becomes

$$\eta(x,t)=\begin{cases}0 & \text{for } 0\leqslant t<\frac{x}{v}\\ \eta_0\left[e^{-\{\omega(t-x/v)+\beta x/v\}}+\beta\frac{x}{v}\int_{x/v}^{t}e^{-\{\omega(t-\tau)+\beta\tau\}}.\frac{I_1(\beta R)}{R}\,d\tau\right] & \text{for } t>\frac{x}{v}\end{cases} \quad ...(25)$$

Solution for large values of time may be obtained in this case from (23) on replacing ω by $i\omega$. Thus we have

$$\eta(x,t)\simeq\eta_0 e^{-\beta x/v}\left[e^{-\omega\{t-x/v(1-\beta^2/2\omega^2)\}}\right] \quad ...(26)$$

on shifting the origin back to the point $x=-a$ and replacing x by $x'-a$ the above result yields

$$\eta(x,t)\simeq\eta_0 e^{-\beta(x'-a)/v}\left[e^{-\omega\{t-(x'-a)/v(1-\beta^2/2\omega^2)\}}\right] \quad ...(27)$$

which gives value of η for large values of time in an explicit form.

It may be pointed out that Baral (1969) has obtained the corresponding result for large values of time in an integral form.

An Analogous Problem

If an electrical conductor is so large that its constants (resistance etc.) can not be thought of as being concentrated in a single place, then the voltage and current depend not only on t but also on the length coordinate x. Let us consider a uniform semi infinite transmission line, $x>0$, with zero initial current and charge and let V be the potential and I, the current at the point x of the line at time t. If R, L, C be the resistance, inductance and capacitance per unit length of the line, then V and I have to satisfy the differential equations.

$$L\frac{\partial I}{\partial t}+RI=-\frac{\partial V}{\partial x} \quad ...(28)$$

$$C\frac{\partial V}{\partial t}=-\frac{\partial I}{\partial x} \quad ...(29)$$

With the following initial and boundary conditions

$$(i)\ I=0,\ t<0,\ 0\leqslant x \qquad (ii)\ V=0,\ t<0,\ 0\leqslant x \quad \ldots(30)$$

$$V=f(t),\ \text{at}\ x=0,\ t>0 \quad \ldots(31)$$

$$V\to 0,\quad \text{as}\ x\to\infty,\ t>0. \quad \ldots(32)$$

Comparing the equations of this section with those under 'Formulation of the Problem', it may be seen that the problem of hydromagnetic disturbances in a conducting shallow liquid due to transient exciting force is analogous to that of a uniform semi-infinite transmission line, $x>0$, with zero initial current and charge when E.M.E. $f(t)$ is applied at $x=0$ for $t>0$. Hence the solution to this problem is identical with that of section 4 of the present study.

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