SOME AXISYMMETRIC BOUNDARY VALUE PROBLEMS FOR A THICK INFINITE PLATE CONTAINING A CIRCULAR HOLE

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The paper presents the eigen function solutions for the axisymmetric stress and displacement boundary value problems associated with a thick infinite plate containing a circular hole. Numerical results are presented for some particular stress and displacement problems.

INTRODUCTION

In a recent paper, Ramachandra Rao *et al.* (1976) have presented the eigen function solutions for the axisymmetric stress, displacement and mixed boundary value problems associated with a circular disc or a short cylinder. The radial eigen functions of the biharmonic equation have been developed in cylindrical polar coordinates and the prescribed boundary functions are satisfied using a biorthogonality relation. These eigen functions can also be effectively used to obtain the solutions of the various axisymmetric boundary value problems associated with a thick infinite plate containing a circular hole. The present paper therefore deals with the stress and displacement problems for a thick infinite plate with a hole. For the stress problem we study the end effects when a self equilibrating boundary load is prescribed on the circular hole. For the displacement problem, we study the problem of a thick infinite plate containing a built-in hole and subject to tension at infinity. The displacement problem also takes into account of the stress singularities which exist along the circumferential plane ends of the hole.

The mathematical analysis for the title problem follows on the lines indicated in (Ramachandra Rao *et al.* 1976). For this reason, we shall directly give some basic relations required for the solution of the stress and displacement boundary value problems.

STATEMENT OF PROBLEMS

We will consider a thick infinite plate with a circular hole occupying the region $r \geqslant a$, $| z | \leqslant h$. We assume that the plane faces $z = \pm h$ are stress free, i.e.,

$$\sigma_z = \tau = 0 \quad \text{on} \quad z = \pm h, r > a \qquad \ldots(1)$$

On the circular boundary $r = a$, we assume that any one of the following pairs of boundary functions are prescribed :

Stress Problem : $\sigma_r = \sigma_{rb}(z);$ $\tau = \tau_b(z);$ ...(2)

and

Displacement Problem : $u = \frac{1+\mu}{E} u_b(z);$ $W = \frac{1+\mu}{E} W_b(z),$ ...(3)

where the subscript b indicates a specified function on the boundary. The constants μ and E represent the Poisson's ratio and Young's modulus respectively.

The governing differential equations for the axisymmetric deformation may be written in terms of the Love strain function χ as

$$\Delta\Delta\chi = 0, \quad r \geqslant a, \quad |z| \leqslant \pm h, \qquad ...(4)$$

where

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

We record the following relations expressing the stresses and displacement in terms of the function χ, viz.,

$$\sigma_r = \frac{\partial}{\partial z}\left(\mu\Delta - \frac{\partial^2}{\partial r^2}\right)\chi, \qquad ...(5)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left(\mu\Delta - \frac{1}{r}\frac{\partial}{\partial r}\right)\chi, \qquad ...(6)$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\mu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi, \qquad ...(7)$$

$$\tau = \frac{\partial}{\partial r}\left[(1-\mu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi, \qquad ...(8)$$

$$\frac{E}{1+\mu} u = -\frac{\partial^2 \chi}{\partial r\,\partial z} \qquad ...(9)$$

and

$$\frac{E}{1+\mu} W = \left[2(1-\mu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi, \qquad ...(10)$$

where σ_r, σ_θ, σ_z represent the normal stresses, τ the non-zero shear stress, u the radial displacement and W the axial displacement.

General Formulation

Defining the functions

$$f^{(1)} = \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r}, \quad f^{(2)} = \frac{\partial^2\chi}{\partial z^2}, \quad f^{(3)} = \frac{\partial^2\chi}{\partial r\,\partial z} \qquad ...(11a—11c)$$

(11*a*) (11*b*) (11*c*)

and introducing a fourth function $f^{(4)}$ such that the biharmonic equation (1) is identically satisfied, we arrive at the following system of four partial differential equations of the first order :

$$\frac{\partial f^{(1)}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}[r f^{(3)}]; \quad \ldots(12a)$$

$$\frac{\partial f^{(2)}}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}[r f^{(3)} + r f^{(4)}]; \quad \ldots(12b)$$

$$\frac{\partial f^{(3)}}{\partial z} = \frac{\partial f^{(2)}}{\partial r}; \quad \ldots(12c)$$

and

$$\frac{\partial f^{(4)}}{\partial z} = \frac{\partial}{\partial r}[f^{(1)} + f^{(2)}]. \quad \ldots(12d)$$

Using relations (2) and (3), the relations connecting the boundary values of the functions $f_b^{(i)}$ with the boundary stresses and displacements may be expressed as follows:

$$f_b^{(1)} = \frac{\mu}{1-\mu} W_b - \frac{1-2\mu}{1-\mu}\int^z (\sigma_{rb} + u_b/a)\, dz. \quad \ldots(13a)$$

$$f_b^{(2)} = W_b + 2\int^z (\sigma_{rb} + u_b/a)\, dz. \quad \ldots(13b)$$

$$f_b^{(3)} = -u_b \quad \ldots(13c)$$

$$f_b^{(4)} = \frac{1}{1-\mu}\left[\int^z \tau_b\, dz - u_b\right] \quad \ldots(13d)$$

We seek the eigen function expansions of the functions $f^{(i)}$, $(i=1, 2, 3, 4)$ in the form

$$\begin{bmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \\ f^{(4)} \end{bmatrix} = \sum \frac{d_n}{K_1(\lambda_n a/h)} \begin{bmatrix} K_0(\lambda_n r/h) z_n^{(1)} \\ K_0(\lambda_n r/h) z_n^{(2)} \\ K_1(\lambda_n r/h) z_n^{(3)} \\ K_1(\lambda_n r/h) z_n^{(4)} \end{bmatrix} \quad \begin{matrix} \ldots(14a) \\ \ldots(14b) \\ \ldots(14c) \\ \ldots(14d) \end{matrix}$$

Substituting eqns. (14) in eqns. (12), it follows that the vector $\bar{z}_n$ satisfies the vector differential equation

$$\bar{z}'_n = -\frac{\lambda_n}{h} U . \bar{z}_n , \qquad \ldots(15a)$$

where prime denotes differentiation with respect to z and the matrix U is given by

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \ldots(15b)$$

For the stress free plane faces, the components of $\bar{z}_n$ may be written as

$$z_n^{(1)} = F_n ; \qquad \ldots(16a)$$

$$z_n^{(2)} = \left(\frac{h}{\lambda_n}\right)^2 F_n'' ; \qquad \ldots(16b)$$

$$z_n^{(3)} = -\frac{h}{\lambda_n} F_n' ; \qquad \ldots(16c)$$

and

$$z_n^{(4)} = \left(\frac{h}{\lambda_n}\right)^3 \left[F_n''' + \frac{\lambda_n^2}{h^2} F_n'\right], \qquad \ldots(16d)$$

where the solutions of F_n which are even in z are given by

$$_eF_n = (2\mu \cos \lambda_n h - \lambda_n h \sin \lambda_n h) \cos \lambda_n z + \lambda_n z \cos \lambda_n h \sin \lambda_n z \qquad \ldots(17a)$$

$$\sin 2\lambda_n h + 2\lambda_n h = 0 \qquad \ldots(17b)$$

and the solutions odd in z by

$$_oF_n = (2\mu \sin \lambda_n h + \lambda_n h \cos \lambda_n h) \sin \lambda_n z - \lambda_n z \sin \lambda_n h \cos \lambda_n z \qquad \ldots(18a)$$

$$\sin 2\lambda_n h - 2\lambda_n h = 0. \qquad \ldots(18b)$$

The roots of (17*b*) are given by Hilman and Salzer (1943) and that of (18*b*) by Robbins and Smith (1948).

Using (15), it can be shown that the vectors $\bar{z}_n$ admit the following biorthogonality relation viz.,

$$\int_{-h}^{+h} W_m^t . U . z_n \, dz = 0 \quad (m \neq n), \qquad \ldots(19)$$

where the components of the vector $\overline{W}_m$ are given by

$$W_n^{(1)} = G_n^{*'''} + 2\lambda_n^{*2} G_n^{*'} ; \quad W_n^{(2)} = \lambda_n^{*2} G_n^{*'}$$
$$(a) \qquad\qquad (b)$$

$$W_n^{(3)} = -\lambda_n^{*} G_n^{*'''} - \lambda_n^{*3} G_n^{*} \; W_n^{(4)} = \lambda_n^{*3} G_n^{*} \qquad \ldots(20a—20d)$$
$$(c) \qquad\qquad (d)$$

The even and odd solutions of G_n are given by

$${}_eG_n = [2(1-\mu)\cos\lambda_n h - \lambda_n h \sin\lambda_n h]\cos\lambda_n z + \lambda_n z \cos\lambda_n h \sin\lambda_n z \qquad \ldots(21a)$$

$${}_oG_n = [2(1-\mu)\sin\lambda_n h + \lambda_n h \cos\lambda_n h]\sin\lambda_n z - \lambda_n z \sin\lambda_n h \cos\lambda_n z \qquad \ldots(21b)$$

It may be noted that $_*$ denotes the complex conjugate and t the conjugate transpose. The summation in eqns. (14) is extended over the eigen values whose real part is positive. Using (19) it can be shown that the constants d_n involved in the expansions (14) are given by

$$d_n = \frac{1}{N_n}\int_{-h}^{+h} \Big[f_b^{(1)} W_n^{*(4)} + f_b^{(2)} \Big(W_n^{*(3)} + W_n^{*(4)} \Big)$$
$$+ f_b^{(3)} \Big(W_n^{*(1)} - W_n^{*(3)} \Big) - f_b^{(4)} W_n^{*(2)} \Big] dz , \qquad \ldots(22)$$

where

$$N_n = (1-\mu)\Big[1 + \frac{K_0(\lambda_n a/h)}{K_1(\lambda_n a/h)}\Big] \int_{-h}^{h} \Big[F_n G_n - \frac{h^4}{\lambda_n^4} F_n'' G_n'' \Big] dz.$$

Using the relations (13) and (20), the constants d_n may be written as

$$d_n = -\frac{(h/\lambda_n)^4}{N_n}\int_{-h}^{h} \Big[\Big\{ \Big(1-2\mu\Big) G_n''' - 2\mu \Big(\lambda_n/h\Big)^2 G_n' \Big\} \Big(\sigma_{rb} + u_b/a\Big)$$
$$+ \Big(\lambda_n/h\Big)^3 G_n \tau_b + \Big(\lambda_n/h\Big)^2 \Big\{ \Big(1-\mu\Big) G_n'' - \mu \Big(\lambda_n/h\Big)^2 G_n \Big\} W_b$$
$$- \Big(\lambda_n/h\Big) \Big\{ \Big(1-\mu\Big) G_n''' - \mu \Big(\lambda_n/h\Big)^2 G_n' \Big\} u_b \Big] dz \qquad \ldots(23)$$
$$(Re\lambda_n > 0).$$

Using the relation (23), solutions of the stress, displacement and mixed boundary value problems can be obtained. We shall consider here some particular boundary loadings for the stress and displacement problems.

Stress Problem

If the boundary stresses σ_{rb} and τ_b are prescribed, then the unknown displacements are expanded in the form (using eqn. 13).

$$u_b = - \Sigma d_n \, z_n^{(3)} \qquad \ldots(24)$$

$$W_b = -2 \int^{z} \sigma_{rb} dz + \Sigma d_n \frac{K_0(\lambda_n a/h)}{K_1(\lambda_n a/h)} z_n^{(2)} + \frac{2}{a} \Sigma d_n \int^{z} z_n^{(3)} dz. \qquad \ldots(25)$$

Using (23), the determination of the constants d_n can be directly reduced to solving a system of linear algebraic equations in infinitely many unknowns. To illustrate the numerical convergence of the expansions involved we consider the following self-equilibrating load on the boundary $r=a$ viz.,

$$\sigma_{rb} = 7\left(\frac{z}{h}\right)^5 - 10\left(\frac{z}{h}\right)^3 + 3\left(\frac{z}{h}\right) ; \; \tau_b = 5\left(\frac{z}{h}\right)^4 - 6\left(\frac{z}{h}\right)^2 + 1. \qquad \ldots(26)$$

The aspect ratio a/h is set equal to 3.0 and the Poisson's ratio $\mu=0.3$. The resultant system of algebraic equations has been solved using the first five eigen values. The reproduction of the boundary stresses via the eigen function series (14) is given in Table I. The decay of the stresses σ_r and σ_θ is illustrated in Fig. 1 and the decay of the shear stress in Fig. 2. These stresses completely decay out at a distance r equal to $5.0h$.

TABLE I

Numerical Convergence of the Expansions (14) *for the load* (26)

z/h	Specified σ_{rb}	Reproduced σ_r	Specified τ_b	Reproduced τ
0.0	0.0000	0.0000	1.0000	1.1152
0.2	0.5222	0.5654	0.7680	0.6787
0.4	0.6317	0.5630	0.1680	0.2106
0.6	0.1843	0.1968	—0.5120	—0.5165
0.8	—0.4262	—0.4337	—0.7920	—0.7568
1.0	0.0000	—0.1253	0.0000	0.0000

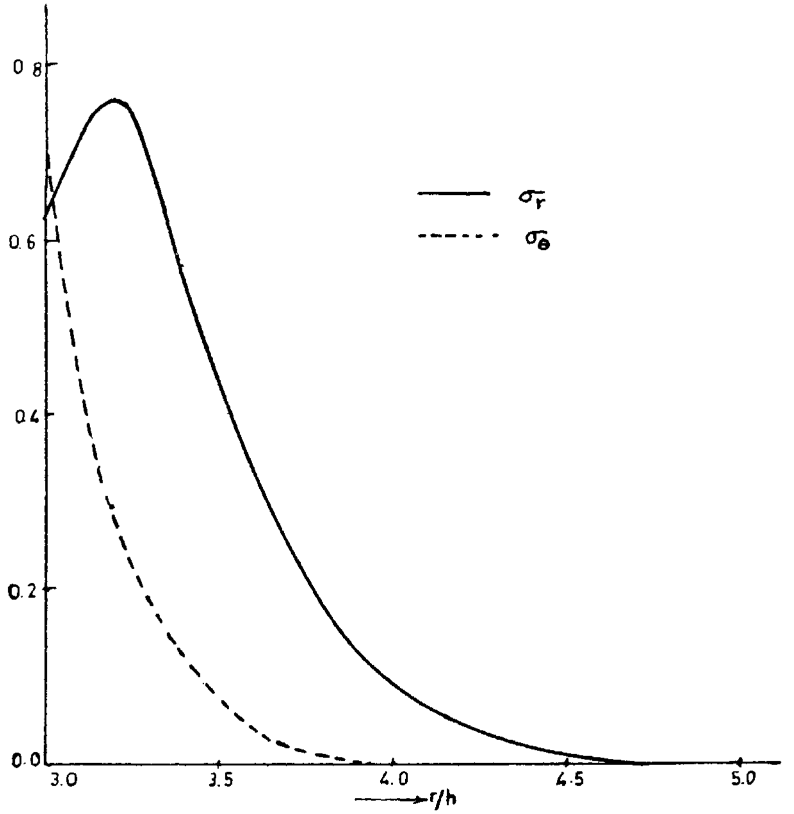

FIG. 1. The decay of the stresses σ_r and σ_θ.

DISPLACEMENT PROBLEM

If the boundary displacements are prescribed, then the unknown stresses σ_{rb} and τ_b can be expanded in the form (using 13).

$$\sigma_{rb} = \frac{1}{2} \sum d_n \frac{K_0(\lambda_n a/h)}{K_1(\lambda_n a/h)} \; z_n^{(2)'} - \frac{1}{2} W_b' - \frac{1}{a} u_b \qquad \text{...(27)}$$

and

$$\tau_b = (1-\mu) \sum d_n z_n^{(4)'} + u_b. \qquad \text{...(28)}$$

Using (23), the determination of the constants d_n can be directly reduced to solving a system of linear algebraic equations in infinitely many unknowns. However, such attempts to obtain the constants d_n poses convergence difficulties. It is extremely important to note that in the displacement problem the stresses and $r=a$, $z=\pm h$ become unbounded on account of the singularities which exist at the circumferential plane ends of the hole. The attention to the existence of such singularities arising from the various boundary conditions at the corners of plates in extension was drawn by Williams (1952). The character of these

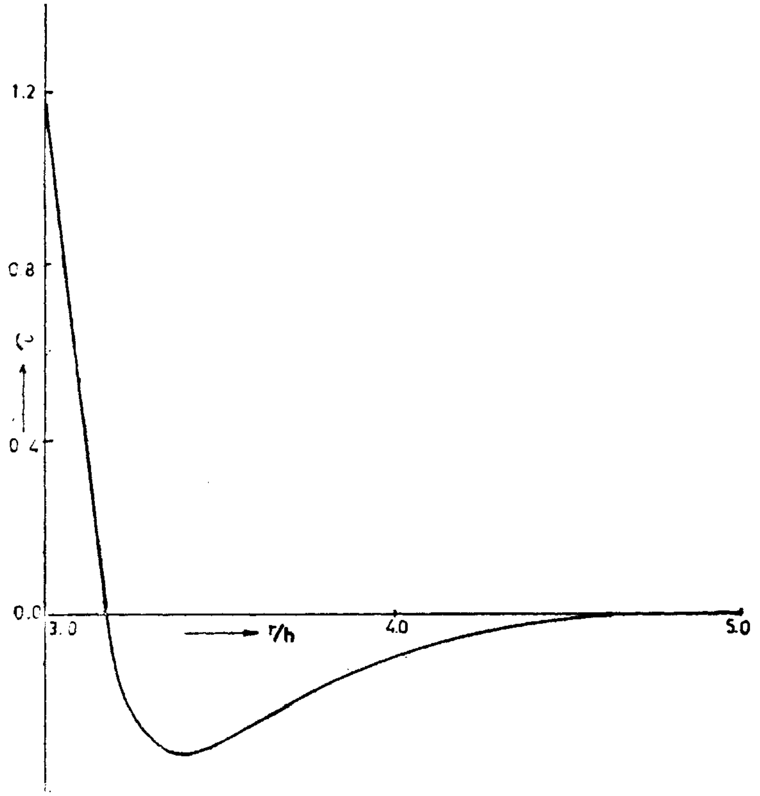

FIG. 2. The decay of the stear stress.

singularities are of the type

$$\sigma = \text{Real}\,(c\,\rho^{-\lambda_0}) \text{ as } \rho \to 0,\ 0 < \text{Re}\,\lambda_0 < 1, \qquad \text{...(29)}$$

where σ is a stress along the plane end of the plate, ρ the distance from the plane end, λ_0 the exponent of the singularity and c the strength of the singularity. If these singularities are not taken into account then whatever be the method for the solution of the problem it is not possible to get the correct stress distribution along the built-in boundary. Benthem (1963) and Benthem and Minderhoud (1972) have shown the method of introducing these singularities in the analysis of the problems.

Thick Infinite Plate with a Built-in Circular Hole under Tension at Infinity

A particular solution for the biharmonic function X which satisfies the stress-free conditions along the plane face $z = \pm h$ and gives rise to all round tension T, is given by

$$\chi_p = \frac{T}{6\,(1+\mu)}\,[2\,(2-\mu)\,z^3 - 2\,(1-\mu)\,r^2\,z]. \qquad \text{...(30)}$$

Since the boundary $r = a$ is built-in, we have the conditions

$$u = W = 0 \text{ on } r = a, \ |z| \leqslant h. \qquad \ldots(31)$$

The displacements on the boundary arising out of (30) are given by

$$\frac{E}{1+\mu} u_p = \frac{1-\mu}{1+\mu} a\, T \, ; \ \frac{E}{1+\mu} W_p = - \frac{2\mu}{1+\mu} T. \qquad \ldots(32)$$

In order to satisfy the conditions (31), the boundary functions u_b and W_b will be taken as

$$u_b = - \frac{1-\mu}{1+\mu} aT; \quad W_b = \frac{2\mu}{1+\mu} zT. \qquad \ldots(33)$$

The unknown boundary stresses σ_{rb} and τ_b will now be taken in the form (cf. Benthem 1963; and Benthem & Minderhoud 1972)

$$\sigma_{rb} = c_1 \left[(1 + z/h)^{-\lambda_0} + (1 - z/h)^{-\lambda_0} - 2^{-\lambda_0}\right] + \sum_{1,3,5} a_m \cos \frac{m\pi z}{2h} - T \qquad \ldots(34)$$

$$\tau_b = + c_2 \left[(1 - z/h)^{-\lambda_0} - (1 + z/h)^{-\lambda_0} + \frac{z}{h} 2^{-\lambda_0}\right] + \sum_{2,4,6} b_n \sin \frac{n\pi z}{2h}, \qquad \ldots(35)$$

where
$$c_2 = - \frac{\lambda_0 + 1 - 2\mu}{\lambda_0 - 2 + 2\mu} \tan\left(\lambda_0 \frac{\pi}{2}\right) c_1.$$

Using (33—35) and the relation (23), it follows after considerable simplification

$$d_n = - \frac{(h/\lambda_n)^4}{N_n} \int_{-h}^{+h} dz \left[(\lambda_n/h)^3 G_n \tau_b + \left\{(1 - 2\mu) G_n''' - 2\mu (\lambda_n/h)^2 G_n' \right\} \sigma_b\right], \ \mathrm{Re}\,\lambda_n > 0, \qquad \ldots(36)$$

where
$$\sigma_b = \sigma_{rb} + T.$$

The condition that the stresses and displacements evaluated from the eigen function series (14) (remembering that this series is composed from the non-zero eigen values) tend to zero for large values of r imposes the restriction that $\mathrm{Re}\,\lambda_n > 0$. It, therefore, follows that

$$d_n(\lambda_n) = 0, \quad \text{if } \mathrm{Re}\,\lambda_n < 0 \qquad \ldots(37)$$

It may be noted that every n in (37) delivers to equations arising from its real and imaginary parts. The integrals in (37) resulting from the singularities in the stresses are developed into a power series or in asymptotic series. The unknowns c_1, a_m, b_n appearing in the system (37) are determined taking into account the self equilibrating condition

$$\int_{-h}^{h} \sigma_{rb}\, dz = 0 \qquad \ldots(38)$$

Calculations are performed taking $\mu = 0.25$. The number of unknowns are increased successively from 3, 5.... up to 11. In the last case the unknowns are $c_1, b_1, a_1, b_2, a_2, \ldots b_5, a_5$. Using these in the expansions (35—36) the stresses σ_{rb} and τ_b are computed. The successive approximations are given in Tables II and III.

It may be noted that the expansions (34 — 35) separate into singular and regular parts. The regular part described the Fourier series is rapidly convergent. The expansions become over complete if a large number of terms of the Fourier series are used and the system of equations become ill-conditioned. However, if the number of terms in the Fourier series is not large the linear independency of the terms with the singular part ensure rapid convergence of the expansions.

The determination of the boundary stresses enables us to calculate the complex constants d_n using the relation (36). With the knowledge of the constant d_n the stress and displacement distribution in the plate is determined using the relations (11), (14) and (5 to 10).

TABLE II

The stress ratios σ_b/T at $r = a$ ('a' arbitrary) for $\mu = 0.25$

$\sigma_b = \sigma_{rb} + T$; $\lambda_0 = 0.25525$; $\rho = 1 - z/h$

z/h	Number of unknowns used				
	3	5	7	9	11
0.000	0.9374	0.9231	0 9239	0.9270	0.9239
0.200	0.9334	0.9263	0.9265	0.9244	0.9274
0.400	0.9256	0.9331	0.9325	0 9321	0.9290
0.600	0.9286	0.9435	0.9433	0.9462	0.9490
0.800	0.9925	0.9967	0.9970	0.9934	0.9909
0.990	1.9476	1.9112	1.9108	1.9179	1.9249
0.995	2.3215	2.2772	2.2767	2.2856	2.2946
0.999	3.4981	3.4304	3.4296	3.4435	3.4577
1·000	$0.59982\rho^{-\lambda_0}$	$0.58819\rho^{-\lambda_0}$	$0.58806\rho^{-\lambda_0}$	$0.59046\rho^{-\lambda_0}$	$0.59292\rho^{-\lambda_0}$

TABLE III

The stress ratios τ_b/T at $r = a$ ('a' arbitrary) for $\mu = 0.25$

z/h	$\lambda_0 = 0.25525$; $\rho = 1 - z/h$				
	Number of unknowns used				
	3	5	7	9	11
0.000	0.0000	0.0000	0.0000	0.0000	0.0000
0.200	− 0.0370	− 0.0375	− 0.0356	− 0.0390	− 0.0387
0.400	− 0.0799	− 0.0783	− 0.0799	− 0.0750	− 0.0759
0.600	− 0.1277	− 0.1271	− 0.1288	− 0.1317	− 0.1300
0.800	− 0.1984	− 0.1956	− 0.1940	− 0.1936	− 0.1947
0.990	− 0.4981	− 0.4885	− 0.4882	− 0.4898	− 0.4915
0.995	− 0.5957	− 0.5842	− 0.5839	− 0.5861	− 0.5884
0.999	− 0.8994	− 0.8820	− 0.8818	− 0.8854	− 0.8890
1.000	$-0.15428\rho^{-\lambda_0}$	$-0.15129\rho^{-\lambda_0}$	$-0.15125\rho^{-\lambda_0}$	$-0.15187\rho^{-\lambda_0}$	$-0.15250\rho^{-\lambda_0}$

CONCLUSIONS

The numerical results presented in Tables I and II clearly indicate the effectiveness of the eigen function analysis presented here for obtaining the solutions of the stress and displacement boundary value problems. A study of the decay of the stresses shows that at a distance $r/h = 5.0$ that is, when the distance from the boundary of the hole is $2/3 \times$ aspect ratio (a/h), the stresses completely decay out.

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